Short recap: σ-algebras generated by maps

1) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f\colon X \to Y$ be a map. Then we define the *pull back σ-algebra* as

$$f^{-1}(\mathcal{B}) := \{f^{-1}(B)\colon B \in \mathcal{B}\}$$

This is the smallest σ-algebra on X such that f is measurable. In particular, f is $(\mathcal{A}, \mathcal{B})$-measurable if and only if $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$.

2) If $(f_i)_{i\in I}$ is a family of maps $f_i\colon X \to Y_i$, then

$$\sigma(f_i, i \in I) := \sigma\Big(\bigcup_{i\in I} f_i^{-1}(\mathcal{B}_i)\Big)$$

is the *σ-algebra generated by* $(f_i)_{i\in I}$. Please note that $\bigcup_{i\in I} f_i^{-1}(\mathcal{B}_i)$ is not a σ-algebra in general, and as in 1) we can easily prove that $\sigma(f_i, i \in I)$ is the smallest σ-algebra on X such that all f_i are measurable.

Exercise 4 ($\sigma(X)$-measurable random variables)

Let $(\Omega, \mathcal{A})$ be a measurable space.

a) Let $a_1, \dots, a_n \in \mathbb{R}$ be pairwise distinct and $A_1, \dots, A_n \in \mathcal{A}$ be pairwise disjoint with $\Omega = \bigcup_{j=1}^n A_j$. Let $X := \sum_{j=1}^n a_j \mathbb{1}_{A_j}$ and $\sigma(X)$ be the σ-algebra generated by X.

   i) Determine $\sigma(X)$.

   ii) Let Y be a $\sigma(X)$-measurable random variable. Show that Y is constant on each set A_j, $j = 1, \dots, n$.

   iii) Show that Y can be written as a function of X.

b) Let $X\colon \Omega \to \mathbb{R}$ be an arbitrary random variable and let $Y\colon \Omega \to \mathbb{R}$ be $\sigma(X)$-measurable. Show that there exists a function Φ such that

$$Y = \Phi(X).$$

Hint: algebraic induction.

Solution: a) i) In this particular case we obtain

$$\sigma(X) = X^{-1}(\mathcal{B}(\mathbb{R})) = \sigma\big(\{A_1, \dots, A_n\}\big) = \Big\{\bigcup_{j\in J} A_j\colon J \subseteq \{1, \dots, n\}\Big\}.$$

a) ii) We have $Y^{-1}(B) \in \sigma(X)$ for all $B \in \mathcal{B}(\mathbb{R})$, in particular $Y^{-1}(\{z\}) \in \sigma(X)$ for all $z \in \mathbb{R}$. Now assume that $Y^{-1}(\{z\}) \neq \emptyset$ for some $z \in \mathbb{R}$. Then there exists a set $A \in \sigma(X) \setminus \{\emptyset\}$ such that $Y^{-1}(\{z\}) = A = \bigcup_{j\in J} A_j$ for some index set $J \subseteq \{1, \dots, n\}$, i.e. $Y(\omega) = z$ for all $\omega \in A$. This means, that Y is constant on each set A_j, $j = 1, \dots, n$.

a) iii) By part ii) we can find $z_1, \dots, z_n \in \mathbb{R}$ such that

$$Y = \sum_{i=1}^n z_i \mathbb{1}_{A_i} = \sum_{i=1}^n z_i \mathbb{1}_{\{X=a_i\}}.$$

By defining the map $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = \sum_{i=1}^{n} z_i \mathbb{1}_{\{a_i\}}(x)$ we finally obtain

$$g(X(\omega)) = \sum_{i=1}^{n} z_i \mathbb{1}_{\{X=a_i\}}(\omega) = Y(\omega).$$

b) 1) Let Y be a simple function, i.e $Y = \sum_{i=1}^{n} z_i \mathbb{1}_{A_i}$ for some $(z_i) \subseteq \mathbb{R}$ and $(A_i) \subseteq \sigma(X)$. Then there exist sets $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X^{-1}(B_i)$, $i = 1, \dots, n$. Now define

$$\varphi(x) := \sum_{i=1}^{n} z_i \mathbb{1}_{B_i}(x), \quad x \in \mathbb{R}.$$

