Stochastic differential equations

Exercise sheet 4

Exercise 10  (Properties of a Brownian motion)

Let \((B_t)_{t \geq 0}\) be a real-valued Brownian motion. Show that:

a) \(\mathbb{E}B_t^2 = \frac{(2k)t^k}{2^k k!}\) for \(t \geq 0, \ k \in \mathbb{N}\);

b) \(\lambda(\{t \geq 0: B_t = u\}) = 0\) \(\mathbb{P}\)-almost surely for all \(u \in \mathbb{R}\);

c) \(\lim_{n \to \infty} \frac{1}{n} B_n = 0\) \(\mathbb{P}\)-almost surely.

Hint for part c): Use the Lemma of Borel-Cantelli.

Exercise 11  (Brownian motion once again)

Let \((B_t)_{t \geq 0}\) be a real-valued Brownian motion. Show that

a) \((X_t)_{t \geq 0}\) defined by \(X_t := \frac{1}{a}B_{at}\) for any \(a \neq 0\) (Brownian scaling), and

b) \((Y_t)_{t \geq 0}\) defined by \(Y_t := tB_{\frac{1}{t}}\) for \(t > 0\) and \(Y_0 = 0\) (time inversion),

are Brownian motions.

Exercise 12  (Integrated Brownian motion)

Let \((B_t)_{t \geq 0}\) be a real-valued Brownian motion and let \((X_t)_{t \geq 0}\) be the stochastic process defined by \(X_t := \int_0^t B_s \, ds\). Show that:

a) \(\mathbb{E}X_t^2 = \frac{1}{3} t^3\) for \(t \geq 0\);

b) \(\mathbb{E} \exp(\lambda X_t) = \exp(\frac{\lambda^2 t^3}{6})\) for \(t \geq 0, \lambda \in \mathbb{R}\).