Short recap: Wiener integral

Let $f(t) = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{(t_{n-1},t_n]}(t)$ be a step function. Then we define the random variable

$$\Big(\int_0^T f(t)\,\mathrm{d}B_t\Big)(\omega) := \sum_{n=1}^{N} \alpha_n \big(B_{t_n}(\omega) - B_{t_{n-1}}(\omega)\big), \quad \omega \in \Omega.$$

In this case, we get

$$\Big\|\underbrace{\int_0^T f(t)\,\mathrm{d}B_t}_{=:I(f)}\Big\|^2_{L^2(\Omega)} = \sum_{n=1}^{N} \alpha_n^2 \mathrm{Var}(B_{t_n} - B_{t_n-1}) = \sum_{n=1}^{N} \alpha_n^2 (t_n - t_{n-1}) = \|f\|^2_{L^2[0,T]}.$$

This means: $\mathbb{E}\big(\int_0^T f(t)\,\mathrm{d}B_t\big)^2 = \int_0^T |f(t)|^2\,\mathrm{d}t$ (Itô isometry). In particular, this implies

$$\langle I(f), I(g)\rangle_{L^2(\Omega)} = \langle f, g\rangle_{L^2[0,T]}, \quad f, g \in L^2[0,T].$$

Using that step functions are dense in $L^2[0,T]$, we can find for any function $f \in L^2[0,T]$ a sequence of step functions $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ in $L^2[0,T]$. Therefore, by Itô's isometry the limit

$$\int_0^T f(t)\,\mathrm{d}B_t := \lim_{n\to\infty} \int_0^T f_n(t)\,\mathrm{d}B_t \quad \text{in } L^2(\Omega)$$

is well-defined, and independent of the approximating sequence.

Properties:

a) The Wiener integral is linear (in particular, the linear operator $I\colon L^2[0,T] \to L^2(\Omega)$ is continuous).

b) $\int_0^T a\,\mathrm{d}B_t = aB_T$, by definition.

c) $\int_0^t f(s)\,\mathrm{d}B_s = \int_0^T \mathbb{1}_{[0,t]}(s) f(s)\,\mathrm{d}B_s$, $t \in [0,T]$.

Exercise 13 (Integration by parts)

Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$ and $\phi \in C^1[0,T]$. Proof that:

$$\int_0^T \phi(t)\,\mathrm{d}B_t = \phi(T)B_T - \int_0^T \phi'(t) B_t\,\mathrm{d}t \qquad \text{almost surely.}$$

Solution: Wlog let $\phi(0) = 0$ (else take $\psi(t) = \phi(t) - \phi(0)$).

1) We start with a general observation: Let $0 \leq t_0 < \ldots < t_N = T$, $(c_n)_n \subseteq \mathbb{R}$, and take

$$g(t) := \sum_{n=1}^{N} c_n \mathbb{1}_{(t_{n-1},t_n]}(t), \qquad G(t) := \sum_{n=1}^{N}\sum_{m=1}^{n} c_m (t_m - t_{m-1}) \mathbb{1}_{(t_{n-1},t_n]}(t).$$

Then

$$\begin{aligned}
\int_0^T g(t)B_t\,\mathrm{d}t &= \sum_{n=1}^{N} c_n \int_{t_{n-1}}^{t_n} B_t\,\mathrm{d}t \\
\int_0^T G(t)\,\mathrm{d}B_t &= \sum_{n=1}^{N}\sum_{m=1}^{n} c_m(t_m - t_{m-1})(B_{t_n} - B_{t_{n-1}}) \\
&= \sum_{m=1}^{N}\sum_{n=m}^{N} c_m(t_m - t_{m-1})(B_{t_n} - B_{t_{n-1}}) \\
&= \sum_{m=1}^{N} c_m(t_m - t_{m-1})(B_T - B_{t_{m-1}}) \\
&= G(T)B_T - \sum_{m=1}^{N} c_m(t_m - t_{m-1})B_{t_{m-1}}.
\end{aligned}$$

2) Let $\varepsilon > 0$. For $k \in \mathbb{N}$ choose a partition $0 = t_0^k < \ldots < t_{N_k}^k = T$ of $[0,T]$ such that $\sup_n |t_n^k - t_{n-1}^k| \to 0$ as $k \to \infty$. Now we define

$$g_k(t) := \sum_{n=1}^{N_k} \phi'(t_{n-1}^k)\mathbb{1}_{(t_{n-1}^k, t_n^k]}(t), \qquad G_k(t) := \sum_{n=1}^{N_k}\sum_{m=1}^{n} \phi'(t_{m-1}^k)(t_m^k - t_{m-1}^k)\mathbb{1}_{(t_{n-1}^k, t_n^k]}(t).$$

