KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT)
Institute for Analysis
Prof. Dr. Lutz Weis
Dr. Markus Antoni

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Stochastic differential equations

Exercise sheet 9

Exercise 25 (Risk of ruin of a Brownian motion)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$.

a) Let $A, B > 0$ and $\tau := \min\{t \geq 0 \colon B_t = A \text{ or } B_t = -B\}$. Proof that:

   i) $\mathbb{P}(\tau < \infty) = 1$;
   ii) $\mathbb{P}(B_\tau = A) = \frac{B}{A+B}$;
   iii) $\mathbb{E}\tau = AB$.

b) Let $a \in \mathbb{R} \setminus \{0\}$ and $\tau_a := \min\{t \geq 0 \colon B_t = a\}$. Proof that:

   i) $\mathbb{P}(\tau_a < \infty) = 1$;
   ii) $\mathbb{E}\exp(-\lambda\tau_a) = \exp\big(-|a|\sqrt{2\lambda}\big)$ for $\lambda \geq 0$;
   iii) $\mathbb{E}\tau_a = \infty$;
   iv) $\mathbb{E}\tau_a^{-1} = \frac{1}{a^2}$.

Exercise 26 (Conditional Itô isometry)

Let $b \in \mathcal{H}^2[0,T]$ and $(\mathcal{F}_t)_{t\geq 0}$ be the Brownian filtration.

a) Show that

$$\mathbb{E}\Big[\Big(\int_s^t b(u)\,\mathrm{d}B_u\Big)^2 \,\Big|\, \mathcal{F}_s\Big] = \mathbb{E}\Big[\int_s^t |b(u)|^2\,\mathrm{d}u \,\Big|\, \mathcal{F}_s\Big].$$

   if $0 \leq s < t \leq T$.

b) Conclude that

$$M_t := \Big(\int_0^t b(u)\,\mathrm{d}B_u\Big)^2 - \int_0^t |b(u)|^2\,\mathrm{d}u, \quad t \geq 0,$$

   is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 27 (a local martingale which is not a martingale)

Let $(B_t)_{t\geq 0}$ be a 3-dimensional Brownian motion, $(\mathcal{F}_t)_{t\geq 0}$ be the corresponding Brownian filtration, and $f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{|x|}$.

a) Show that $M_t := f(B_t)$, $t \geq 1$, is a local martingale with respect to $(\mathcal{F}_t)_{t\geq 1}$.

b) Calculate $\mathbb{E}M_t^2$ for $t \geq 1$.

c) Conclude that $(M_t)_{t\geq 1}$ is not a martingale with respect to $(\mathcal{F}_t)_{t\geq 1}$.