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2^{nd} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

25. Some Linear Algebra:
Let $A, B \in \mathbb{R}^{n\times n}$ symmetric. Show that, if $A = (a_{jk})$ is positive definite and $B = (b_{jk})$ is positive semi-definite, then holds
$$\sum_{j,k=1}^{n} a_{jk} b_{jk} \geq 0.$$

26. Let $\Omega \subset \mathbb{R}^3 \setminus \{0\}$, $u : \Omega \to \mathbb{R}$ harmonic. Show that
$$v(x_1, x_2, x_3) := \frac{1}{|x|} u\Big(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}, \frac{x_3}{|x|^2}\Big)$$
is harmonic on
$$\Omega' := \Big\{x \in \mathbb{R}^3 : \Big(\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}, \frac{x_3}{|x|^2}\Big) \in \Omega\Big\}.$$

27. (Schwarz' reflection principle) Let $\Omega^+ \subset \{x \in \mathbb{R}^n : x_n > 0\}$ such that
$$\Sigma := \partial\Omega^+ \cap \{x \in \mathbb{R}^n : x_n = 0\} \neq \emptyset.$$
Let u harmonic in Ω^+, continuous on $\Omega^+ \cup \Sigma$ and $u(x) = 0$ for $x \in \Sigma$.
We define
$$\bar{u}(x) := \begin{cases} u(x_1, \ldots, x_{n-1}, x_n) & \text{for } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{for } x_n < 0. \end{cases}$$
Show that $\bar{u}$ is harmonic in the interior of $\Omega^+ \cup \Sigma \cup \Omega^-$ with
$$\Omega^- := \{(x_1, \ldots, x_{n-1}, -x_n) : x \in \Omega^+\}.$$
Hint: Use some mean-value characterization for harmonic functions. Maybe you will learn the necessary theory not before the lecture on Monday.

28. Find Green's function for the Laplace equation on the halfspace
$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$
Hint: Take the fundamental solution and copy the idea of Schwarz' reflection principle.