KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT) winter term 2018/2019
Institute for Analysis January 24, 2019
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Stochastic differential equations
Exercise sheet 13

Exercise 37 (Modelling the evolution of populations)

The nonlinear stochastic differential equation

$$\mathrm{d}X_t = rX_t(K - X_t)\,\mathrm{d}t + \beta X_t\,\mathrm{d}B_t, \quad X_0 = x > 0,$$

is used to model the growth of a population X_t in a domain with carrying capacity $K > 0$ and with environmental noise $\beta X_t\,\mathrm{d}B_t$. In this case, $r \in \mathbb{R}$ is a rate measuring the level of quality of the domain and $\beta \in \mathbb{R}$ measures the noise in the system.

Show that

$$X_t = \frac{x\exp\big((rK - \frac{1}{2}\beta^2)t + \beta B_t\big)}{1 + rx\int_0^t \exp\big((rK - \frac{1}{2}\beta^2)s + \beta B_s\big)\,\mathrm{d}s}, \quad t \geq 0,$$

is the unique solution of this equation.

Hint: Exercise 36, Bernoulli differential equations.

Exercise 38 (Geometric mean reversion process)

In 1995 Jostein Tvedt used the differential equation

$$\mathrm{d}X_t = \kappa\big(\alpha - \log(X_t)\big)X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t, \quad X_0 = x > 0,$$

in his PhD thesis „Market Structure, Freight Rates and Assets in Bulk Shipping“ to model the spot freight rate in shipping. In this case we assume $\kappa, \alpha, \sigma > 0$.

a) Show that the solution of this system is given by

$$X_t = \exp\Big(e^{-\kappa t}\log(x) + (\alpha - \tfrac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\,\mathrm{d}B_s\Big), \quad t \geq 0.$$

Hint: Use the substitution $Y_t := \log(X_t)$.

b) Show that

$$\mathbb{E}X_t = \exp\Big(e^{-\kappa t}\log(x) + (\alpha - \tfrac{\sigma^2}{2\kappa})(1 - e^{-\kappa t}) + \frac{\sigma^2(1 - e^{-2\kappa t})}{4\kappa}\Big), \quad t \geq 0.$$

Hint: Exercise 14.

— please turn over —

Exercise 39 (Linearized predator-prey model)

Considering two populations $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ acting as prey and predator we get the 2-dimensional differential equation

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}x &= ax - bxy, \quad x(0) = x_0,\\ \tfrac{\mathrm{d}}{\mathrm{d}t}y &= -cy + dxy, \quad y(0) = y_0,\end{aligned}$$

with $a, b, c, d, x_0, y_0 > 0$. Linearizing this equation at the steady state $(\frac{c}{d}, \frac{a}{b})$ by the first order Taylor polynomial leads to

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}x &= -\frac{bc}{d}\Big(y - \frac{a}{b}\Big), \quad x(0) = x_0,\\ \tfrac{\mathrm{d}}{\mathrm{d}t}y &= \frac{ad}{b}\Big(x - \frac{c}{d}\Big), \quad y(0) = y_0.\end{aligned}$$

By defining $u_t := x_t - \frac{c}{d}$ and $v_t := y_t - \frac{a}{b}$ we get the linear equation

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}u &= -\frac{bc}{d}v, \quad u(0) = u_0 := x_0 - \frac{c}{d},\\ \tfrac{\mathrm{d}}{\mathrm{d}t}v &= \frac{ad}{b}u, \quad v(0) = v_0 := y_0 - \frac{a}{b}.\end{aligned}$$

We next add a Gaussian noise to the parameters a and c of the form $a+\sigma_1 B'_t$ and $c+\sigma_2 B'_t$, where $\sigma_1, \sigma_2 \geq 0$ and $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}$. This leads to the stochastic differential equation

$$\begin{aligned}\mathrm{d}u_t &= -\frac{bc}{d}v_t\,\mathrm{d}t - \frac{\sigma_1 b}{d}v_t\,\mathrm{d}B_t, \quad u(0) = u_0,\\ \mathrm{d}v_t &= \frac{ad}{b}u_t\,\mathrm{d}t + \frac{\sigma_2 d}{b}u_t\,\mathrm{d}B_t, \quad v(0) = v_0.\end{aligned}$$

(which corresponds to the inhomogeneous stochastic differential equation

$$\begin{aligned}\mathrm{d}x_t &= -\frac{c}{d}(a - by_t)\,\mathrm{d}t + \frac{\sigma_1}{d}(a - by_t)\,\mathrm{d}B_t, \quad x(0) = x_0,\\ \mathrm{d}y_t &= \frac{a}{b}(dx_t - c)\,\mathrm{d}t + \frac{\sigma_2}{b}(dx_t - c)\,\mathrm{d}B_t, \quad y(0) = y_0.)\end{aligned}$$

a) Define $Z_t := (u_t, v_t)$ and rewrite this equation as a vector valued equation of the form

$$\mathrm{d}Z_t = AZ_t\,\mathrm{d}t + CZ_t\,\mathrm{d}B_t, \quad Z(0) = Z_0,$$

for certain matrices $A, C \in \mathbb{R}^{2\times 2}$.

b) Solve the vector valued equation by introducing the integrating factor

$$F_t := \exp(-B_t C + \frac{1}{2}tC^2), \quad t \geq 0,$$

(see also Exercise 36) and assuming that $c\sigma_2 = a\sigma_1$.

c) Finally, calculate the solutions u_t and v_t explicitly (or x_t and y_t, respectively).