(These are just the functions g and G from **1)** with $c_n = \phi'(t_{n-1}^k)$). Using that ϕ' is uniformly continuous on $[0,T]$, we get

$$\begin{aligned}
|g_k(t) - \phi'(t)| &= |\phi'(t_{n-1}^k) - \phi'(t)| < \varepsilon \qquad \text{for } t \in (t_{n-1}^k, t_n^k] \text{ and } k \text{ large enough, and} \\
|G_k(t) - \phi(t)| &= \Big| \underbrace{\sum_{m=1}^{n} \int_{t_{m-1}^k}^{t_m^k} \phi'(t_{m-1}^k)\,\mathrm{d}s - \sum_{m=1}^{n} \int_{t_{m-1}^k}^{t_m^k} \phi'(s)\,\mathrm{d}s}_{|\cdot| \le \varepsilon t_n^k, \text{ if } k \text{ is large enough}} + \int_t^{t_n^k} \phi'(s)\,\mathrm{d}s \Big| \\
&\le \varepsilon t_n^k + (t_n^k - t_{n-1}^k)\|\phi'\|_\infty \le \varepsilon T + \varepsilon\|\phi'\|_\infty \qquad \text{for } k \text{ sufficiently large.}
\end{aligned}$$

Please observe that the estimates above are actually independent of t because of the uniform continuity of ϕ'. This implies that $g_k \to \phi'$ and $G_k \to \phi$ uniformly as $k \to \infty$. This convergence also yields $g_k \to \phi'$ and $G_k \to \phi$ in $L^2[0,T]$ as $k \to \infty$.

3) The established convergences of part **2)** imply

$$\begin{aligned}
\int_0^T G_k(t)\,\mathrm{d}B_t &\to \int_0^T \phi(t)\,\mathrm{d}B_t \quad \text{in } L^2(\Omega) \text{ as } k \to \infty \text{ by Itô's isometry;} \\
G_k(T)B_T &\to \phi(T)B_T \quad \text{almost surely and in } L^2(\Omega) \text{ as } k \to \infty; \\
\int_0^T g_k(t)B_t\,\mathrm{d}t &\to \int_0^T \phi'(t)B_t\,\mathrm{d}t \quad \text{almost surely and in } L^2(\Omega) \text{ as } k \to \infty.
\end{aligned}$$

Now we use that $t \mapsto B_t$ is almost surely uniformly continuous on $[0,T]$ to get

$$\underbrace{\Big|\int_0^T g_k(t)B_t\,\mathrm{d}t - \Big(G_k(T)B_T - \int_0^T G_k(t)\,\mathrm{d}B_t\Big)\Big|}_{=:\Psi_k}$$

$$\overset{1)}{=} \Big|\sum_{n=1}^{N_k} \phi'(t_{n-1}^k)\int_{t_{n-1}^k}^{t_n^k} B_t\,\mathrm{d}t - \sum_{n=1}^{N_k} \phi'(t_{n-1}^k)(t_n^k - t_{n-1}^k)B_{t_{n-1}^k}\Big|$$

$$\le \sum_{n=1}^{N_k} |\phi'(t_{n-1}^k)| \int_{t_{n-1}^k}^{t_n^k} \underbrace{|B_t - B_{t_{n-1}^k}|}_{\le\varepsilon \text{ a.s. for } k \text{ large}}\,\mathrm{d}t \le \varepsilon T\|\phi'\|_\infty.$$

Putting all things together, we almost surely obtain

$$\int_0^T \phi(t)\,\mathrm{d}B_t - \Big(\phi(T)B_T - \int_0^T \phi'(t)B_t\,\mathrm{d}t\Big) = \lim_{k\to\infty}\Psi_k = 0.$$

Exercise 14 (Properties of the Wiener integral process)

a) Let $(B_t)_{t\ge 0}$ be a Brownian motion in $\mathbb{R}$ and $f \in L^2[0,T]$. Then we define the stochastic process $(X_t)_{t\in[0,T]}$ by

$$X_t := \int_0^t f(s)\,\mathrm{d}B_s, \quad t \in [0,T].$$

Show the following properties:

i) $X_t \sim N\big(0, \int_0^t f(s)^2\,\mathrm{d}s\big)$, $t \in [0,T]$;

ii) $(X_t)_{t\in[0,T]}$ has independent increments.

b) Let $(B_t)_{t\ge 0}$ be an n-dimensional Brownian motion and $F \in L^2([0,T], \mathbb{R}^{m\times n})$. Proof that

$$Y_t := \int_0^t F(s)\,\mathrm{d}B_s, \quad t \in [0,T],$$

has an m-dimensional normal distribution, and determine its covariance matrix.

