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Stochastic differential equations
Exercise sheet 8

Exercise 22 (Stopping times)

a) Let τ_1, τ_2 be two stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Proof that $\min(\tau_1, \tau_2)$, $\max(\tau_1, \tau_2)$, $\tau_1 + \tau_2$, and $\alpha\tau_1$ for $\alpha \geq 1$ are stopping times with respect to $(\mathcal{F}_t)_{t\geq 0}$ as well.

b) Give an example of stopping times τ_1, τ_2, such that $\tau_1 \leq \tau_2$, but $\tau_2 - \tau_1$ is not a stopping time.

Exercise 23 (Hitting times of a Brownian motion)

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ be the augmented Brownian filtration, i.e. for all $t \geq 0$ the σ-algebra $\mathcal{F}_t = \sigma(\mathcal{F}_t^B \cup \mathcal{N})$, where $(\mathcal{F}_t^B)_{t\geq 0}$ is the natural Brownian filtration and $\mathcal{N}$ is the set of all $\mathbb{P}$-null sets. Then we define for any $U \in \mathcal{B}(\mathbb{R})$

$$\tau_U(\omega) := \inf\{t \geq 0 \colon B_t(\omega) \in U\}, \quad \omega \in \Omega.$$

Give a proof of the following assertions:

a) If U is open, then
$$\{\tau_U < t\} \in \mathcal{F}_t \qquad \text{for all } t > 0.$$

b) If U is closed, then
$$\{\tau_U \leq t\} \in \mathcal{F}_t \qquad \text{for all } t > 0.$$

Exercise 24 (Progressive measurability)

Let $(X_t)_{t\geq 0}$ be a stochastic process and $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then we call $(X_t)_{t\geq 0}$ *progressively measurable* with respect to $(\mathcal{F}_t)_{t\geq 0}$, if for all $t \geq 0$ the map

$$X : [0,t] \times \Omega \to \mathbb{R}, \quad X(s,\omega) := X_s(\omega),$$

is $(\mathcal{B}[0,t] \otimes \mathcal{F}_t)$-measurable. Show the following statements:

a) If $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, then $(X_t)_{t\geq 0}$ is adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$.

b) If $(X_t)_{t\geq 0}$ is adapted with respect to $(\mathcal{F}_t)_{t\geq 0}$ and if $(X_t)_{t\geq 0}$ has left or right continuous paths, then $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$.