KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

winter term 2018/2019
December 20, 2018

Stochastic differential equations
Exercise sheet 10

Exercise 28 (Itô processes)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$.

a) Show that you can rewrite the process $X_t := \frac{B_t}{1+t}$, $t \geq 0$, as

$$\mathrm{d}X_t = -\frac{1}{1+t} X_t \,\mathrm{d}t + \frac{1}{1+t}\,\mathrm{d}B_t, \quad X_0 = 0.$$

b) Show that the process $X_t := \tan(t + B_t)$, $t < \tau := \inf\{s \geq 0 \colon s + B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}]\}$, is the solution of the following stochastic differential equation

$$\mathrm{d}X_t = (1 + X_t)(1 + X_t^2)\,\mathrm{d}t + (1 + X_t^2)\,\mathrm{d}B_t, \quad X_0 = 0.$$

c) Show that the process $X_t := \sin(B_t)$, $t < \tau := \inf\{s \geq 0 \colon B_s \notin [-\frac{\pi}{2}, \frac{\pi}{2}]\}$, can be written as

$$\mathrm{d}X_t = -\frac{1}{2} X_t \,\mathrm{d}t + \sqrt{1 - X_t^2}\,\mathrm{d}B_t, \quad X_0 = 0.$$

d) Show that the process $X_t := \arctan(B_t)$, $t \geq 0$, solves the following stochastic differential equation

$$\mathrm{d}X_t = -\sin(X_t)\cos^3(X_t)\,\mathrm{d}t + \cos^2(X_t)\,\mathrm{d}B_t, \quad X_0 = 0.$$

Exercise 29 (Homogeneous linear stochastic differential equations)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ with augmented Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$. Let X_0 be $\mathcal{F}_0$-measurable, $\mu, \sigma \colon [0,T] \times \Omega \to \mathbb{R}$ be measurable and $(\mathcal{F}_t)_{t\geq 0}$-adapted with $\mu(\cdot, \omega) \in L^1[0,T]$ for $\mathbb{P}$-almost all $\omega \in \Omega$ and $\sigma \in \mathcal{L}^2_{\mathrm{loc}}[0,T]$.

a) Show that the process

$$X_t := X_0 \exp\Big(\int_0^t \big(\mu(s) - \frac{1}{2}\sigma(s)^2\big)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s\Big), \quad t \in [0,T],$$

is a solution of the stochastic differential equation

$$\mathrm{d}X_t = \mu(t) X_t \,\mathrm{d}t + \sigma(t) X_t \,\mathrm{d}B_t$$

with initial value X_0.

b) Let $\tilde{X}$ be another solution of the same equation satisfying $\tilde{X}_0 = X_0$. Proof that $\mathbb{P}(\tilde{X}_t = X_t, \ \forall t) = 1$.

— please turn over —

Exercise 30 (Bessel process)

Let $(B_t)_{t\geq 0} = (B_t^{(1)}, \ldots, B_t^{(n)})_{t\geq 0}$ be an n-dimensional Brownian motion, $n \geq 2$. Then we define the Bessel process by

$$R_t := |B_t| = \Big(\sum_{j=1}^{n} |B_t^{(j)}|^2\Big)^{\frac{1}{2}}, \quad t \geq 0.$$

a) Show that $V_t := R_t^2$ satisfies

$$\mathrm{d}V_t = n\,\mathrm{d}t + 2B_t \cdot \mathrm{d}B_t.$$

b) Proof that $(R_t)_{t\geq 0}$ is a solution of the stochastic Bessel equation, i.e.

$$\mathrm{d}R_t = \frac{n-1}{2R_t}\,\mathrm{d}t + \frac{1}{R_t} B_t \cdot \mathrm{d}B_t.$$

We wish you a Merry Christmas

and a Happy New Year 2019!