Before we start with the proof we show two auxiliary results:

Claim A: For any random variable $X\colon \Omega \to \mathbb{R}^n$ we have

X has an n-dim. normal distribution $\iff$ $a^\top X$ has a 1-dim. normal distribution for all $a \in \mathbb{R}^n$.

Proof: If $X \sim N(m,C)$ for $m \in \mathbb{R}^n$ and a positive definite $C \in \mathbb{R}^{n\times n}$, then

$$\phi_{a^\top X}(t) = \phi_X(ta) = \exp(ia^\top mt)\exp\big(-\tfrac{1}{2}(a^\top Ca)t^2\big), \quad t \in \mathbb{R},$$

i.e. $a^\top X \sim N(a^\top m, a^\top Ca)$ (see Exercise 3c)).

Conversely, if $a^\top X$ has a 1-dimensional normal distribution for all $a \in \mathbb{R}^n$, we obtain for $m := \mathbb{E}X$ and $C := \mathrm{Cov}(X)$

$$\mathbb{E}(a^\top X) = a^\top m \quad \text{und} \quad \mathrm{Var}(a^\top X) = a^\top Ca,$$

and therefore

$$\phi_X(t) = \phi_{t^\top X}(1) = \exp(it^\top m)\exp(-\tfrac{1}{2}t^\top Ct), \quad t \in \mathbb{R}^n.$$

This implies $X \sim N(m, C)$.

Claim B: A process $(B_t)_{t\geq 0} = (B_t^{(1)}, \dots, B_t^{(n)})_{t\geq 0}$ in $\mathbb{R}^n$ is an n-dimensional Brownian motion if and only if the components $(B_t^{(1)})_{t\geq 0}, \dots, (B_t^{(n)})_{t\geq 0}$ are independent 1-dimensional Brownian motions.

Proof: If the components are independent 1-dimensional Brownian motions, then we have seen in the lectures that $(B_t^{(1)}, \dots, B_t^{(n)})_{t\geq 0}$ satisfies the properties of an n-dimensional Brownian motion.

Conversely, assume that $(B_t)_{t\geq 0} = (B_t^{(1)}, \dots, B_t^{(n)})_{t\geq 0}$ is an n-dimensional Brownian motion. Then we know by Exercise 9 that each component is a 1-dimensional Brownian motion. So, it remains to prove that the processes $(B_t^{(1)})_{t\geq 0}, \dots, (B_t^{(n)})_{t\geq 0}$ are independent.

Since for $0 = t_0 < t_1 < \ldots < t_N$ the random variables

$$B_{t_1} - B_{t_0}, \dots, B_{t_N} - B_{t_{N-1}}$$

are independent and have an n-dimensional normal distribution, the random variable

$$X := (B_{t_1} - B_{t_0}, \dots, B_{t_N} - B_{t_{N-1}})$$

has an nN-dimensional normal distribution.

Now let $(\lambda_k)_{k=1}^N \subset \mathbb{R}^n$. Then

$$\begin{aligned}(\lambda_1 + \dots + \lambda_N, \lambda_2 + \dots + \lambda_N, \dots, \lambda_N)^\top X &= \sum_{m=1}^N \Big(\sum_{k=m}^N \lambda_k^\top\Big)(B_{t_m} - B_{t_{m-1}}) = \sum_{k=1}^N \lambda_k^\top B_{t_k} \\ &= (\lambda_1, \dots, \lambda_N)^\top (B_{t_1}, \dots, B_{t_N}).\end{aligned}$$

So, by Claim A $(B_{t_1}, \dots, B_{t_N})$ has an nN-dimensional normal distribution.

For $j \in \{1, \dots, n\}$ we choose $0 \leq t_1^{(j)} < \ldots < t_{N_j}^{(j)}$ for some $N_j \in \mathbb{N}$. Then by the argument above

$$(B_{t_1^{(1)}}^{(1)}, \dots, B_{t_{N_1}^{(1)}}^{(1)}, \dots\dots, B_{t_1^{(n)}}^{(n)}, \dots, B_{t_{N_n}^{(n)}}^{(n)})$$

has a $\sum_{j=1}^n N_j$-dimensional normal distribution. For $j \neq k$ and $t_m^{(j)} \leq t_p^{(k)}$ the independence of $B_{t_m^{(j)}}$ and $B_{t_p^{(k)}} - B_{t_m^{(j)}}$ implies that $B_{t_m^{(j)}}^{(j)}$ and $B_{t_p^{(k)}}^{(k)} - B_{t_m^{(j)}}^{(k)}$ are independent. This and $B_t \sim N(0, tI)$ then gives

