Exercise 30 b) Let $(B_t)_{t\geq 0} = (B_t^{(1)}, \dots, B_t^{(n)})_{t\geq 0}$ be an n-dimensional Brownian motion, $n \geq 2$. Then we define the Bessel process by

$$R_t := |B_t| = \Big(\sum_{j=1}^n |B_t^{(j)}|^2\Big)^{\frac{1}{2}}, \quad t \geq 0.$$

Proof that $(R_t)_{t\geq 0}$ is a solution of the stochastic Bessel equation, i.e.

$$\mathrm{d}R_t = \frac{n-1}{2R_t}\,\mathrm{d}t + \frac{1}{R_t} B_t \cdot \mathrm{d}B_t.$$

Solution: Since $|\cdot| \notin C^2(\mathbb{R}^n)$, we cannot apply Itô's formula directly. We circumvent this problem by a suitable approximation.

Let $g(y) := \sqrt{y}$. Then $R_t = g(V_t)$, and $V_t = R_t^2$. For g and any $\varepsilon > 0$ we define

$$g_\varepsilon(y) := \begin{cases} \frac{3}{8}\sqrt{\varepsilon} + \frac{3}{4\sqrt{\varepsilon}} y - \frac{1}{8\varepsilon\sqrt{\varepsilon}} y^2, & \text{if } y < \varepsilon, \\ \sqrt{y}, & \text{if } y \geq \varepsilon, \end{cases}$$

where the first line is just a function $f(x) = ax^2 + bx + c$ such that $f(\varepsilon) = g(\varepsilon), f'(\varepsilon) = g'(\varepsilon)$ and $f''(\varepsilon) = g''(\varepsilon)$. Then $g_\varepsilon \in C^2(\mathbb{R})$ (by construction) and $g_\varepsilon(y) \searrow g(y)$ as $\varepsilon \to 0$ and all $y \geq 0$. Moreover, we have

$$g'_\varepsilon(y) = \begin{cases} \frac{3}{4\sqrt{\varepsilon}} - \frac{1}{4\varepsilon\sqrt{\varepsilon}} y, & \text{if } y < \varepsilon, \\ \frac{1}{2\sqrt{y}}, & \text{if } y \geq \varepsilon, \end{cases}$$

$$g''_\varepsilon(y) = \begin{cases} -\frac{1}{4\varepsilon^{3/2}}, & \text{if } y < \varepsilon, \\ -\frac{1}{4y^{3/2}}, & \text{if } y \geq \varepsilon. \end{cases}$$

Applying now Itô's formula, we obtain

$$g_\varepsilon(V_t) = \frac{3}{8}\sqrt{\varepsilon} + \int_0^t g'_\varepsilon(V_s)\,\mathrm{d}V_s + \frac{1}{2}\int_0^t g''_\varepsilon(V_s)(\,\mathrm{d}V_s)(\,\mathrm{d}V_s).$$

By part a) we have $\mathrm{d}V_t = n\,\mathrm{d}t + 2B_t \cdot \mathrm{d}B_t$. Therefore,

$$\begin{aligned} g_\varepsilon(V_t) &= \frac{3}{8}\sqrt{\varepsilon} + \sum_{j=1}^n \int_0^t \Big[\mathbb{1}_{\{V_s \geq \varepsilon\}} \frac{1}{\sqrt{V_s}} + \mathbb{1}_{\{V_s < \varepsilon\}} \frac{1}{2\sqrt{\varepsilon}}\Big(3 - \frac{V_s}{\varepsilon}\Big)\Big] B_s^{(j)}\,\mathrm{d}B_s^{(j)} \\ &\quad + \int_0^t \mathbb{1}_{\{V_s \geq \varepsilon\}} \Big(\frac{n}{2\sqrt{V_s}} - \frac{1}{2}\frac{1}{4V_s^{3/2}} 4|B_t|^2\Big)\,\mathrm{d}s \\ &\quad + \int_0^t \mathbb{1}_{\{V_s < \varepsilon\}} \Big(\frac{3n}{4\sqrt{\varepsilon}} - \frac{n}{4\varepsilon\sqrt{\varepsilon}} V_s - \frac{1}{2}\frac{1}{4\varepsilon\sqrt{\varepsilon}} 4|B_t|^2\Big)\,\mathrm{d}s \\ &= \frac{3}{8}\sqrt{\varepsilon} + \sum_{j=1}^n \underbrace{\int_0^t \Big[\mathbb{1}_{\{V_s \geq \varepsilon\}} \frac{1}{R_s} + \mathbb{1}_{\{V_s < \varepsilon\}} \frac{1}{2\sqrt{\varepsilon}}\Big(3 - \frac{V_s}{\varepsilon}\Big)\Big] B_s^{(j)}\,\mathrm{d}B_s^{(j)}}_{=:I_t^{(j)}(\varepsilon)} \\ &\quad + \underbrace{\int_0^t \mathbb{1}_{\{V_s \geq \varepsilon\}} \frac{n-1}{2R_s}\,\mathrm{d}s}_{=:J_t(\varepsilon)} + \underbrace{\int_0^t \mathbb{1}_{\{V_s < \varepsilon\}} \frac{1}{4\sqrt{\varepsilon}}\Big(3n - (n+2)\frac{V_s}{\varepsilon}\Big)\,\mathrm{d}s}_{=:K_t(\varepsilon)} \\ &= \frac{3}{8}\sqrt{\varepsilon} + \sum_{j=1}^n I_t^{(j)}(\varepsilon) + J_t(\varepsilon) + K_t(\varepsilon). \end{aligned}$$