Then we obtain

$$\varphi(X(\omega)) = \sum_{i=1}^{n} z_i \underbrace{\mathbb{1}_{B_i}(X(\omega))}_{=\mathbb{1}_{X^{-1}(B_i)}(\omega)} = \sum_{i=1}^{n} z_i \mathbb{1}_{A_i} = Y.$$

2) Let $Y\colon \Omega \to \mathbb{R}$ be a positive function. Then there exists a sequence of simple functions Y_n such that $0 \leq Y_n \nearrow Y$ pointwise, in particular $Y = \sup_n Y_n$. By part 1) we can find a sequence of functions φ_n such that $\varphi_n(X) = Y_n$, $n \in \mathbb{N}$. Now take $\varphi := \sup_{n\in\mathbb{N}} \varphi_n$. Then we get

$$\varphi(X) = \sup_n \varphi_n(X) = \sup_n Y_n = Y.$$

3) Let Y be a general measurable map. Then $Y = Y_+ - Y_-$ with $Y_+, Y_- \geq 0$. By part 2) we can construct functions φ_+ and φ_- such that $Y_+ = \varphi_+(X)$ and $Y_- = \varphi_-(X)$. Defining $\varphi := \varphi_+ - \varphi_-$ we finally get

$$\varphi(X) = \varphi_+(X) - \varphi_-(X) = Y_+ - Y_- = Y.$$

Moreover, we obtain the following result:

Lemma T2. Let $f_i\colon (\Omega, \mathcal{A}) \to (\Omega_i, \mathcal{B}_i)$, $i \in I$, and $\varphi\colon (\Omega_0, \mathcal{A}_0) \to (\Omega, \mathcal{A})$ be maps. Then we have

$$\varphi \text{ is } (\mathcal{A}_0, \sigma(f_i, i \in I))\text{-measurable} \iff f_i \circ \varphi \text{ is } (\mathcal{A}_0, \mathcal{B}_i)\text{-measurable for all } i \in I.$$

Proof: First assume the left-hand side. Since φ is $(\mathcal{A}_0, \sigma(f_i))$-measurable and f_i is $(\sigma(f_i), \mathcal{B}_i)$-measurable for any fixed $i \in I$, we trivially obtain that $f_i \circ \varphi$ is $(\mathcal{A}_0, \mathcal{B}_i)$-measurable for all $i \in I$.

Now assume the converse. Let $\mathcal{M} := \bigcup_{i\in I} f_i^{-1}(\mathcal{B}_i)$ be the generating system of $\sigma(f_i, i \in I)$ and $A \in \mathcal{M}$ be any set. Then there exists an $i \in I$ and $B_i \in \mathcal{B}_i$ with $A = f_i^{-1}(B_i)$ satisfying

$$\varphi^{-1}(A) = \varphi^{-1}(f_i^{-1}(B_i)) = (f_i \circ \varphi)^{-1}(B_i) \in \mathcal{A}_0.$$

Since $A \in \mathcal{M}$ was arbitrary, the claim follows.

Exercise 5 (Product probability spaces)

Let $(\Omega_j, \mathcal{A}_j, \mathbb{P}_j)$, $j \in \mathbb{N}$, be probability spaces,

$$\Omega := \bigtimes_{j=1}^{\infty} \Omega_j, \quad \text{and} \quad \mathcal{A} := \bigotimes_{j=1}^{\infty} \mathcal{A}_j := \sigma(\pi_j, j \in \mathbb{N})$$

the product σ-algebra generated by the projections $\pi_j \colon \Omega \to \Omega_j$ ($\pi_j(\omega) = \omega_j$, $j \in \mathbb{N}$). Show that there exists a uniquely determined probability measure $\mathbb{P}$ on $\mathcal{A}$ satisfying

$$\mathbb{P}\big(A_1 \times \ldots \times A_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j\big) = \prod_{j=1}^{n} \mathbb{P}_j(A_j)$$

for any sets $A_j \in \mathcal{A}_j$, $j = 1, \ldots, n$.

