Exercise 12 Let $(B_t)_{t\geq 0}$ be a real-valued Brownian motion and let $(X_t)_{t\geq 0}$ be the stochastic process defined by $X_t := \int_0^t B_s \,\mathrm{d}s$. Show that:

a) $\mathbb{E}X_t^2 = \frac{1}{3}t^3$ for $t \geq 0$;

b) $\mathbb{E}\exp(\lambda X_t) = \exp(\frac{\lambda^2 t^3}{6})$ for $t \geq 0, \lambda \in \mathbb{R}$.

Solution:

a) Since $\mathrm{Cov}(B_s, B_r) = s \wedge r$ for $r, s \geq 0$ we obtain for $t \geq 0$

$$\begin{aligned}\mathbb{E}\Big(\int_0^t B_s \,\mathrm{d}s\Big)^2 &= \int_\Omega \int_0^t \int_0^t B_s B_r \,\mathrm{d}s \,\mathrm{d}r \,\mathrm{d}\mathbb{P} \overset{\text{Fub}}{=} \int_0^t \int_0^t \mathrm{Cov}(B_s, B_r) \,\mathrm{d}s \,\mathrm{d}r \\ &= \int_0^t \int_0^t s \wedge r \,\mathrm{d}s \,\mathrm{d}r = \int_0^t \Big(\int_0^r s \,\mathrm{d}s + \int_r^t r \,\mathrm{d}s\Big) \,\mathrm{d}r \\ &= \int_0^t \frac{1}{2}r^2 + (t-r)r \,\mathrm{d}r = \frac{1}{6}t^3 + \frac{1}{2}t^3 - \frac{1}{3}t^3 = \frac{1}{3}t^3.\end{aligned}$$

b) Choose $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that $t \mapsto B_t(\omega)$ is continuous for all $\omega \in \Omega_0$. Let $\omega \in \Omega_0$ be fixed and take $t_n^N := n\frac{t}{N}$. Then the set $\{t_0^N, \ldots, t_N^N\}$ is a partition of $[0,t]$ with $\lim_{N\to\infty} \sup_{n=1,\ldots,N} |t_n^N - t_{n-1}^N| = \lim_{N\to\infty} \frac{t}{N} = 0$. Because $t \mapsto B_t(\omega)$ is continuous, it is Riemann-integrable, i.e. the Riemann sum

$$S_t^N(\omega) := \sum_{n=1}^N B_{t_n^N}(\omega)(t_n^N - t_{n-1}^N)$$

converges to $X_t(\omega) = \int_0^t B_s(\omega) \,\mathrm{d}s$ as $N \to \infty$. This means

$$\lim_{N\to\infty} S_t^N = X_t \qquad \text{almost surely.}$$

Moreover, we have

$$\lim_{N\to\infty} \mathbb{E}S_t^N = 0 = \mathbb{E}X_t,$$

and

$$\begin{aligned}\|S_t^N - X_t\|_{L^2(\Omega)} &= \Big\|\int_0^t \sum_{n=1}^N \mathbb{1}_{[t_{n-1}^N, t_n^N]}(s)(B_{t_n^N} - B_s) \,\mathrm{d}s\Big\|_{L^2(\Omega)} \\ &\leq \int_0^t \Big\|\sum_{n=1}^N \mathbb{1}_{[t_{n-1}^N, t_n^N]}(s)(B_{t_n^N} - B_s)\Big\|_{L^2(\Omega)} \,\mathrm{d}s \\ &\leq \sum_{n=1}^N \int_{t_{n-1}^N}^{t_n^N} \|B_{t_n^N} - B_s\|_{L^2(\Omega)} \,\mathrm{d}s = \sum_{n=1}^N \int_{t_{n-1}^N}^{t_n^N} (t_n^N - s)^{\frac{1}{2}} \,\mathrm{d}s \\ &\leq N \sup_{n=1,\ldots,N} |t_n^N - t_{n-1}^N|^{\frac{3}{2}} = \frac{t^{\frac{3}{2}}}{N^{\frac{1}{2}}},\end{aligned}$$

which implies $\lim_{N\to\infty} \|S_t^N - X_t\|_{L^2(\Omega)} = 0$. In particular, we get

$$\lim_{N\to\infty} \mathrm{Var}(S_t^N) = \mathrm{Var}(X_t) \overset{a)}{=} \frac{1}{3}t^3.$$

Since $(B_t)_{t\geq 0}$ is a Gaussian process, $(B_{t_1^N}, \ldots, B_{t_N^N})$ has an N-dimensional normal distribution. And this implies that S_t^N has a 1-dimensional normal distribution.

Using that S_t^N converges to X_t almost surely and $|e^{iaS_t^N}| \leq 1$ for $a \in \mathbb{R}$, the dominated convergence theorem implies that

$$\varphi_{X_t}(a) = \lim_{N\to\infty} \varphi_{S_t^N}(a) \overset{\text{Ex. 3}}{=} \lim_{N\to\infty} e^{-\frac{1}{2}a^2 \mathrm{Var}(S_t^N)} = e^{-\frac{1}{2}a^2\frac{t^3}{3}}.$$

As a consequence, $X_t \sim N(0, \frac{1}{3}t^3)$.

By that, we finally obtain

$$\begin{aligned}
\mathbb{E}\exp(\lambda X_t) &= \frac{1}{\sqrt{2\pi\frac{1}{3}t^3}} \int_{\mathbb{R}} \exp(\lambda x) \exp\big(-\tfrac{3}{2t^3}x^2\big)\,\mathrm{d}x \\
&\overset{y=\sqrt{\frac{3}{t^3}}x}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\big(\lambda\sqrt{\tfrac{t^3}{3}}y\big) \exp\big(-\tfrac{1}{2}y^2\big)\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\big(-\tfrac{1}{2}\big(y - \lambda\sqrt{\tfrac{t^3}{3}}\big)^2\big)\,\mathrm{d}y\ \exp\big(\tfrac{\lambda^2 t^3}{6}\big) \\
&\overset{z=y-\lambda\sqrt{\frac{t^3}{3}}}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\tfrac{1}{2}z^2)\,\mathrm{d}z\ \exp\big(\tfrac{\lambda^2 t^3}{6}\big) \\
&= \exp\big(\tfrac{\lambda^2 t^3}{6}\big).
\end{aligned}$$