Short recap: Wiener integral

Let \( f(t) = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{(t_{n-1}, t_n]}(t) \) be a step function. Then we define the random variable

\[
\left( \int_0^T f(t) \, dB_t \right)(\omega) := \sum_{n=1}^{N} \alpha_n (B_{t_n}(\omega) - B_{t_{n-1}}(\omega)), \quad \omega \in \Omega.
\]

In this case, we get

\[
\left\| \int_0^T f(t) \, dB_t \right\|^2_{L^2(\Omega)} = \sum_{n=1}^{N} \alpha_n^2 \text{Var}(B_{t_n} - B_{t_{n-1}}) = \sum_{n=1}^{N} \alpha_n^2 (t_n - t_{n-1}) = \|f\|^2_{L^2[0,T]}.
\]

This means: \( \mathbb{E}(\int_0^T f(t) \, dB_t)^2 = \int_0^T |f(t)|^2 \, dt \) (Itô isometry). In particular, this implies

\[
\langle I(f), I(g) \rangle_{L^2(\Omega)} = \langle f, g \rangle_{L^2[0,T]}, \quad f, g \in L^2[0,T].
\]

Using that step functions are dense in \( L^2[0,T] \), we can find for any function \( f \in L^2[0,T] \) a sequence of step functions \((f_n)_{n\in\mathbb{N}}\) such that \( \lim_{n\to\infty} f_n = f \) in \( L^2[0,T] \). Therefore, by Itô’s isometry the limit

\[
\int_0^T f(t) \, dB_t := \lim_{n\to\infty} \int_0^T f_n(t) \, dB_t \quad \text{in } L^2(\Omega)
\]

is well-defined, and independent of the approximating sequence.

Properties:

a) The Wiener integral is linear (in particular, the linear operator \( I: L^2[0,T] \to L^2(\Omega) \) is continuous).

b) \( \int_0^T a \, dB_t = a B_T \), by definition.

c) \( \int_0^T f(s) \, dB_s = \int_0^T \mathbb{1}_{[0,t]}(s) f(s) \, dB_s, \quad t \in [0,T] \).

Exercise 13 (Integration by parts)

Let \((B_t)_{t \geq 0}\) be a Brownian motion in \( \mathbb{R} \) and \( \phi \in C^1[0,T] \). Proof that:

\[
\int_0^T \phi(t) \, dB_t = \phi(T) B_T - \int_0^T \phi'(t) B_t \, dt \quad \text{almost surely.}
\]

Solution: Wlog let \( \phi(0) = 0 \) (else take \( \psi(t) = \phi(t) - \phi(0) \)).

1) We start with a general observation: Let \( 0 \leq t_0 < \ldots < t_N = T \), \((c_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \), and take

\[
g(t) := \sum_{n=1}^{N} c_n \mathbb{1}_{(t_{n-1}, t_n]}(t), \quad G(t) := \sum_{n=1}^{N} \sum_{m=1}^{n} c_m (t_m - t_{m-1}) \mathbb{1}_{(t_{n-1}, t_n]}(t).
\]
Then
\[ \int_0^T g(t) B_t \, dt = \sum_{n=1}^N c_n \int_{t_{n-1}}^{t_n} B_t \, dt \]
\[ \int_0^T G(t) \, dB_t = \sum_{n=1}^N \sum_{m=1}^n c_m(t_m - t_{m-1})(B_{t_n} - B_{t_{n-1}}) \]
\[ = \sum_{m=1}^N \sum_{n=m}^N c_m(t_m - t_{m-1})(B_{t_n} - B_{t_{n-1}}) \]
\[ = \sum_{m=1}^N c_m(t_m - t_{m-1})(B_T - B_{t_{m-1}}) \]
\[ = G(T) B_T - \sum_{m=1}^N c_m(t_m - t_{m-1}) B_{t_{m-1}}. \]