Notation: $\mathbb{P} = \bigotimes_{j=1}^{\infty} \mathbb{P}_j$ and $\bigotimes_{j=1}^{\infty}(\Omega_j, \mathcal{A}_j, \mathbb{P}_j) := (\bigtimes_{j=1}^{\infty} \Omega_j, \bigotimes_{j=1}^{\infty} \mathcal{A}_j, \bigotimes_{j=1}^{\infty} \mathbb{P}_j)$.

Hints for the proof:

1) For $n \in \mathbb{N}$ define

$$\mathcal{F}_n := \big\{B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j \colon B_n \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n\big\}$$

and show that this is an ascending sequence of σ-algebras.

2) For $\mathcal{F} := \bigcup_{n=1}^{\infty} \mathcal{F}_n$ show that $\mathcal{F} \subset \mathcal{A}$ and $\sigma(\mathcal{F}) = \mathcal{A}$.

3) On $\mathcal{F}_n$ define the probability measure

$$\mu_n\big(B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j\big) := \mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n(B_n)$$

and on $\mathcal{F}$ the map $\mathbb{P}$ by

$$\mathbb{P}(A) := \mu_n(A) \quad \text{if } A \in \mathcal{F}_n.$$

4) Show that $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}$ is σ-additive.

5) Finally, use Caratheodory's extension theorem as well as the measure uniqueness theorem to conclude the proof.

Before we start with the proof of this result we shortly recall Caratheodory's extension theorem and the measure uniqueness theorem:

Caratheodory's extension theorem. Let $\mathcal{R}$ be a semi-ring, $\mu \colon \mathcal{R} \to [0, \infty]$ be a set function satisfying

1) $\mu(\emptyset) = 0$

2) μ is σ-additive

Then there exists a measure $\tilde{\mu}$ on $\sigma(\mathcal{R})$ with $\mu(A) = \tilde{\mu}(A)$ for all $A \in \mathcal{R}$.

Measure uniqueness theorem. Let $\mathcal{M} \subseteq \mathbb{P}(\Omega)$ be an intersection stable generator of the σ-algebra $\mathcal{A}$ and let μ_1, μ_2 be measures on $\mathcal{A}$ such that

$$\mu_1(A) = \mu_2(A) \quad \text{for all } A \in \mathcal{M}.$$

If there exists an exhausting sequence of sets $(A_n)_{n\geq 1} \subseteq \mathcal{M}$ with $\Omega = \bigcup_{n\geq 1} A_n$ with $\mu_1(A_n) = \mu_2(A_n) < \infty$ for all $n \in \mathbb{N}$, then $\mu_1 = \mu_2$.

Solution: 1) Let $\mathcal{F}_n := \big\{B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j \colon B_n \in \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n\big\}$, $n \in \mathbb{N}$. Using that $\otimes_{j=1}^{n} A_j$ is a σ-algebra, it can be easily verified that $\mathcal{F}_n$ is a σ-algebra as well. Moreover, since $B_n \times \Omega_{n+1} \in \otimes_{j=1}^{n+1} \mathcal{A}_j$ for any set $B_n \in \otimes_{j=1}^{n} \mathcal{A}_j$ we trivially obtain $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$.

2) Now define $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ (which is not a σ-algebra in general). Then $\mathcal{F}$ is an algebra:

i) Since $\Omega \in \mathcal{F}_n$ for all $n \in \mathbb{N}$, we obtain $\Omega \in \mathcal{F}$.

ii) Let $A \in \mathcal{F}$. Then there is an $n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. Using that $\mathcal{F}_n$ is a σ-algebra we get $A^C \in \mathcal{F}_n$, i.e. $A^C \in \mathcal{F}$.

iii) Let $A, B \in \mathcal{F}$. Then there are $m, n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$ and $B \in \mathcal{F}_m$. Without loss of generality we may assume that $n < m$. Then $A, B \in \mathcal{F}_m$ by part 1). Using again that $\mathcal{F}_m$ is a σ-algebra, we get $A \cup B \in \mathcal{F}_m$, and this implies $A \cup B \in \mathcal{F}$ by definition.

