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Stochastic differential equations
Exercise sheet 11

Exercise 31 (Uniqueness)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$. We consider the stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t, \quad X_0 = a \in \mathbb{R},$$

for $t \geq 0$. In this case we assume that the coefficients b and σ satisfy the following conditions for any $t \geq 0$ and $x, y \in \mathbb{R}$:

$$|b(t,x) - b(t,y)| \leq K|x-y| \quad \text{and} \quad |\sigma(t,x) - \sigma(t,y)| \leq h(|x-y|),$$

where $K \geq 0$ and $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is strictly monotonically increasing with $h(0) = 0$ such that for all $\varepsilon > 0$

$$\int_0^\varepsilon \frac{1}{h(s)^2}\,\mathrm{d}s = \infty.$$

Show that in this case the equation above has at most one solution.

Hint: Grönwall's lemma.

Exercise 32 (Existence and uniqueness)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$.

a) Show that the stochastic differential equation

$$\mathrm{d}X_t = \ln(1 + X_t^2)\,\mathrm{d}t + \mathbb{1}_{\{X_t>0\}} X_t\,\mathrm{d}B_t, \quad X_0 = a \in \mathbb{R},$$

has a unique solution for $t \geq 0$.

b) For any $\alpha > 0$ and $t \geq 0$ we consider the differential equations

(1) $$\mathrm{d}X_t = |X_t|^\alpha\,\mathrm{d}B_t, \quad X_0 = 0,$$

and

(2) $$\mathrm{d}X_t = |X_t|^\alpha\,\mathrm{d}t, \quad X_0 = 0.$$

i) Show that for $\alpha \geq \frac{1}{2}$ there is a unique solution of (1) and determine this solution.

ii) Show that for $\alpha \in (0,1)$ there is more than one solution of (2) and determine at least two different solutions.

— please turn over —

Exercise 33 **(L^p estimates)**

Let $(B_t)_{t\geq 0}$ be an m-dimensional Brownian motion. We consider the stochastic differential equation

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t, \quad X_0 = \xi,$$

where $\mu\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the assumptions of the existence and uniqueness theorem and

$$\mathbb{E}|\xi|^{2k} < \infty, \quad k \in \mathbb{N}.$$

Now let $(X_t)_{t\geq 0}$ be the corresponding unique solution. Show that in this case we can find for any $T > 0$ and $k \in \mathbb{N}$ a constant $C > 0$ such that

$$\mathbb{E}|X_t|^{2k} \leq (1 + \mathbb{E}|\xi|^{2k})e^{Ct}, \quad t \in [0, T].$$

Hint: Grönwall's lemma.