Exercise 37  (Modelling the evolution of populations)

The nonlinear stochastic differential equation
\[ \text{d}X_t = rX_t(K - X_t) \text{d}t + \beta X_t \text{d}B_t, \quad X_0 = x > 0, \]
is used to model the growth of a population \( X_t \) in a domain with carrying capacity \( K > 0 \) and with environmental noise \( \beta X_t \text{d}B_t \). In this case, \( r \in \mathbb{R} \) is a rate measuring the level of quality of the domain and \( \beta \in \mathbb{R} \) measures the noise in the system.

Show that
\[ X_t = \frac{x \exp((rK - \frac{1}{2}\beta^2)t + \beta B_t)}{1 + rx \int_0^t \exp((rK - \frac{1}{2}\beta^2)s + \beta B_s) \text{d}s}, \quad t \geq 0, \]
is the unique solution of this equation.

Hint: Exercise 36, Bernoulli differential equations.

Exercise 38  (Geometric mean reversion process)

In 1995 Jostein Tvedt used the differential equation
\[ \text{d}X_t = \kappa(\alpha - \log(X_t))X_t \text{d}t + \sigma X_t \text{d}B_t, \quad X_0 = x > 0, \]
in his PhD thesis „Market Structure, Freight Rates and Assets in Bulk Shipping“ to model the spot freight rate in shipping. In this case we assume \( \kappa, \alpha, \sigma > 0 \).

a) Show that the solution of this system is given by
\[ X_t = \exp\left( e^{-\kappa t} \log(x) + (\alpha - \frac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \text{d}B_s \right), \quad t \geq 0. \]

Hint: Use the substitution \( Y_t := \log(X_t) \).

b) Show that
\[ \mathbb{E}X_t = \exp\left( e^{-\kappa t} \log(x) + (\alpha - \frac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \frac{\sigma^2(1 - e^{-2\kappa t})}{4\kappa} \right), \quad t \geq 0. \]

Hint: Exercise 14.
Exercise 39  (Linearized predator-prey model)

Considering two populations \((x_t)_{t \geq 0}\) and \((y_t)_{t \geq 0}\) acting as prey and predator we get the 2-dimensional differential equation

\[
\begin{align*}
\frac{dx}{dt} &= ax - bxy, \quad x(0) = x_0, \\
\frac{dy}{dt} &= -cy + dxy, \quad y(0) = y_0,
\end{align*}
\]

with \(a, b, c, d, x_0, y_0 > 0\). Linearizing this equation at the steady state \((\frac{c}{d}, \frac{a}{b})\) by the first order Taylor polynomial leads to

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{bc}{d} \left( y - \frac{a}{b} \right), \quad x(0) = x_0, \\
\frac{dy}{dt} &= \frac{ad}{b} \left( x - \frac{c}{d} \right), \quad y(0) = y_0.
\end{align*}
\]

By defining \(u_t := x_t - \frac{c}{d}\) and \(v_t := y_t - \frac{a}{b}\) we get the linear equation

\[
\begin{align*}
\frac{du}{dt} &= -\frac{bc}{d} v, \quad u(0) = u_0 := x_0 - \frac{c}{d}, \\
\frac{dv}{dt} &= \frac{ad}{b} u, \quad v(0) = v_0 := y_0 - \frac{a}{b}.
\end{align*}
\]

We next add a Gaussian noise to the parameters \(a\) and \(c\) of the form \(a + \sigma_1 B_t^t\) and \(c + \sigma_2 B_t^t\), where \(\sigma_1, \sigma_2 \geq 0\) and \((B_t^t)_{t \geq 0}\) is a Brownian motion in \(\mathbb{R}\). This leads to the stochastic differential equation

\[
\begin{align*}
\frac{du}{dt} &= -\frac{bc}{d} v_t \, dt - \frac{\sigma_1 b}{d} v_t \, dB_t, \quad u(0) = u_0, \\
\frac{dv}{dt} &= \frac{ad}{b} u_t \, dt + \frac{\sigma_2 d}{b} u_t \, dB_t, \quad v(0) = v_0.
\end{align*}
\]

(which corresponds to the inhomogeneous stochastic differential equation

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{c}{d} (a - by_t) \, dt + \frac{\sigma_1}{d} (a - by_t) \, dB_t, \quad x(0) = x_0, \\
\frac{dy}{dt} &= \frac{a}{b} (dx_t - c) \, dt + \frac{\sigma_2}{b} (dx_t - c) \, dB_t, \quad y(0) = y_0.
\end{align*}
\]

a) Define \(Z_t := (u_t, v_t)\) and rewrite this equation as a vector valued equation of the form

\[
\frac{dZ}{dt} = AZ \, dt + CZ \, dB_t, \quad Z(0) = Z_0,
\]

for certain matrices \(A, C \in \mathbb{R}^{2 \times 2}\).

b) Solve the vector valued equation by introducing the integrating factor

\[
F_t := \exp(-B_t C + \frac{1}{2} t C^2), \quad t \geq 0,
\]

(see also Exercise 36) and assuming that \(c\sigma_2 = a\sigma_1\).

c) Finally, calculate the solutions \(u_t\) and \(v_t\) explicitly (or \(x_t\) and \(y_t\), respectively).