In particular, $\mathcal{F}$ is an intersection stable semi-ring. Consider now the maps

$$p_n \colon \Omega \to \Omega_1 \times \ldots \times \Omega_n, \quad p_n(\omega) = (\omega_1, \ldots, \omega_n).$$

Then p_n is $(\mathcal{A}, \otimes_{i=1}^{n} \mathcal{A}_i)$-measurable. This can be seen by considering the maps

$$\pi_j^n \colon \bigtimes_{i=1}^{n} \Omega_i \to \Omega_j, \quad \pi_j^n((\omega_1, \ldots, \omega_n)) = \omega_j, \quad j = 1, \ldots, n$$

which has the property $\pi_j^n \circ p_n = \pi_j$, and this is $(\mathcal{A}, \mathcal{A}_j)$-measurable for each $j = 1, \ldots, n$. By Lemma T2 p_n is $(\mathcal{A}, \underbrace{\sigma(\pi_j^n, j = 1, \ldots, n)}_{=\otimes_{j=1}^{n} \mathcal{A}_j})$-measurable.

For any set $A \in \mathcal{F}$ we can find an $n \in \mathbb{N}$ such that $A = B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j$ for some $B_n \in \otimes_{j=1}^{n} \mathcal{A}_j$. Using the measurability of p_n above we obtain

$$A = p_n^{-1}(B_n) \in \mathcal{A}.$$

This implies $\mathcal{F} \subseteq \mathcal{A}$.

Define the sets $\mathcal{R}_n := \{A_1 \times \ldots A_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j \colon A_j \in \mathcal{A}_j\}$, $n \in \mathbb{N}$. Then we have

$$\bigcup_{j=1}^{\infty} \pi_j^{-1}(\mathcal{A}_j) \subseteq \bigcup_{j=1}^{\infty} \mathcal{R}_j \subseteq \mathcal{F} \subseteq \mathcal{A},$$

which implies that $\sigma(\mathcal{F}) = \mathcal{A}$.

3) On $\mathcal{F}_n$ we define the probability measure μ_n by

$$\mu_n\Big(B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j\Big) := \otimes_{j=1}^{n} \mathbb{P}_j(B_n)$$

and the set function $\mathbb{P}$ on $\mathcal{F}$ by

$$\mathbb{P}(A) := \mu_n(A) \quad \text{if } A \in \mathcal{F}_n.$$

4) By construction $\mathbb{P}$ is well-defined and it holds that

$$\mathbb{P}(\Omega) = \mu_1(\Omega_1) = \mathbb{P}_1(\Omega_1) = 1.$$

Now let $A_1, \ldots, A_n \in \mathcal{F}$ be disjoint sets. Then there is an $m \in \mathbb{N}$ such that $A_1, \ldots, A_n \in \mathcal{F}_m$. This implies

$$\mathbb{P}\Big(\bigcup_{j=1}^{n} A_j\Big) = \mu_m\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \mu_m(A_j) = \sum_{j=1}^{n} \mathbb{P}(A_j).$$

which means that $\mathbb{P}$ is finitely additive. We next show that $\mathbb{P}$ is continuous in $\emptyset$ (and hence σ-additive). To prove that let $(C_n)_n \subseteq \mathcal{F}$ be a decreasing sequence of sets such that $\lim_{n\to\infty} \mathbb{P}(C_n) > 0$. In the remaining of this part we want to show that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ (which then proves the continuity of $\mathbb{P}$).

We may assume that $C_n = B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j$, $B_n \in \mathcal{A}_{\leq n}$. Then

$$\lim_{n\to\infty} \mu_n(C_n) = \lim_{n\to\infty} \mathbb{P}(C_n) > 0.$$

Now define

$$\begin{aligned} f_n^1(\omega_1) &:= \int_{\Omega_2 \times \ldots \times \Omega_n} \mathbb{1}_{B_n}(\omega_1, \omega_2, \ldots, \omega_n)\, \mathrm{d}(\mathbb{P}_2 \otimes \ldots \otimes \mathbb{P}_n)(\omega_2, \ldots, \omega_n) \\ &\geq \int_{\Omega_2 \times \ldots \times \Omega_{n+1}} \mathbb{1}_{B_{n+1}}(\omega_1, \omega_2, \ldots, \omega_{n+1})\, \mathrm{d}(\mathbb{P}_2 \otimes \ldots \otimes \mathbb{P}_{n+1})(\omega_2, \ldots, \omega_{n+1}) \\ &= f_{n+1}^1(\omega_1), \end{aligned}$$