2) Let \( \varepsilon > 0 \). For \( k \in \mathbb{N} \) choose a partition \( 0 = t_0^k < \ldots < t_N^k = T \) of \([0, T]\) such that \( \sup_n |t_n^k - t_{n-1}^k| \to 0 \) as \( k \to \infty \). Now we define
\[ g_k(t) := \sum_{n=1}^{N_k} \phi'(t_{n-1}^k) 1_{(t_{n-1}^k, t_n^k)}(t), \quad G_k(t) := \sum_{n=1}^{N_k} \sum_{m=1}^n \phi'(t_{m-1}^k)(t_m^k - t_{m-1}^k) 1_{(t_{m-1}^k, t_m^k)}(t). \]
(These are just the functions \( g \) and \( G \) from 1) with \( c_n = \phi'(t_{n-1}^k) \)). Using that \( \phi' \) is uniformly continuous on \([0, T]\), we get
\[ |g_k(t) - \phi'(t)| = |\phi'(t_{n-1}^k) - \phi'(t)| \leq \varepsilon \quad \text{for } t \in (t_{n-1}^k, t_n^k] \text{ and } k \text{ large enough, and} \]
\[ |G_k(t) - \phi(t)| = \left| \sum_{m=1}^n \int_{t_{m-1}^k}^{t_m^k} \phi'(s) \, ds - \sum_{m=1}^n \int_{t_{m-1}^k}^{t_m^k} \phi'(s) \, ds + \int_t^{t_n^k} \phi'(s) \, ds \right| \]
\[ \leq \varepsilon t_n^k + \int_{t_{n-1}^k}^{t_n^k} \|\phi'\|_\infty \leq \varepsilon T + \varepsilon \|\phi'\|_\infty \quad \text{for } k \text{ sufficiently large}. \]
Please observe that the estimates above are actually independent of \( t \) because of the uniform continuity of \( \phi' \). This implies that \( g_k \to \phi' \) and \( G_k \to \phi \) uniformly as \( k \to \infty \). This convergence also yields \( g_k \to \phi' \) and \( G_k \to \phi \) in \( L^2[0, T] \) as \( k \to \infty \).

3) The established convergences of part 2) imply
\[ \int_0^T G_k(t) \, dB_t \to \int_0^T \phi(t) \, dB_t \quad \text{in } L^2(\Omega) \text{ as } k \to \infty \text{ by Itô's isometry;} \]
\[ G_k(T) B_T \to \phi(T) B_T \quad \text{almost surely and in } L^2(\Omega) \text{ as } k \to \infty; \]
\[ \int_0^T g_k(t) B_t \, dt \to \int_0^T \phi'(t) B_t \, dt \quad \text{almost surely and in } L^2(\Omega) \text{ as } k \to \infty. \]
Now we use that $t \mapsto B_t$ is almost surely uniformly continuous on $[0, T]$ to get
\[
\left| \int_0^T g_k(t) B_t \, dt - \left( G_k(T) B_T - \int_0^T G_k(t) \, dB_t \right) \right| =: \Psi_k
\]
\[
\leq \frac{1}{N_k} \sum_{n=1}^{N_k} \phi'(t_{n-1}^k) \int_{t_{n-1}^k}^{t_n^k} B_t \, dt - \frac{1}{N_k} \sum_{n=1}^{N_k} \phi'(t_{n-1}^k)(t_n^k - t_{n-1}^k) B_{t_{n-1}^k}
\leq T \| \phi' \|_{\infty} \epsilon.
\]

Putting all things together, we almost surely obtain
\[
\int_0^T \phi(t) \, dB_t - \left( \phi(T) B_T - \int_0^T \phi'(t) B_t \, dt \right) = \lim_{k \to \infty} \Psi_k = 0.
\]

**Exercise 14 (Properties of the Wiener integral process)**

a) Let $(B_t)_{t \geq 0}$ be a Brownian motion in $\mathbb{R}$ and $f \in L^2([0, T])$. Then we define the stochastic process $(X_t)_{t \in [0, T]}$ by
\[
X_t := \int_0^t f(s) \, dB_s, \quad t \in [0, T].
\]
Show the following properties:

i) $X_t \sim \mathcal{N}(0, \int_0^t f(s)^2 \, ds)$, $t \in [0, T]$;

ii) $(X_t)_{t \in [0, T]}$ has independent increments.

b) Let $(B_t)_{t \geq 0}$ be an $n$-dimensional Brownian motion and $F \in L^2([0, T], \mathbb{R}^{m \times n})$. Proof that
\[
Y_t := \int_0^t F(s) \, dB_s, \quad t \in [0, T],
\]
has an $m$-dimensional normal distribution, and determine its covariance matrix.

Before we start with the proof we show two auxiliary results:

**Claim A:** For any random variable $X : \Omega \to \mathbb{R}^n$ we have $X$ has an $n$-dim. normal distribution $\iff a^\top X$ has a 1-dim. normal distribution for all $a \in \mathbb{R}^n$.

