KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

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Stochastic differential equations

Exercise sheet 5

Exercise 13 (Integration by parts)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ and $\phi \in C^1[0,T]$. Proof that:

$$\int_0^T \phi(t)\,\mathrm{d}B_t = \phi(T)B_T - \int_0^T \phi'(t)B_t\,\mathrm{d}t \qquad \text{almost surely.}$$

Exercise 14 (Properties of the Wiener integral process)

a) Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ and $f \in L^2[0,T]$. Then we define the stochastic process $(X_t)_{t\in[0,T]}$ by

$$X_t := \int_0^t f(s)\,\mathrm{d}B_s, \quad t \in [0,T].$$

Show the following properties:

i) $X_t \sim N\big(0, \int_0^t f(s)^2\,\mathrm{d}s\big)$, $t \in [0,T]$;

ii) $(X_t)_{t\in[0,T]}$ has independent increments.

b) Let $(B_t)_{t\geq 0}$ be an n-dimensional Brownian motion and $F \in L^2([0,T],\mathbb{R}^{m\times n})$. Proof that

$$Y_t := \int_0^t F(s)\,\mathrm{d}B_s, \quad t \in [0,T],$$

has an m-dimensional normal distribution, and determine its covariance matrix.

Exercise 15 (Stochastic version of Fubini's theorem)

Let $\phi\colon \Omega\times[0,T]\times[0,T]\to\mathbb{R}$ be measurable, $(\phi(s,t))_{s\in[0,T]}$ be adapted with respect to the Brownian filtration for all $t\in[0,T]$, and

$$\int_0^T \Big(\int_0^T \mathbb{E}|\phi(s,t)|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\,\mathrm{d}t < \infty.$$

Proof that:

a) $(\omega,s)\mapsto\phi(\omega,s,t)\in L^2(\Omega\times[0,T])$ for almost all $t\in[0,T]$ and $t\mapsto\int_0^T\phi(s,t)\,\mathrm{d}B_s\in L^1([0,T],L^2(\Omega))$;

b) $t\mapsto\phi(\omega,s,t)\in L^1[0,T]$ for almost all $(\omega,s)\in\Omega\times[0,T]$, $(\omega,s)\mapsto\int_0^T\phi(s,t)\,\mathrm{d}t\in L^2(\Omega\times[0,T])$ and $\big(\int_0^T\phi(s,t)\,\mathrm{d}t\big)_{s\in[0,T]}$ is adapted with respect to the Brownian filtration;

c) We have

$$\int_0^T\int_0^T\phi(s,t)\,\mathrm{d}B_s\,\mathrm{d}t = \int_0^T\int_0^T\phi(s,t)\,\mathrm{d}t\,\mathrm{d}B_s \quad \text{in } L^2(\Omega).$$