where the inequality comes from the fact that $C_{n+1} \subseteq C_n$ and $B_{n+1} \subseteq B_n \times \Omega_{n+1}$. This implies that $(f_n^1)_{n\geq 2}$ is a decreasing sequence. By monotone convergence this implies

$$\int_{\Omega_1} \inf_{n\geq 2} f_n^1(\omega_1)\, \mathrm{d}\mathbb{P}_1(\omega_1) = \inf_{n\geq 2} \int_{\Omega_1} f_n^1(\omega_1)\, \mathrm{d}\mathbb{P}_1(\omega_1) \overset{\text{def}}{=} \inf_{n\geq 2} \mu_n(C_n) > 0.$$

Hence, there exists an $\bar{\omega}_1 \in \Omega_1$ such that $\inf_{n\geq 2} f_n^1(\bar{\omega}_1) > 0$. Inductively, we can define for any $k \in \mathbb{N}$ and $n \geq k+1$ the function

$$f_n^k(\bar{\omega}_1, \ldots, \bar{\omega}_{k-1}, \omega_k) := \int_{\Omega_{k+1} \times \ldots \Omega_n} \mathbb{1}_{B_n}(\bar{\omega}_1, \ldots, \bar{\omega}_{k-1}, \omega_k, \ldots, \omega_n)\, \mathrm{d}(\mathbb{P}_{k+1} \otimes \ldots \otimes \mathbb{P}_n)(\omega_{k+1}, \ldots, \omega_n)$$

and with the same argument as above we get an $\bar{\omega}_k \in \Omega_k$ satisfying

$$\inf_{n\geq k+1} f_n^k(\bar{\omega}_1, \ldots, \bar{\omega}_k) > 0.$$

In particular, $f_{k+1}^k(\bar{\omega}_1, \ldots, \bar{\omega}_k) = \int_{\Omega_{k+1}} \mathbb{1}_{B_{k+1}}(\bar{\omega}_1, \ldots, \bar{\omega}_k, \omega_{k+1})\, \mathrm{d}\mathbb{P}_{k+1}(\omega_{k+1}) > 0$. This implies that the set

$$\{\omega_{k+1} \in \Omega_{k+1} \colon (\bar{\omega}_1, \ldots, \bar{\omega}_k, \omega_{k+1}) \in B_{k+1}\}$$

is not empty for all $k \in \mathbb{N}$. Now using that $B_{k+1} \subseteq B_k \times \Omega_{k+1}$ we obtain

$$(\bar{\omega}_1, \ldots, \bar{\omega}_k) \in B_k \quad \text{for all } k \in \mathbb{N}.$$

Eventually, $\bar{\omega} := (\bar{\omega}_j)_{j\geq 1} \in \bigcap_{n=1}^{\infty} C_n$.

5) By part 4) the set function $\mathbb{P}$ is σ-additive on the intersection stable semi-ring $\mathcal{F}$. By Caratheodory's extension theorem and the uniqueness theorem for measures we obtain a unique probability measure $\tilde{\mathbb{P}}$ on $\sigma(\mathcal{F}) = \mathcal{A}$ such that

$$\tilde{\mathbb{P}}(A) = \mathbb{P}(A) \quad \text{for all } A \in \mathcal{F}.$$

Having this result at hand we can prove the following Corollary:

Corollary T3. Let $X_i \colon \Omega \to \Omega_i$, $i \in \mathbb{N}$, be random variables. Then we have

$$X = (X_i)_{i\in\mathbb{N}} \text{ is independent } \iff \mathbb{P}^X = \otimes_{i=1}^{\infty} \mathbb{P}^{X_i}.$$

Proof: Let $(X_i)_{i\in\mathbb{N}}$ be independent. Then, for any $n \in \mathbb{N}$ we have $\mathbb{P}^{(X_1,\ldots,X_n)} = \otimes_{i=1}^{n} \mathbb{P}^{X_i}$. Using this, we get

$$\begin{aligned}\mathbb{P}^X(B_1 \times \ldots \times B_n \times \bigtimes_{j=n+1}^{\infty} \Omega_j) &= \mathbb{P}(X_1 \in B_1, \ldots, X_n \in B_n, X_j \in \Omega_j, j \geq n+1) \\ &= \mathbb{P}(X_1 \in B_1, \ldots, X_n \in B_n) = \prod_{j=1}^{n} \mathbb{P}^{X_j}(B_j).\end{aligned}$$

By Exercise 5, the only measure satisfying this is $\otimes_{i=1}^{\infty} \mathbb{P}^{X_i}$, i.e. $\mathbb{P}^X = \otimes_{i=1}^{\infty} \mathbb{P}^{X_i}$.