**Proof:** If $X \sim \mathcal{N}(m, C)$ for $m \in \mathbb{R}^n$ and a positive definite $C \in \mathbb{R}^{n \times n}$, then
\[
\phi_a(t) = \phi_X(ta) = \exp(i \langle a, mt \rangle) \exp(-\frac{1}{2} \langle a, Ca \rangle t^2), \quad t \in \mathbb{R},
\]
i.e. $a^\top X \sim \mathcal{N}(a^\top m, a^\top Ca)$ (see Exercise 3c)).

Conversely, if $a^\top X$ has a 1-dimensional normal distribution for all $a \in \mathbb{R}^n$, we obtain for $m := \mathbb{E}X$ and $C := \text{Cov}(X)$
\[
\mathbb{E}(a^\top X) = a^\top m \quad \text{und} \quad \text{Var}(a^\top X) = a^\top Ca,
\]
3
and therefore
\[ \phi_X(t) = \phi_{t^\top X}(1) = \exp(it^\top m) \exp(-\frac{1}{2} t^\top C t), \quad t \in \mathbb{R}^n. \]
This implies \( X \sim N(m, C) \).

**Claim B:** A process \((B_t)_{t \geq 0} = (B_t^{(1)}, \ldots, B_t^{(n)})_{t \geq 0}\) in \( \mathbb{R}^n \) is an \( n \)-dimensional Brownian motion if and only if the components \((B_t^{(1)})_{t \geq 0}, \ldots, (B_t^{(n)})_{t \geq 0}\) are independent 1-dimensional Brownian motions.

**Proof:** If the components are independent 1-dimensional Brownian motions, then we have seen in the lectures that \((B_t^{(1)}, \ldots, B_t^{(n)})_{t \geq 0}\) satisfies the properties of an \( n \)-dimensional Brownian motion.

Conversely, assume that \((B_t)_{t \geq 0} = (B_t^{(1)}, \ldots, B_t^{(n)})_{t \geq 0}\) is an \( n \)-dimensional Brownian motion. Then we know by Exercise 9 that each component is a 1-dimensional Brownian motion. So, it remains to prove that the processes \((B_t^{(1)})_{t \geq 0}, \ldots, (B_t^{(n)})_{t \geq 0}\) are independent.

Since for \( 0 = t_0 < t_1 < \ldots < t_N \) the random variables
\[ B_{t_1} - B_{t_0}, \ldots, B_{t_N} - B_{t_{N-1}} \]
are independent and have an \( n \)-dimensional normal distribution, the random variable
\[ X := (B_{t_1} - B_{t_0}, \ldots, B_{t_N} - B_{t_{N-1}}) \]
has an \( nN \)-dimensional normal distribution.

Now let \((\lambda_k)_{k=1}^N \in \mathbb{R}^n\). Then
\[
(\lambda_1 + \cdots + \lambda_N, \lambda_2 + \cdots + \lambda_N, \ldots, \lambda_N) \top X = \sum_{m=1}^N \left( \sum_{k=m}^N \lambda_k^\top \right) (B_{t_m} - B_{t_{m-1}}) = \sum_{k=1}^N \lambda_k^\top B_{t_k}
= (\lambda_1, \ldots, \lambda_N)^\top (B_{t_1}, \ldots, B_{t_N}).
\]
So, by Claim A \((B_{t_1}, \ldots, B_{t_N})\) has an \( nN \)-dimensional normal distribution.

For \( j \in \{1, \ldots, n\} \) we choose \( 0 \leq t_1^{(j)} < \ldots < t_N^{(j)} \) for some \( N_j \in \mathbb{N} \). Then by the argument above
\( (B_t^{(1)}, \ldots, B_t^{(n)})_{t \in \mathbb{R}} \)
has a \( \sum_{j=1}^n N_j \)-dimensional normal distribution. For \( j \neq k \) and \( t_m^{(j)} \leq t_p^{(k)} \) the independence of \( B_{t_m^{(j)}} \) and \( B_{t_p^{(k)}} \) implies that \( B_{t_m^{(j)}} \) and \( B_{t_p^{(k)}} \) are independent. This and \( B_t \sim N(0, tI) \) then gives
\[
\text{Cov}(B_{t_m^{(j)}}, B_{t_p^{(k)}}) = \mathbb{E}(B_{t_m^{(j)}}(B_{t_p^{(k)}} - B_{t_p^{(k)}})) + \mathbb{E}(B_{t_m^{(j)}}B_{t_p^{(k)}})
= \mathbb{E}B_{t_m^{(j)}}\mathbb{E}(B_{t_p^{(k)}} - B_{t_p^{(k)}}) + \text{Cov}(B_{t_m^{(j)}}, B_{t_p^{(k)}}) = 0.
\]

