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Stochastic differential equations
Exercise sheet 3

Exercise 7 (Sums of independent random variables)

Let X, Y be $\mathbb{R}$-valued and independent random variables with densities f_X and f_Y, respectively.

a) Show that the random variable $Z := X + Y$ has the density

$$f_Z(z) = \int_{\mathbb{R}} f_X(x) f_Y(z-x)\,\mathrm{d}x = (f_X * f_Y)(z), \qquad z \in \mathbb{R}.$$

b) Calculate the characteristic function of Z (see Exercise 2).

Exercise 8 (Gaussian sums)

Let $1 \le p < \infty$, $(\gamma_i)_{i=1}^n$ be a Gaussian sequence, and $(\alpha_i)_{i=1}^n \subset \mathbb{R}$. Show that there exists a costant $C_p > 0$ such that

$$\Big(\mathbb{E}\Big|\sum_{i=1}^n \alpha_i \gamma_i\Big|^p\Big)^{\frac{1}{p}} = C_p \Big(\sum_{i=1}^n |\alpha_i|^2\Big)^{\frac{1}{2}}.$$

Hint: Determine the distribution of the random variable $\sum_{i=1}^n \alpha_i \gamma_i$, for example by calculating its characteristic function (see Exercise 2 and 3).

Exercise 9 (Brownian motion)

Let $(B_t)_{t\ge 0} = (B_t^{(1)}, \ldots, B_t^{(n)})_{t \ge 0}$ be an n-dimensional Brownian motion. Show the following properties:

a) For any fixed $s \ge 0$ the process $(X_t)_{t\ge 0} := (B_{s+t} - B_s)_{t \ge 0}$ is again a Brownian motion;

b) For each $x \in \mathbb{R}^n$ with $|x| = 1$ the process $(x^\top B_t)_{t\ge 0}$ is a Brownian motion in $\mathbb{R}$;

c) For any choice of $0 \le t_1 < \ldots < t_m$, $m \in \mathbb{N}$, and any $i \in \{1, \ldots, n\}$ we have

$$(B_{t_1}^{(i)}, \ldots, B_{t_m}^{(i)}) \sim N(0, (\min\{t_k, t_l\})_{k,l=1,\ldots,m}).$$