Now assume the converse and let $J \subseteq \mathbb{N}$ be a finite subset. Then there is an $N \in \mathbb{N}$ such that $J \subseteq \{1, \ldots, N\}$ and we have (let $B_j = \Omega_j$ for $j \notin J$)

$$\begin{aligned}\mathbb{P}(X_j \in B_j, j \in J) = \mathbb{P}^X(B_1 \times \ldots \times B_N \times \bigtimes_{j=N+1}^{\infty} \Omega_j) &= \prod_{j=1}^{N} \mathbb{P}^{X_j}(B_j) \\ = \prod_{j\in J} \mathbb{P}^{X_j}(B_j) = \prod_{j\in J} \mathbb{P}(X_j \in B_j).\end{aligned}$$

which implies that $(X_i)_{i\in\mathbb{N}}$ are independent.

Exercise 6 (Infinite and independent coin toss)

Let $(\Omega, \mathcal{A}, \mathbb{P}) = \bigotimes_{j=1}^{\infty} \big(\{0,1\}, \mathcal{P}(\{0,1\}), \mathbb{P}_j\big)$ with $\mathbb{P}_j(\{0\}) = p$ and $\mathbb{P}_j(\{1\}) = 1 - p$, $0 < p < 1$, $j \in \mathbb{N}$, the stochastic model of an infinite sequence of independent coin tosses with a not necessarily fair coin. We now continuously flip the coin and stop when the coin shows head (head $= 1$). Calculate the probabilities of the following events:

a) A: we flip the coin at least k times ($k \in \mathbb{N}$);

b) B: the amount of coin flips is even.

Solution: Let $X\colon \Omega \to \mathbb{N}$ be the random variable counting the amounts of flips of this experiment. By independence we get

$$\begin{aligned}
\mathbb{P}(X=1) &= \mathbb{P}_1(\{1\}) = 1-p,\\
\mathbb{P}(X=2) &= \mathbb{P}_1(\{0\})\mathbb{P}_2(\{1\}) = p(1-p),\\
\mathbb{P}(X=3) &= \mathbb{P}_1(\{0\})\mathbb{P}_2(\{0\})\mathbb{P}_3(\{1\}) = p^2(1-p),\\
&\vdots\\
\mathbb{P}(X=k) &= \mathbb{P}_1(\{0\})\dots\mathbb{P}_{k-1}(\{0\})\mathbb{P}_k(\{1\}) = p^{k-1}(1-p), \quad k\in\mathbb{N}.
\end{aligned}$$

a) Using this observation we obtain

$$\begin{aligned}
\mathbb{P}(A) = \mathbb{P}(X\geq k) = 1-\mathbb{P}(X\leq k-1) &= 1-\sum_{j=1}^{k-1} p^{j-1}(1-p)\\
&= 1-(1-p)\sum_{j=0}^{k-2} p^j = 1-(1-p)\frac{1-p^{k-1}}{1-p}\\
&= p^{k-1}.
\end{aligned}$$

b) Similarly we get

$$\begin{aligned}
\mathbb{P}(B) &= \sum_{k=1}^{\infty}\mathbb{P}(X=2k) = \sum_{k=1}^{\infty} p^{2k-1}(1-p)\\
&= p(1-p)\sum_{j=0}^{\infty} p^{2j} = p(1-p)\frac{1}{1-p^2}\\
&= \frac{p}{1+p}.
\end{aligned}$$

If we have a fair coin ($p=\frac{1}{2}$) we get $\mathbb{P}(B)=\frac{1}{3}$, which means that the probability of X being odd is twice as much as X being even.