By (a vector valued version of) Exercise 3 e) we obtain that the random vectors
\[ (B_t^{(1)})(t_1^{(1)}), \ldots, (B_t^{(n)})(t_n^{(n)}) \]
are independent, but this means that the processes \( B^{(1)}, \ldots, B^{(n)} \) are independent.
Solution: a) i) Let \((f_n)_n\) be step functions such that \(\lim_{n \to \infty} f_n = f\) in \(L^2[0,T]\). Then

\[
X_t^n := \int_0^t f_n(s) \, dB_s \to X_t \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty
\]

by Itô’s isometry. Moreover,

\[
\mathbb{E}X_t^n = 0 \to 0 \quad \text{as } n \to \infty,
\]

\[
\text{Var}(X_t^n) = \lim_{n \to \infty} \|f_n\|_{L^2[0,T]}^2 \to \|f\|^2_{L^2[0,T]} \quad \text{as } n \to \infty,
\]

and \(X_t^n\) has a normal distribution because it is a linear combination of independent Gaussians. Using these results as well as the dominated convergence theorem we obtain

\[
\phi_{X_t}(s) = \lim_{n \to \infty} \phi_{X_t^n}(s) = \lim_{n \to \infty} \exp\left(-\frac{1}{2} s^2 \|f_n\|_{L^2}^2\right) = \exp\left(-\frac{1}{2} s^2 \|f\|_{L^2}^2\right).
\]

By uniqueness of the characteristic function we finally get \(X_t \sim N(0, \|f\|_{L^2}^2)\).

a) ii) Let \((f_k)_k\) be a sequence of step functions approximating \(f\) in \(L^2[0,T]\). Then we obtain for \(0 < t_1 < \ldots < t_N\) and for a suitable choice of \(\alpha^{(n)}_i\)'s, \(n = 1, \ldots, N:\)

\[
X^{(k)} := \begin{pmatrix}
\int_0^{t_1} f_k \, dB_t \\
\vdots \\
\int_0^{t_N} f_k \, dB_t
\end{pmatrix} = \begin{pmatrix}
\sum_{i=1}^M \alpha^{(1)}_i (B_{s_i} - B_{s_{i-1}}) \\
\vdots \\
\sum_{i=1}^M \alpha^{(N)}_i (B_{s_i} - B_{s_{i-1}})
\end{pmatrix} = A_{f_k} \begin{pmatrix}
B_{s_1} - B_{s_0} \\
\vdots \\
B_{s_M} - B_{s_{M-1}}
\end{pmatrix},
\]

where \(A_{f_k} = (a_{n,m})_{n,m} = (\alpha^{(n)}_m)_{n,m}\). So \(X^{(k)}\) has an \(N\)-dimensional normal distribution. Since \(\int_0^{t_n} f_k \, dB_t \to \int_0^{t_n} f \, dB_t\) in \(L^2(\Omega)\) as \(k \to \infty\), we obtain

\[
X^{(k)} \to X := \left( \int_0^{t_1} f \, dB_t, \ldots, \int_0^{t_N} f \, dB_t \right) = (X_{t_1}, \ldots, X_{t_N}) \quad \text{in } L^2(\Omega; \mathbb{R}^N).
\]

In the same way as in part i) this implies that \(X \sim N(0, \text{Cov}(X))\). In particular, the random vector \((X_{t_2} - X_{t_1}, \ldots, X_{t_N} - X_{t_{N-1}})\) has a normal distribution. Finally, we use Itô’s isometry to get

\[
\text{Cov}(X_{t_j} - X_{t_{j-1}}, X_{t_i} - X_{t_{i-1}}) = \langle X_{t_j} - X_{t_{j-1}}, X_{t_i} - X_{t_{i-1}} \rangle_{L^2(\Omega)}
\]

\[
= \langle 1_{[t_{j-1}, t_j]} f, 1_{[t_{i-1}, t_i]} f \rangle_{L^2[0,T]} = 0 \quad \text{for } i \neq j,
\]

which implies that \(X_t\) has independent increments by Exercise 3 e).

