KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

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# Stochastic differential equations

## Exercise sheet 7

### Exercise 19 (Properties of conditional expectations)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{A}$ be a sub-$\sigma$-algebra of $\mathcal{A}$, and $X \in L^1(\Omega)$. Proof that:

a) If $X \geq 0$, then $\mathbb{E}[X|\mathcal{G}] \geq 0$.

b) If $\phi\colon \mathbb{R} \to \mathbb{R}$ is a convex function such that $\phi(X) \in L^1(\Omega)$, then
$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}] \qquad \text{almost surely.}$$

c) For $1 \leq p \leq \infty$ and $X \in L^p(\Omega)$ we have:
$$\big\|E[X|\mathcal{G}]\big\|_{L^p(\Omega)} \leq \|X\|_{L^p(\Omega)}.$$

### Exercise 20 (Discrete-time martingales)

a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and integrable random variables, and $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$, $n \in \mathbb{N}$. Proof the following statements:

  i) If $\mathbb{E}X_n = 0$ for all $n \in \mathbb{N}$, then $S_n := \sum_{k=1}^n X_k$, $n \in \mathbb{N}$, is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

  ii) If $\mathbb{E}X_n = 0$ and $\operatorname{Var}(X_n) = \sigma^2$ for all $n \in \mathbb{N}$, then $M_n := S_n^2 - n\sigma^2$, $n \in \mathbb{N}$, is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

  iii) If $X_n \geq 0$ and $\mathbb{E}X_n = 1$ for all $n \in \mathbb{N}$, then $P_n := \prod_{k=1}^n X_k$, $n \in \mathbb{N}$, is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

  iv) If all $X_n$ have the same distribution with $\phi(\lambda) := \mathbb{E}(\exp(\lambda X_1)) < \infty$ for some $\lambda \in \mathbb{R}$, then $L_n := \frac{1}{\phi(\lambda)^n}\exp(\lambda S_n)$, $n \in \mathbb{N}$, is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

b) Let $(M_n)_{n\in\mathbb{N}}$ be a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, and let $(v_n)_{n\in\mathbb{N}}$ be a sequence of bounded random variables such that $v_n$ is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$ (a sequence like that is usually called *predictable* with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$). Show that
$$(v * M)_n := \sum_{k=1}^n v_k(M_k - M_{k-1}), \quad n \in \mathbb{N},$$
is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ (in this case we let $M_0 := 0$ and $\mathcal{F}_0 := \{\varnothing, \Omega\}$).

— please turn over —

**Exercise 21 (Brownian motions and martingales)**

Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}$-valued Brownian motion and $\mathcal{F}_t := \sigma(B_s, s \leq t)$, $t \geq 0$. Show that the following processes are martingales with respect to $(\mathcal{F}_t)_{t\geq 0}$:

a) $\big(\exp(\alpha B_t - \frac{\alpha^2}{2} t)\big)_{t\geq 0}$ for any $\alpha \in \mathbb{R}$;

b) $(B_t^3 - 3tB_t)_{t\geq 0}$;

c) $\big(\cos(\alpha B_t)\exp(\frac{\alpha^2}{2} t)\big)_{t\geq 0}$ and $\big(\sin(\alpha B_t)\exp(\frac{\alpha^2}{2} t)\big)_{t\geq 0}$ for $\alpha \in \mathbb{R}$.