KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

winter term 2018/2019
October 18, 2018

Stochastic differential equations
Exercise sheet 1

Exercise 1 (Weierstrass approximation theorem)

For any function $f \in C[0,1]$ we define the Bernstein polynomial $b_{n,f}$ by

$$b_{n,f}(x) := \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1], n \in \mathbb{N}.$$

Proof that

$$\lim_{n\to\infty} \|f - b_{n,f}\|_\infty = 0,$$

using Markov's inequality.

Hint: If a random variable Z has a binomial distribution $B(n,x)$, then $\mathbb{E}\big(f(\frac{Z}{n})\big) = b_{n,f}(x)$.

Exercise 2 (Characteristic function)

Let $(H, (\cdot,\cdot))$ be a real Hilbert space and $X\colon \Omega \to H$ be a random variable. The we call the map

$$\varphi_X\colon H \to \mathbb{C}, \quad \varphi_X(h) := \mathbb{E}\big[\exp\big(i\,(h,X)\big)\big] = \int_H \exp\big(i\,(h,x)\big)\,\mathrm{d}\mathbb{P}^X(x),$$

the *characteristic function* of X (or $\mathbb{P}^X$, respectively). Show that:

a) $|\varphi_X(h)| \le \varphi_X(0) = 1$ for all $h \in H$;

b) $\varphi_{-X}(h) = \varphi_X(-h) = \overline{\varphi_X}(h)$ for all $h \in H$;

c) $\varphi_{(h,X)}(t) = \varphi_X(th)$ for all $t \in \mathbb{R}$ and $h \in H$;

d) $\varphi_{CX+b}(h) = \exp\big(i\,(h,b)\big)\varphi_X(C^*h)$ for all $h \in H$, $C \in \mathcal{B}(H)$, and $b \in H$;

e) φ_X is uniformly continuous.

Exercise 3 (n-dimensional normal distribution)

In the following let $m \in \mathbb{R}^n$, and $C \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix.

If X is an n-dimensional real random variable with density function

$$f\colon \mathbb{R}^n \to \mathbb{R}, \quad f(x) := \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\Big(-\frac{(x-m)^\top C^{-1}(x-m)}{2}\Big),$$

then X is called *normally distributed*, in short: $X \sim N(m,C)$.

— please turn over —

Show that:

a) If $Y \sim N(0, I)$, then there exists an invertible matrix $B \in \mathbb{R}^{n \times n}$ such that

$$BY + m \sim N(m, C).$$

b) Conversely, if $X \sim N(m, C)$, then there exists an invertible matrix $B \in \mathbb{R}^{n \times n}$ such that

$$B^{-1}(X - m) \sim N(0, I).$$

c) Calculate the characteristic function of an $N(m, C)$ distributed random variable X.

d) If $X = (X_1, \ldots, X_n) \sim N(m, C)$, we have

$$X_k \sim N(m_k, c_{kk}) \quad \text{and} \quad \mathrm{Cov}(X)_{jk} := \mathrm{Cov}(X_j, X_k) = c_{jk}$$

for $j, k = 1, \ldots, n$.

e) If $X \sim N(m, C)$, then it holds that

$$X_1, \ldots, X_n \text{ are independent} \iff c_{jk} = 0 \text{ for all } j \neq k.$$

Hint: In part d) and e) you can use (without proof) that probability measures are uniquely determined by their characteristic function.