Exercise 19 (Properties of conditional expectations)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{A}$ be a sub-σ-algebra of $\mathcal{A}$, and $X \in L^1(\Omega)$. Proof that:

c) For $1 \leq p \leq \infty$ and $X \in L^p(\Omega)$ we have:

$$\big\|E[X|\mathcal{G}]\big\|_{L^p(\Omega)} \leq \|X\|_{L^p(\Omega)}.$$

Solution: c) We first consider the case $p \in [1, \infty)$. Then the function $\phi\colon \mathbb{R} \to \mathbb{R}$, $\phi(t) = |t|^p$ is convex and $\phi(X) \in L^1(\Omega)$ by assumption. This then implies

$$\big\|\mathbb{E}[X|\mathcal{G}]\big\|_{L^p}^p = \mathbb{E}\big(\phi(\mathbb{E}[X|\mathcal{G}])\big) \overset{\text{b)}}{\leq} \mathbb{E}\mathbb{E}[\phi(X)|\mathcal{G}] = \mathbb{E}\phi(X) = \|X\|_{L^p}^p.$$

If $p = \infty$, we use the fact that $0 \leq |X| \leq \|X\|_{L^\infty}$ almost surely, which yields

$$0 \leq \mathbb{E}[|X|\,|\mathcal{G}] \leq \|X\|_{L^\infty}\mathbb{E}[\mathbb{1}_\Omega|\mathcal{G}] = \|X\|_{L^\infty} \quad \text{almost surely.}$$

Therefore, $\mathbb{E}[|X|\,|\mathcal{G}] \in L^\infty(\Omega)$ and $\|\mathbb{E}[X|\mathcal{G}]\|_{L^\infty} \leq \|X\|_{L^\infty}$.

Exercise 21 (Brownian motions and martingales)

Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}$-valued Brownian motion and $\mathcal{F}_t := \sigma(B_s, s \leq t)$, $t \geq 0$. Show that the following processes are martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$:

c) $\big(\cos(\alpha B_t)\exp(\frac{\alpha^2}{2}t)\big)_{t\geq 0}$ and $\big(\sin(\alpha B_t)\exp(\frac{\alpha^2}{2}t)\big)_{t\geq 0}$ for $\alpha \in \mathbb{R}$.

Solution: c) We show two different ways to prove that. In the following let

$$M_t^{(1)} := \cos(\alpha B_t)\exp(\tfrac{\alpha^2}{2}t) \quad \text{and} \quad M_t^{(2)} := \sin(\alpha B_t)\exp(\tfrac{\alpha^2}{2}t).$$

1) We check the properties of the definition of a martingale.

i) **Integrability:** Since $|M_t^{(j)}| \leq \exp(\frac{\alpha^2}{2}t)$ almost surely for each fixed $t \geq 0$ and $j \in \{1, 2\}$, we get $M_t^{(j)} \in L^1(\Omega)$.

ii) **Adaptedness:** Trivially, B_t is $\mathcal{F}_t$-measurable (see also Exercise 4), and that implies that $M_t^{(j)}$ is $\mathcal{F}_t$-measurable as a composition of measurable functions.

iii) **Projection:** We define

$$Y_\alpha(t) := \exp(i\alpha B_t + \tfrac{\alpha^2}{2}t), \quad t \geq 0.$$

Then, similarly as in i) and ii), $Y_\alpha(t)$ is integrable and $(\mathcal{F}_t)$-adapted. Moreover, using that $\mathbb{E}\exp(it\gamma) = \exp(-\frac{1}{2}t^2)$ if $\gamma \sim N(0, 1)$ we obtain as in part a)

$$\begin{aligned}\mathbb{E}[Y_\alpha(t)|\mathcal{F}_s] &= \exp(\tfrac{\alpha^2}{2}t)\exp(i\alpha B_s)\mathbb{E}\big(\exp(i\alpha(B_t - B_s))\big) \\ &= \exp(\tfrac{\alpha^2}{2}t)\exp(i\alpha B_s)\exp(-\tfrac{\alpha^2}{2}(t - s)) \\ &= \exp(\tfrac{\alpha^2}{2}s)\exp(i\alpha B_s).\end{aligned}$$

This then implies

$$\begin{aligned}\mathbb{E}[M_t^{(1)}|\mathcal{F}_s] &= \frac{1}{2}\Big(\mathbb{E}[Y_\alpha(t)|\mathcal{F}_s] + \mathbb{E}[Y_{-\alpha}(t)|\mathcal{F}_s]\Big) = \frac{1}{2}\exp(\tfrac{\alpha^2}{2}s)\big(\exp(i\alpha B_s) + \exp(-i\alpha B_s)\big) \\ &= \exp(\tfrac{\alpha^2}{2}s)\cos(\alpha B_s).\end{aligned}$$

Similarly for $M_t^{(2)}$.

2) If we define $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by $f(t,x) = \cos(\alpha x)\exp(\frac{\alpha^2}{2}t)$, then $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and Itô's formula implies

$$\begin{aligned}
M_t^{(1)} &= f(t,B_t) = f(0,0) + \int_0^t \partial_t f(s,B_s) + \tfrac{1}{2}\partial_{xx} f(s,B_s)\,\mathrm{d}s + \int_0^t \partial_x f(s,B_s)\,\mathrm{d}B_s \\
&= 1 + \int_0^t \tfrac{\alpha^2}{2}\cos(\alpha B_s)\exp(\tfrac{\alpha^2}{2}s) - \tfrac{1}{2}\alpha^2\cos(\alpha B_s)\exp(\tfrac{\alpha^2}{2}s)\,\mathrm{d}s \\
&\quad - \alpha\int_0^t \sin(\alpha B_s)\exp(\tfrac{\alpha^2}{2}s)\,\mathrm{d}B_s \\
&= 1 - \alpha\int_0^t \sin(\alpha B_s)\exp(\tfrac{\alpha^2}{2}s)\,\mathrm{d}B_s.
\end{aligned}$$

Since the integrand $\sin(\alpha B_t)\exp(\frac{\alpha^2}{2}t)(= M_t^{(1)})$ is adapted (see part I)) and

$$\begin{aligned}
\left\| t \mapsto \sin(\alpha B_t)\exp(\tfrac{\alpha^2}{2}t)\right\|_{L^2([0,T]\times\Omega)} &\leq \|\exp(\tfrac{\alpha^2}{2}\cdot)\|_{L^2[0,T]} \\
&= \begin{cases} \frac{1}{|\alpha|}(\exp(\alpha^2 T) - 1)^{\frac{1}{2}} & \text{if } \alpha \neq 0, \\ T^{\frac{1}{2}} & \text{if } \alpha = 0, \end{cases}
\end{aligned}$$

we get that the integrand is actually in $\mathcal{H}^2[0,T]$. By a result proved in the lectures, this implies that $(\int_0^t \sin(\alpha B_s)\exp(\frac{\alpha^2}{2}s)\,\mathrm{d}B_s)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. So $M^{(1)}$ is a martingale as well.

In the same way we can prove that $M^{(2)}$ is a martingale.