**Exercise 3** **d)** If $X = (X_1, \dots, X_n) \sim N(m, C)$, we have

$$X_k \sim N(m_k, c_{kk}) \quad \text{and} \quad \mathrm{Cov}(X)_{jk} := \mathrm{Cov}(X_j, X_k) = c_{jk}$$

for $j, k = 1, \dots, n$.

**Solution:** For any $t \in \mathbb{R}$ we calculate

$$\varphi_{X_k}(t) = \varphi_X(0, \dots, 0_{(k-1)}, t, 0_{(k+1)}, \dots, 0) = e^{itm_k} e^{-\frac{1}{2}t^2 c_{kk}}.$$

By part c) and uniqueness of the characteristic function we obtain $X_k \sim N(m_k, c_{kk})$.

Now let $Y_1, \dots, Y_n$ be independent standard Gaussians and $B = (b_{k\ell})_{k,\ell=1}^n$ as in the previous parts. Then $X \overset{d}{=} BY + m$ and $X_k \overset{d}{=} \sum_{\ell=1}^n b_{k\ell} Y_\ell + m_k$. Using this, we finally get

$$\begin{aligned}
\mathbb{E}X_j X_k &= \mathbb{E}\Big(\sum_{\ell,m=1}^n b_{jm} b_{k\ell} Y_j Y_\ell\Big) + m_j m_k + m_j \underbrace{\mathbb{E}\Big(\sum_{\ell=1}^n b_{k\ell} Y_\ell\Big)}_{=0} + m_k \underbrace{\mathbb{E}\Big(\sum_{m=1}^n b_{jm} Y_m\Big)}_{=0} \\
&= \sum_{\ell,m=1}^n b_{jm} b_{k\ell} \underbrace{\mathbb{E}Y_j Y_\ell}_{=\delta_{j\ell}} + m_j m_k = \sum_{\ell=1}^n b_{j\ell} b_{k\ell} + m_j m_k \\
&= c_{jk} + m_j m_k,
\end{aligned}$$

which implies the claim.

**e)** If $X \sim N(m, C)$, then it holds that

$$X_1, \dots, X_n \text{ are independent} \iff c_{jk} = 0 \text{ for all } j \neq k.$$

**Solution:** If $X_1, \dots, X_n$ are independent, we get for $j \neq k$ by part d)

$$c_{jk} = \mathrm{Cov}(X_j, X_k) = \mathbb{E}X_j X_k - \mathbb{E}X_j \mathbb{E}X_k = \mathbb{E}X_j \mathbb{E}X_k - \mathbb{E}X_j \mathbb{E}X_k = 0.$$

Conversely, if $c_{jk} = 0$ for $j \neq k$, then $C$ is of diagonal form and for the characteristic function of $\mathbb{P}^X$ we obtain

$$\begin{aligned}
\varphi_X(t) &\overset{\text{c)}}{=} e^{it^\top m} e^{-\frac{1}{2}\sum c_{kk} t_k^2} = \prod_{k=1}^n e^{it_k m_k} e^{-\frac{1}{2}c_{kk} t_k^2} \overset{\text{d)}}{=} \prod_{k=1}^n \varphi_{X_k}(t_k) \\
&= \prod_{k=1}^n \int_{\mathbb{R}} e^{it_k x_k} \, \mathrm{d}\mathbb{P}^{X_k}(x_k) \overset{\text{Fub.}}{=} \int_{\mathbb{R}^n} e^{it^\top x} \, \mathrm{d}(\mathbb{P}^{X_1} \otimes \ldots \otimes \mathbb{P}^{X_n})(x).
\end{aligned}$$

Since the characteristic function determines the probability measure uniquely, we conclude that

$$\mathbb{P}^X = \bigotimes_{k=1}^n \mathbb{P}^{X_k},$$

which means that $X_1, \dots, X_n$ are independent.