$$\begin{aligned}\mathrm{Cov}(B_{t_m^{(j)}}^{(j)}, B_{t_p^{(k)}}^{(k)}) &= \mathbb{E}\big(B_{t_m^{(j)}}^{(j)}(B_{t_p^{(k)}}^{(k)} - B_{t_m^{(j)}}^{(k)})\big) + \mathbb{E}\big(B_{t_m^{(j)}}^{(j)} B_{t_m^{(j)}}^{(k)}\big) \\ &= \mathbb{E}B_{t_m^{(j)}}^{(j)}\,\mathbb{E}\big(B_{t_p^{(k)}}^{(k)} - B_{t_m^{(j)}}^{(k)}\big) + \mathrm{Cov}(B_{t_m^{(j)}}^{(j)}, B_{t_m^{(j)}}^{(k)}) = 0.\end{aligned}$$

By (a vector valued version of) Exercise 3 e) we obtain that the random vectors

$$(B_{t_k^{(1)}}^{(1)})_{k=1}^{N_1}, \dots, (B_{t_k^{(n)}}^{(n)})_{k=1}^{N_n}$$

are independent, but this means that the processes $B^{(1)}, \dots, B^{(n)}$ are independent.

Solution: a) i) Let $(f_n)_n$ be step functions such that $\lim_{n\to\infty} f_n = f$ in $L^2[0,T]$. Then

$$X_t^n := \int_0^t f_n(s)\,\mathrm{d}B_s \to X_t \quad \text{in } L^2(\Omega) \text{ as } n\to\infty$$

by Itô's isometry. Moreover,

$$\mathbb{E}X_t^n = 0 \to 0 \quad \text{as } n\to\infty,$$
$$\operatorname{Var}(X_t^n) \overset{\text{Itô}}{=} \|f_n\|^2_{L^2[0,T]} \to \|f\|^2_{L^2[0,T]} \quad \text{as } n\to\infty,$$

and X_t^n has a normal distribution because it is a linear combination of independent Gaussians. Using these results as well as the dominated convergence theorem we obtain

$$\phi_{X_t}(s) = \lim_{n\to\infty}\phi_{X_t^n}(s) = \lim_{n\to\infty}\exp\big(-\tfrac{1}{2}s^2\|f_n\|^2_{L^2}\big) = \exp\big(-\tfrac{1}{2}s^2\|f\|^2_{L^2}\big).$$

By uniqueness of the characteristic function we finally get $X_t \sim N(0,\|f\|^2_{L^2})$.

a) ii) Let $(f_k)_k$ be a sequence of step functions approximating f in $L^2[0,T]$. Then we obtain for $0 \le t_1 < \ldots < t_N$ and for a suitable choice of $\alpha_i^{(n)}$'s, $n = 1,\ldots,N$:

$$X^{(k)} := \begin{pmatrix} \int_0^{t_1} f_k\,\mathrm{d}B_t \\ \vdots \\ \int_0^{t_N} f_k\,\mathrm{d}B_t \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^M \alpha_i^{(1)}(B_{s_i}-B_{s_{i-1}}) \\ \vdots \\ \sum_{i=1}^M \alpha_i^{(N)}(B_{s_i}-B_{s_{i-1}}) \end{pmatrix} = \underbrace{A_{f_k}}_{\in\mathbb{R}^{N\times M}} \begin{pmatrix} B_{s_1}-B_{s_0} \\ \vdots \\ B_{s_M}-B_{s_{M-1}} \end{pmatrix},$$

where $A_{f_k} = (a_{n,m})_{n,m} = (\alpha_m^{(n)})_{n,m}$. So $X^{(k)}$ has an N-dimensional normal distribution. Since $\int_0^{t_n} f_k\,\mathrm{d}B_t \to \int_0^{t_n} f\,\mathrm{d}B_t$ in $L^2(\Omega)$ as $k\to\infty$, we obtain

$$X^{(k)} \to X := \Big(\int_0^{t_1} f\,\mathrm{d}B_t,\ldots,\int_0^{t_N} f\,\mathrm{d}B_t\Big) = (X_{t_1},\ldots,X_{t_N}) \quad \text{in } L^2(\Omega;\mathbb{R}^N).$$

