KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

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Stochastic differential equations
Exercise sheet 2

Exercise 4 ($\sigma(X)$-measurable random variables)

Let $(\Omega, \mathcal{A})$ be a measurable space.

a) Let $a_1, \dots, a_n \in \mathbb{R}$ be pairwise distinct and $A_1, \dots, A_n \in \mathcal{A}$ be pairwise disjoint with $\Omega = \bigcup_{j=1}^n A_j$. Let $X := \sum_{j=1}^n a_j \mathbb{1}_{A_j}$ and $\sigma(X)$ be the σ-algebra generated by X.
   i) Determine $\sigma(X)$.
   ii) Let Y be a $\sigma(X)$-measurable random variable. Show that Y is constant on each set A_j, $j = 1, \dots, n$.
   iii) Show that Y can be written as a function of X.

b) Let $X\colon \Omega \to \mathbb{R}$ be an arbitrary random variable and let $Y\colon \Omega \to \mathbb{R}$ be $\sigma(X)$-measurable. Show that there exists a function Φ such that

$$Y = \Phi(X).$$

   Hint: algebraic induction.

Exercise 5 (Product probability spaces)

Let $(\Omega_j, \mathcal{A}_j, \mathbb{P}_j)$, $j \in \mathbb{N}$, be probability spaces,

$$\Omega := \bigtimes_{j=1}^{\infty} \Omega_j, \quad \text{and} \quad \mathcal{A} := \bigotimes_{j=1}^{\infty} \mathcal{A}_j := \sigma(\pi_j, j \in \mathbb{N})$$

the product σ-algebra generated by the projections $\pi_j\colon \Omega \to \Omega_j$ ($\pi_j(\omega) = \omega_j$, $j \in \mathbb{N}$). Show that there exists a uniquely determined probability measure $\mathbb{P}$ on $\mathcal{A}$ satisfying

$$\mathbb{P}\big(A_1 \times \ldots \times A_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j\big) = \prod_{j=1}^{n} \mathbb{P}_j(A_j)$$

for any sets $A_j \in \mathcal{A}_j$, $j = 1, \dots, n$.

Notation: $\mathbb{P} = \bigotimes_{j=1}^\infty \mathbb{P}_j$ and $\bigotimes_{j=1}^\infty (\Omega_j, \mathcal{A}_j, \mathbb{P}_j) := (\bigtimes_{j=1}^\infty \Omega_j, \bigotimes_{j=1}^\infty \mathcal{A}_j, \bigotimes_{j=1}^\infty \mathbb{P}_j)$.

Hints for the proof:

1) For $n \in \mathbb{N}$ define

$$\mathcal{F}_n := \big\{B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j \colon B_n \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n\big\}$$

   and show that this is an ascending sequence of σ-algebras.

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2) For $\mathcal{F} := \bigcup_{n=1}^{\infty} \mathcal{F}_n$ show that $\mathcal{F} \subset \mathcal{A}$ and $\sigma(\mathcal{F}) = \mathcal{A}$.

3) On $\mathcal{F}_n$ define the probability measure

$$\mu_n\big(B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j\big) := \mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n(B_n)$$

and on $\mathcal{F}$ the map $\mathbb{P}$ by

$$\mathbb{P}(A) := \mu_n(A) \quad \text{if } A \in \mathcal{F}_n.$$

4) Show that $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}$ is σ-additive.

5) Finally, use Caratheodory's extension theorem as well as the measure uniqueness theorem to conclude the proof.

Exercise 6 (Infinite and independent coin toss)

Let $(\Omega, \mathcal{A}, \mathbb{P}) = \bigotimes_{j=1}^{\infty} \big(\{0,1\}, \mathcal{P}(\{0,1\}), \mathbb{P}_j\big)$ with $\mathbb{P}_j(\{0\}) = p$ and $\mathbb{P}_j(\{1\}) = 1-p$, $0 < p < 1$, $j \in \mathbb{N}$, the stochastic model of an infinite sequence of independent coin tosses with a not necessarily fair coin. We now continuously flip the coin and stop when the coin shows head (head = 1). Calculate the probabilities of the following events:

a) A: we flip the coin at least k times ($k \in \mathbb{N}$);

b) B: the amount of coin flips is even.