Next, we look at each summand separately:

1) By monotone convergence we obtain

$$\lim_{\varepsilon\to 0} J_t(\varepsilon) = \int_0^t \mathbb{1}_{\{V_s>0\}} \frac{n-1}{2R_s}\,\mathrm{d}s \overset{\text{Ex. 10 b)}}{=} \int_0^t \frac{n-1}{2R_s}\,\mathrm{d}s \quad \text{almost surely.}$$

2.1) For $n \geq 2$ we have

$$\begin{aligned}\mathbb{P}(V_s < \varepsilon) &\leq \mathbb{P}\big(B_s^{(1)^2} + B_s^{(2)^2} < \varepsilon\big) = \frac{1}{2\pi s}\int_{B(0,\sqrt{\varepsilon})} \exp(-\tfrac{1}{2s}|x|^2)\,\mathrm{d}x \\ &= \frac{1}{s}\int_0^{\sqrt{\varepsilon}} r\exp(-\tfrac{1}{2s}r^2)\,\mathrm{d}r.\end{aligned}$$

Fubini's theorem now implies

$$\int_0^t \mathbb{P}(V_s < \varepsilon)\,\mathrm{d}s \leq \int_0^{\sqrt{\varepsilon}}\int_0^t \frac{r}{s}\exp(-\tfrac{1}{2s}r^2)\,\mathrm{d}s\,\mathrm{d}r \overset{\xi=\frac{1}{2s}r^2}{=} \int_0^{\sqrt{\varepsilon}}\int_{\frac{r^2}{2t}}^{\infty} \frac{r}{\xi}e^{-\xi}\,\mathrm{d}\xi\,\mathrm{d}r,$$

and by L'Hospital we finally get

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\int_0^t \mathbb{P}(V_s < \varepsilon)\,\mathrm{d}s \leq \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{\frac{\varepsilon}{2t}}^{\infty} \xi^{-1/2}e^{-\xi}\,\mathrm{d}\xi = 0,$$

since $\lim_{\varepsilon\to 0}\int_{\frac{\varepsilon}{2t}}^{\infty}\xi^{-1/2}e^{-\xi}\,\mathrm{d}\xi = \Gamma(\tfrac{1}{2}) < \infty$.

2.2) On the one hand this yields

$$0 \leq \lim_{\varepsilon\to 0}\mathbb{E}K_t(\varepsilon) \leq \frac{3n}{4}\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\int_0^t \mathbb{P}(V_s < \varepsilon)\,\mathrm{d}s = 0,$$

and on the other hand Itô isometry gives

$$\begin{aligned}\mathbb{E}\Big|\int_0^t \frac{B_s^{(j)}}{R_s}\,\mathrm{d}B_s^{(j)} - I_t^{(j)}(\varepsilon)\Big|^2 &= \mathbb{E}\int_0^t \mathbb{1}_{\{V_s<\varepsilon\}}\Big|\frac{1}{R_s} - \frac{1}{2\sqrt{\varepsilon}}\Big(3 - \frac{V_s}{\varepsilon}\Big)\Big|^2 B_s^{(j)^2}\,\mathrm{d}s \\ &= \mathbb{E}\int_0^t \mathbb{1}_{\{V_s<\varepsilon\}}\underbrace{\Big[1 - \frac{1}{2}\sqrt{\frac{V_s}{\varepsilon}}\Big(3 - \frac{V_s}{\varepsilon}\Big)\Big]^2}_{\leq 1}\underbrace{\frac{B_s^{(j)^2}}{R_s^2}}_{\leq 1}\,\mathrm{d}s \\ &\leq \int_0^t \mathbb{P}(V_s < \varepsilon)\,\mathrm{d}s \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Choosing a suitable subsequence, we finally get

$$\begin{aligned}R_t = g(V_t) &= \lim_{\varepsilon\to 0} g_\varepsilon(V_t) = \lim_{\varepsilon\to 0}\Big(\frac{3}{8}\sqrt{\varepsilon} + \sum_{j=1}^n I_t^{(j)}(\varepsilon) + J_t(\varepsilon) + K_t(\varepsilon)\Big) \\ &= \int_0^t \frac{n-1}{2R_s}\,\mathrm{d}s + \sum_{j=1}^n \int_0^t \frac{B_s^{(j)}}{R_s}\,\mathrm{d}B_s^{(j)} \\ &= \int_0^t \frac{n-1}{2R_s}\,\mathrm{d}s + \int_0^t \frac{1}{R_s}B_s \cdot \mathrm{d}B_s \qquad \text{almost surely.}\end{aligned}$$