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# $9^{th}$ exercise sheet
PARTIAL DIFFERENTIAL EQUATIONS
WS 2006/07

33. Let $P(x_1, x_2) := x_1 x_2$ and $\eta \in C_c^\infty$ with $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$. We set $v := \Delta(\eta P)$ and

$$f(x) := \sum_{k=1}^{\infty} 16^{-k} v(4^k x).$$

Show that $f$ is continuos on $B(0, 2)$, but $\Delta u = f$ has no $C^2$ solution in any neighbourhood of the origin.

Hints: Consider mainly those $x \in \mathbb{R}^2$ where you know the values of the functions.
In a first step, assume you can always exchange summation and differentiation, to get an idea how to solve the problem.
Consider $\partial_{x_1}\partial_{x_2}(\eta P)$ and $\partial_{x_1}\partial_{x_2} u$.

34. Let $\Omega \subset \mathbb{R}^n$ open and $u \in C^2(\Omega)$. Show that $u$ is subharmonic if and olny if

$$\Delta u \geq 0 \qquad \text{on } \Omega.$$

Hint: Use $\int_{B(x_0,r)} \Delta u \; dy = k \frac{\partial}{\partial r}(S_r u(x_0))$ for $r > 0$ and some $k > 0$.

35. a) Let $u : \mathbb{R}^n \supset B(0, R) \to \mathbb{R}$ harmonic and non negative. Show the following version of Harnack's inequality

$$\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0)$$

for all $x \in B(0, R)$.

Hint: You do not need Harnack's inequality. Yet you could use Corollary 6.13 and the spherical mean characterisation of harmonic functions.

b) Let $u : \mathbb{R}^n \to \mathbb{R}$ harmonic and non negative. Show that $u$ is constant.
Hint: use part a).

36. Let $\Omega \subset \mathbb{R}^n$ bounded and open with $C^1$ boundary. Further let $u \in C^2(\bar{\Omega})$ with $u(x) = 0$ for all $x \in \partial\Omega$. Show that for $\varepsilon > 0$ holds

$$2 \int_\Omega |\nabla u(x)|^2 \; dx \leq \varepsilon \int_\Omega (\Delta u(x))^2 \; dx + \frac{1}{\varepsilon} \int_\Omega u^2(x) \; dx.$$