b) Please observe that by Claim B and the definition of the Wiener integral the random variables

\[
\int_0^t g_1(s) \, dB_s^{(1)}, \ldots, \int_0^t g_n(s) \, dB_s^{(n)}
\]

are independent for arbitrary functions \(g_1, \ldots, g_n \in L^2[0,T]\). Using that

\[
\int_0^t F(s) \, dB_s = \begin{pmatrix}
\sum_{k=1}^M \int_0^t F_{1k}(s) \, dB_s^{(k)} \\
\vdots \\
\sum_{k=1}^M \int_0^t F_{mk}(s) \, dB_s^{(k)}
\end{pmatrix}
\]

we get for any \(a \in \mathbb{R}^m\)

\[
a^\top \int_0^t F(s) \, dB_s = \sum_{k=1}^M \int_0^t a^\top F_{k}(s) \, dB_s^{(k)}.
\]
By part a) and the remark above the right-hand side is a sum of \( n \) independent Gaussian random variables, so itself a 1-dimensional normally distributed random variable. By Claim A \( \int_0^t F(s) \, dB_s \) has an \( m \)-dimensional normal distribution. Moreover, by part a)

\[
\mathbb{E} \int_0^t F(s) \, dB_s = 0,
\]

and by independence and Itô’s isometry we get

\[
\text{Cov}(Y_t, Y_{t, \ell}) = \sum_{k=1}^{n} \sum_{\tilde{k}=1}^{n} \left( \int_0^t F_{\ell,k}(s) \, dB_s^{(k)} \right) \left( \int_0^t F_{\tilde{\ell},\tilde{k}}(s) \, dB_s^{(\tilde{k})} \right)_{L^2(\Omega)}
\]

\[
= \sum_{k=1}^{n} \int_0^t F_{\ell,k}(s) \, dB_s^{(k)} \left( \int_0^t F_{\tilde{\ell},\tilde{k}}(s) \, dB_s^{(\tilde{k})} \right)_{L^2(\Omega)}
\]

\[
= \sum_{k=1}^{n} \int_0^t F_{\ell,k}(s) F_{\tilde{\ell},\tilde{k}}(s) \, ds
\]

\[
= \int_0^t (F(s) F(s)^\top)_{\ell, \tilde{\ell}} \, ds.
\]

This means \( \text{Cov}(Y_t) = \int_0^t F(s) F(s)^\top \, ds \).

**Short recap: Itô integral**

The basic idea is to extend the Wiener integral to an integral for processes \( \phi : \Omega \times [0, T] \to \mathbb{R} \). This procedure was done in two steps:

1st step: Define an integral for adapted step processes, i.e.

\[
\phi(\omega, t) = \sum_{n=1}^{N} \alpha_n(\omega) \mathbb{1}_{(t_{n-1}, t_n]}(t),
\]

where \( \alpha_n \) is \( \mathcal{F}_{t_{n-1}} \)-measurable (here, \( \mathcal{F}_t := \sigma(B_s, s \leq t), \ t \geq 0 \)). Then we define

\[
\left( \int_0^T \phi(t) \, dB_t \right)(\omega) := \sum_{n=1}^{N} \alpha_n(\omega) (B_{t_n}(\omega) - B_{t_{n-1}}(\omega)), \ \omega \in \Omega.
\]

In this case, Itô’s isometry still holds in the following way

\[
\| I(\phi) \|_{L^2(\Omega)}^2 = \mathbb{E} \left| \int_0^T \phi(t) \, dB_t \right|^2 = \mathbb{E} \int_0^T |\phi(t)|^2 \, dt = \| \phi \|_{L^2(\Omega \times [0, T])}^2.
\]

2nd step: Define an integral for adapted processes, i.e. a process \( \phi \) satisfying

\[
\phi \in \mathcal{H}^2[0, T] := \{ f \in L^2(\Omega \times [0, T]) : f(t) \text{ is } \mathcal{F}_t \text{ - measurable for all } t \in [0, T] \}.
\]

