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Stochastic differential equations

Exercise sheet 12

Exercise 34 (Vasicek model)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$.

a) Let $\theta, \alpha, x_0 \in \mathbb{R}$. Show that the process $(X_t)_{t\geq 0}$ defined by

$$X_t := \frac{\theta}{\alpha} + e^{-\alpha t}\Big(x_0 - \frac{\theta}{\alpha}\Big) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s}\, \mathrm{d}B_s, \quad t \geq 0$$

is the solution of

$$\mathrm{d}X_t = (\theta - \alpha X_t)\, \mathrm{d}t + \sigma\, \mathrm{d}B_t, \quad X_0 = x_0.$$

b) Let $\theta_1, \theta_2, \sigma_1, \sigma_2, x_0, y_0 \in \mathbb{R}$ and $\kappa_1, \kappa_2 > 0$. Consider the following system of stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t &= \kappa_1(\theta_1 - X_t)\, \mathrm{d}t + \sigma_1\, \mathrm{d}B_t, \quad X_0 = x_0, \\ \mathrm{d}Y_t &= \kappa_2(\theta_2 - Y_t)\, \mathrm{d}t + \sigma_2\, \mathrm{d}B_t, \quad Y_0 = y_0. \end{aligned}$$

1) Find the unique solution of this system.
2) Compute $\mathbb{E}X_t$, $t \geq 0$.
3) The parameter θ_1 is called the *long term mean* and the parameter κ_1 is called the *speed of reversion*. Justify these statements.
4) Determine $\mathrm{Var}(X_t)$ and $\mathrm{Cov}(X_t, Y_t)$.

Exercise 35 (Solving stochastic differential equations I)

Let $U \subset \mathbb{R}$ be open, $\sigma\colon U \to \mathbb{R}$ be continuously differentiable, and $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$. We consider the stochastic differential equation

$$\mathrm{d}X_t = \tfrac{1}{2}\sigma(X_t)\sigma'(X_t)\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}B_t, \quad X_0 = x_0 \in U.$$

a) Define the function $F(t) := \int_0^t \sigma(s)^{-1}\, \mathrm{d}s$ and the process $Y_t := F(X_t)$. Show that almost surely

$$F(X_t) - F(X_0) = \int_{X_0}^{X_t} \sigma(s)^{-1}\, \mathrm{d}s = B_t, \quad t \geq 0.$$

b) Using this approach, solve the following stochastic differential equations:

1) $\mathrm{d}X_t = 1\, \mathrm{d}t + 2\sqrt{X_t}\, \mathrm{d}B_t$, $X_0 = x > 0$;
2) $\mathrm{d}X_t = \frac{1}{2}X_t\, \mathrm{d}t + \sqrt{X_t^2 - 1}\, \mathrm{d}B_t$, $X_0 = x > 1$;
3) $\mathrm{d}X_t = -X_t\big(2\log(X_t) + 1\big)\, \mathrm{d}t - 2X_t\sqrt{-\log(X_t)}\, \mathrm{d}B_t$, $X_0 = x \in (0, 1)$.

— please turn over —

Exercise 36 (Solving stochastic differential equations II)

We consider the stochastic differential equation

$$\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + c(t)X_t\,\mathrm{d}B_t, \quad X_0 = x, \tag{1}$$

where $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $c\colon \mathbb{R}_+ \to \mathbb{R}$ are continuous functions.

a) We define the process

$$F_t := \exp\Big(-\int_0^t c(s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t c(s)^2\,\mathrm{d}s\Big), \quad t \geq 0.$$

Show that we can rewrite (1) as

$$\mathrm{d}(F_t X_t) = F_t f(t, X_t)\,\mathrm{d}t, \quad X_0 = x.$$

b) Moreover, we define

$$Y_t := F_t X_t.$$

Show that for each fixed $\omega \in \Omega$ the function $Y_t(\omega)$ solves the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_t(\omega) = F_t(\omega) f\big(t, F_t^{-1}(\omega) Y_t(\omega)\big), \quad Y_0 = x.$$

This then means that $X_t = F_t^{-1} Y_t$ is a solution of (1).

c) Solve the following stochastic differential equation:

$$\mathrm{d}X_t = \frac{1}{X_t}\,\mathrm{d}t + X_t\,\mathrm{d}B_t, \quad X_0 = x > 0.$$

d) Let $\alpha, \gamma \in \mathbb{R}$. Using the approach above, solve the following stochastic differential equation:

$$\mathrm{d}X_t = X_t^\gamma\,\mathrm{d}t + \alpha X_t\,\mathrm{d}B_t, \quad X_0 = x > 0.$$

Does the solution explode for some $\gamma \in \mathbb{R}$?