KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

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Stochastic differential equations
Exercise sheet 4

Exercise 10 (Properties of a Brownian motion)

Let $(B_t)_{t\geq 0}$ be a real-valued Brownian motion. Show that:

a) $\mathbb{E}B_t^{2k} = \frac{(2k)!t^k}{2^k k!}$ for $t \geq 0$, $k \in \mathbb{N}$;

b) $\lambda\big(\{t \geq 0 \colon B_t = u\}\big) = 0$ $\mathbb{P}$-almost surely for all $u \in \mathbb{R}$;

c) $\lim_{n\to\infty} \frac{1}{n} B_n = 0$ $\mathbb{P}$-almost surely.

Hint for part c): Use the Lemma of Borel-Cantelli.

Exercise 11 (Brownian motion once again)

Let $(B_t)_{t\geq 0}$ be a real-valued Brownian motion. Shot that

a) $(X_t)_{t\geq 0}$ defined by $X_t := \frac{1}{a} B_{a^2 t}$ for any $a \neq 0$ (Brownian scaling), and

b) $(Y_t)_{t\geq 0}$ defined by $Y_t := tB_{\frac{1}{t}}$ for $t > 0$ and $Y_0 = 0$ (time inversion),

are Brownian motions.

Exercise 12 (Integrated Brownian motion)

Let $(B_t)_{t\geq 0}$ be a real-valued Brownian motion and let $(X_t)_{t\geq 0}$ be the stochastic process defined by $X_t := \int_0^t B_s \,\mathrm{d}s$. Show that:

a) $\mathbb{E}X_t^2 = \frac{1}{3}t^3$ for $t \geq 0$;

b) $\mathbb{E}\exp(\lambda X_t) = \exp(\frac{\lambda^2 t^3}{6})$ for $t \geq 0, \lambda \in \mathbb{R}$.