KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT) winter term 2018/2019
Institute for Analysis November 12, 2018
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Stochastic differential equations
Exercise sheet 6

Exercise 16 (Itô integrals)

Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}$-valued Brownian motion.

a) For $t \geq 0$ determine the variances of the following integrals:

   i) $\int_0^t |B_s|^{\frac{1}{2}} \,\mathrm{d}B_s$;

   ii) $\int_0^t (B_s + s)^2 \,\mathrm{d}B_s$.

b) For $t \geq 0$ calculate the following Itô integrals:

   i) $\int_0^t B_s^2 \,\mathrm{d}B_s$;

   ii) $\int_0^t B_s^n \,\mathrm{d}B_s, \quad n \in \mathbb{N}$.

Exercise 17 (Hermite polynomials)

For any $n \in \mathbb{N}_0$ we define the Hermite polynomial by

$$h_n(t,x) := \frac{(-t)^n}{n!} e^{\frac{x^2}{2t}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\frac{x^2}{2t}}, \quad t,x \in \mathbb{R}.$$

Show that:

$$\int_0^t h_n(s,B_s)\,\mathrm{d}B_s = h_{n+1}(t,B_t), \quad t \geq 0, n \in \mathbb{N}_0.$$

Exercise 18 (Kolmogorov)

Let $X\colon [0,1] \times \Omega \to \mathbb{R}$ be a stochastic process with almost surely continuous paths such that

$$\mathbb{E}\big(|X(t) - X(s)|^\beta\big) \leq c\,|t-s|^{1+\alpha}$$

for $\alpha, \beta > 0$, $c \geq 0$, and $s,t \in [0,1]$. Show that for all $\gamma \in (0, \frac{\alpha}{\beta})$ and almost all $\omega \in \Omega$ there is a constant $C = C(\omega)$ such that

$$|X(t,\omega) - X(s,\omega)| \leq C(\omega)|t-s|^\gamma$$

for $s,t \in [0,1]$. This implies that the path $t \mapsto X(t,\omega)$ is γ-Hölder continuous for almost all $\omega \in \Omega$.

— please turn over —

Hints for the proof:

1) For $0 < \gamma < \frac{\alpha}{\beta}$ and $n \in \mathbb{N}$ we consider the set

$$A_n := \left\{ |X(\tfrac{i+1}{2^n}) - X(\tfrac{i}{2^n})| > \tfrac{1}{2^{n\gamma}} \text{ for some } i \in \{0, \dots, 2^n - 1\} \right\}.$$

   Now estimate $\mathbb{P}(A_n)$ appropriately.

2) Show that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, and apply the Lemma of Borel-Cantelli to get

$$|X(\tfrac{i+1}{2^n}, \omega) - X(\tfrac{i}{2^n}, \omega)| \leq K(\omega) \tfrac{1}{2^{n\gamma}}$$

   for some $K(\omega) > 0$, all $i \in \{0, \dots, 2^n - 1\}$, and almost all $\omega \in \Omega$.

3) Using 2), prove that

$$|X(t_1, \omega) - X(t_2, \omega)| \leq C(\omega) |t_1 - t_2|^{\gamma}$$

   for all dyadic rationals $t_1, t_2 \in [0, 1]$ and a constant $C = C(\omega) > 0$.

4) Finally, use that X has almost surely continuous paths and 3) to conclude the proof.