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8^{th} exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

29. Show that the integral kernel of the poisson representation formlua
$$x \mapsto \frac{R^2 - |x|^2}{n\omega_n R}|x-y|^{-n}$$
is harmonic in $B(0,R)$ for fixed $y \in \partial B(0,R)$.

30. Let u harmonic on a convex set $\Omega \subset \mathbb{R}^2$. Show that there exists a conjugate harmonic function $v : \Omega \to \mathbb{R}$ such that the Cauchy-Riemann equations
$$u_x = v_y \qquad u_y = -v_x$$
are satisfied
Hint: for fixed $(x_0, y_0) \in \Omega$ define v by
$$v(x,y) = \int_\gamma u_x \; dy - u_y \; dx,$$
where $\gamma(a) = (x_0, y_0)$ and $\gamma(b) = (x, y)$.
Let $M := \{x + iy : (x,y) \in \Omega\}$. What do we know about about f, which is definied by
$$f : M \to \mathbb{C}, \qquad f(x+iy) := u(x,y) + iv(x,y).$$

31. Let $P(x,y) := xy$ and $\eta \in C_c^\infty$ with $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$.
We set $v := \Delta(\eta P)$ and
$$f(x) := \sum_{k=1}^\infty k^{-1} v(2^k x).$$
Show that f is continuos on $B(0,2)$, but $\Delta u = f$ has no C^2 solution in any neighbourhood of the origin.

32. a) For $n = 2$ we define the function
$$V(x,y) = \frac{1}{8\pi} r^2 \log r, \qquad r = |x-y|.$$
Calculate $\Delta V(x,\cdot)$ and $\Delta^2 V(x,\cdot)$ for $y \neq x$.
b) Let $\Omega \subset \mathbb{R}^n$ with smooth boundary and $u \in C^2(\bar{\Omega})$. Show that for fixed $x \in \Omega$ holds Greens representation formula for Δ^2
$$u(x) = \int_\Omega V \Delta^2 u \; dy - \int_{\partial\Omega} (V \frac{\partial \Delta u}{\partial \nu_y} - \Delta u \frac{\partial V}{\partial \nu_y} + \Delta V \frac{\partial u}{\partial \nu_y} - \frac{\partial \Delta V}{\partial \nu_y}) d\sigma(y).$$
Hint: Apply Greens formula (equation (3) of the lecture) with u replaced by Δu and use Greens representation formula for Δ.