Using again that adapted step processes are dense in \( \mathcal{H}^2[0, T] \) (i.e. for any \( \phi \in \mathcal{H}^2[0, T] \) we can find a sequence of step processes \((\phi_n)_n\) such that \( \lim_{n \to \infty} \phi_n = \phi \) in \( L^2(\Omega \times [0, T]) \)), we can define the Itô integral

\[
\int_0^T \phi(t) \, dB_t := \lim_{n \to \infty} \int_0^T \phi_n(t) \, dB_t \quad \text{in } L^2(\Omega).
\]
Properties:

a) The Itô integral is linear and independent of the approximating sequence.

b) If $\phi$ is independent of $\Omega$, then this integral coincides with the Wiener integral.

c) In general, the Itô integral is no longer Gaussian.

d) We have Itô’s isometry, i.e.

$$
E \left| \int_0^T \phi(t) \, dB_t \right|^2 = E \int_0^T |\phi(t)|^2 \, dt, \quad \phi \in H^2[0, T],
$$
in particular, $I: H^2[0, T] \rightarrow L^2(\Omega)$ is continuous.

Exercise 15 (Stochastic version of Fubini’s theorem)

Let $\phi: \Omega \times [0, T] \times [0, T] \rightarrow \mathbb{R}$ be measurable, $(\phi(s, t))_{s \in [0, T]}$ be adapted with respect to the Brownian filtration for all $t \in [0, T]$, and

$$
\int_0^T \left( \int_0^T E|\phi(s, t)|^2 \, ds \right)^{\frac{1}{2}} \, dt < \infty.
$$

Proof that:

a) $(\omega, s) \mapsto \phi(\omega, s, t) \in L^2(\Omega \times [0, T])$ for almost all $t \in [0, T]$ and $t \mapsto \int_0^T \phi(s, t) \, dB_s \in L^1([0, T], L^2(\Omega));$

b) $t \mapsto \phi(\omega, s, t) \in L^1[0, T]$ for almost all $(\omega, s) \in \Omega \times [0, T]$, $(\omega, s) \mapsto \int_0^T \phi(s, t) \, dt \in L^2(\Omega \times [0, T])$, and $(\int_0^T \phi(s, t) \, dt)_{s \in [0, T]}$ is adapted with respect to the Brownian filtration;

c) We have

$$
\int_0^T \int_0^T \phi(s, t) \, dB_s \, dt = \int_0^T \int_0^T \phi(s, t) \, dt \, dB_s \quad \text{in } L^2(\Omega).
$$

Solution: a) By assumption $\|\phi(\cdot, t)\|_{L^2(\Omega \times [0, T])} < \infty$ for almost all $t \in [0, T]$, and

$$
\int_0^T \left( \int_0^T E|\phi(s, t)|^2 \, ds \right)^{\frac{1}{2}} \, dt < \infty,
$$

which implies that $\int_0^T \phi(s, \cdot) \, dB_s \in L^1([0, T]; L^2(\Omega)).$

b) By Minkowski’s integral inequality we get

$$
\left\| \int_0^T \phi(\cdot, t) \, dt \right\|_{L^2(\Omega \times [0, T])} \leq \int_0^T \left\| \phi(\cdot, t) \right\|_{L^2(\Omega \times [0, T])} \, dt \leq \int_0^T \|\phi(\cdot, t)\|_{L^2(\Omega \times [0, T])} \, dt
$$

$$
= \int_0^T \left( \int_0^T E|\phi(s, t)|^2 \, ds \right)^{\frac{1}{2}} \, dt < \infty,
$$
which implies the first two statements of part b). Finally, since \( \int_0^T \phi(s, t) \, dt \) is a pointwise limit of \( \mathcal{F}_s \)-measurable random variables, it is \( \mathcal{F}_s \)-measurable as well.

e) By assumption and part b) the integral \( \int_0^T \phi(\cdot, t) \, dt \) exists as an \( \mathcal{H}^2[0, T] \)-valued Bochner integral. Using now the continuity of the linear operator \( I: \mathcal{H}^2[0, T] \to L^2(\Omega) \), we get that \( I\phi(t) \) is Bochner-integrable in \( L^2(\Omega) \) by part a) with

\[
\int_0^T \int_0^T \phi(s, t) \, dB_s \, dt = \int_0^T I(\phi(\cdot, t)) \, dt = I\left( \int_0^T \phi(\cdot, t) \, dt \right) = \int_0^T \int_0^T \phi(s, t) \, dt \, dB_s.
\]