Stochastic differential equations

Exercise sheet 8

Exercise 22  (Stopping times)

a) Let $\tau_1, \tau_2$ be two stopping times with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Proof that $\min(\tau_1, \tau_2), \max(\tau_1, \tau_2), \tau_1 + \tau_2$, and $\alpha \tau_1$ for $\alpha \geq 1$ are stopping times with respect to $(\mathcal{F}_t)_{t \geq 0}$ as well.

b) Give an example of stopping times $\tau_1, \tau_2$, such that $\tau_1 \leq \tau_2$, but $\tau_2 - \tau_1$ is not a stopping time.

Exercise 23  (Hitting times of a Brownian motion)

Let $B = (B_t)_{t \geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t \geq 0}$ be the augmented Brownian filtration, i.e. for all $t \geq 0$ the $\sigma$-algebra $\mathcal{F}_t = \sigma(\mathcal{F}^B_t \cup \mathcal{N})$, where $(\mathcal{F}^B_t)_{t \geq 0}$ is the natural Brownian filtration and $\mathcal{N}$ is the set of all $\mathbb{P}$-null sets. Then we define for any $U \in \mathcal{B}(\mathbb{R})$

$$\tau_U(\omega) := \inf\{t \geq 0: B_t(\omega) \in U\}, \quad \omega \in \Omega.$$ 

Give a proof of the following assertions:

a) If $U$ is open, then $\{\tau_U < t\} \in \mathcal{F}_t$ for all $t > 0$.

b) If $U$ is closed, then $\{\tau_U \leq t\} \in \mathcal{F}_t$ for all $t > 0$.

Exercise 24  (Progressive measurability)

Let $(X_t)_{t \geq 0}$ be a stochastic process and $(\mathcal{F}_t)_{t \geq 0}$ be a filtration on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then we call $(X_t)_{t \geq 0}$ progressively measurable with respect to $(\mathcal{F}_t)_{t \geq 0}$, if for all $t \geq 0$ the map

$$X : [0, t] \times \Omega \to \mathbb{R}, \quad X(s, \omega) := X_s(\omega),$$

is $(\mathcal{B}[0, t] \otimes \mathcal{F}_t)$-measurable. Show the following statements:

a) If $(X_t)_{t \geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t \geq 0}$, then $(X_t)_{t \geq 0}$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$.

b) If $(X_t)_{t \geq 0}$ is adapted with respect to $(\mathcal{F}_t)_{t \geq 0}$ and if $(X_t)_{t \geq 0}$ has left or right continuous paths, then $(X_t)_{t \geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t \geq 0}$. 