In the same way as in part i) this implies that $X \sim N(0,\operatorname{Cov}(X))$. In particular, the random vector $(X_{t_2}-X_{t_1},\ldots,X_{t_N}-X_{t_{n-1}})$ has a normal distribution. Finally, we use Itô's isometry to get

$$\begin{aligned}\operatorname{Cov}(X_{t_j}-X_{t_{j-1}}, X_{t_i}-X_{t_{i-1}}) &= \langle X_{t_j}-X_{t_{j-1}}, X_{t_i}-X_{t_{i-1}}\rangle_{L^2(\Omega)} \\ &= \langle \mathbb{1}_{[t_{j-1},t_j]}f, \mathbb{1}_{[t_{i-1},t_i]}f\rangle_{L^2[0,T]} = 0 \quad \text{for } i\neq j,\end{aligned}$$

which implies that X_t has independent increments by Exercise 3 e).

b) Please observe that by Claim B and the definition of the Wiener integral the random variables

$$\int_0^t g_1(s)\,\mathrm{d}B_s^{(1)},\ldots,\int_0^t g_n(s)\,\mathrm{d}B_s^{(n)}$$

are independent for arbitrary functions $g_1,\ldots,g_n \in L^2[0,T]$. Using that

$$\int_0^t F(s)\,\mathrm{d}B_s = \begin{pmatrix} \sum_{k=1}^n \int_0^t F_{1k}(s)\,\mathrm{d}B_s^{(k)} \\ \vdots \\ \sum_{k=1}^n \int_0^t F_{mk}(s)\,\mathrm{d}B_s^{(k)} \end{pmatrix}$$

we get for any $a\in\mathbb{R}^m$

$$a^\top \int_0^t F(s)\,\mathrm{d}B_s = \sum_{k=1}^n \int_0^t a^\top F_{\cdot k}(s)\,\mathrm{d}B_s^{(k)}.$$

By part a) and the remark above the right-hand side is a sum of n independent Gaussian random variables, so itself a 1-dimensional normally distributed random variable. By Claim A $\int_0^t F(s)\,\mathrm{d}B_s$ has an m-dimensional normal distribution. Moreover, by part a)

$$\mathbb{E}\int_0^t F(s)\,\mathrm{d}B_s = 0,$$

and by independence and Itô's isometry we get

$$\begin{aligned}
\mathrm{Cov}(Y_{t,\ell}, Y_{t,\tilde{\ell}}) &= \sum_{k=1}^n \sum_{\tilde{k}=1}^n \Big\langle \int_0^t F_{\ell k}(s)\,\mathrm{d}B_s^{(k)}, \int_0^t F_{\tilde{\ell}\tilde{k}}(s)\,\mathrm{d}B_s^{(\tilde{k})} \Big\rangle_{L^2(\Omega)} \\
&= \sum_{k=1}^n \Big\langle \int_0^t F_{\ell k}(s)\,\mathrm{d}B_s^{(k)}, \int_0^t F_{\tilde{\ell}k}(s)\,\mathrm{d}B_s^{(k)} \Big\rangle_{L^2(\Omega)} \\
&= \sum_{k=1}^n \int_0^t F_{\ell k}(s) F_{\tilde{\ell}k}(s)\,\mathrm{d}s \\
&= \int_0^t (F(s)F(s)^\top)_{\ell\tilde{\ell}}\,\mathrm{d}s.
\end{aligned}$$

This means $\mathrm{Cov}(Y_t) = \int_0^t F(s)F(s)^\top\,\mathrm{d}s$.

Short recap: Itô integral

The basic idea is to extend the Wiener integral to an integral for *processes* $\phi\colon \Omega\times[0,T] \to \mathbb{R}$. This procedure was done in two steps:

1st step: Define an integral for *adapted* step processes, i.e.

$$\phi(\omega, t) = \sum_{n=1}^N \alpha_n(\omega)\mathbb{1}_{(t_{n-1},t_n]}(t),$$

where α_n is $\mathcal{F}_{t_{n-1}}$-measurable (here, $\mathcal{F}_t := \sigma(B_s, s \le t)$, $t \ge 0$). Then we define

$$\Big(\int_0^T \phi(t)\,\mathrm{d}B_t\Big)(\omega) := \sum_{n=1}^N \alpha_n(\omega)\big(B_{t_n}(\omega) - B_{t_{n-1}}(\omega)\big), \quad \omega \in \Omega.$$

In this case, Itô's isometry still holds in the following way

$$\|I(\phi)\|^2_{L^2(\Omega)} = \mathbb{E}\Big|\int_0^T \phi(t)\,\mathrm{d}B_t\Big|^2 = \mathbb{E}\int_0^T |\phi(t)|^2\,\mathrm{d}t = \|\phi\|^2_{L^2(\Omega\times[0,T])}.$$

2nd step: Define an integral for *adapted* processes, i.e. a process ϕ satisfying

$$\phi \in \mathcal{H}^2[0,T] := \{f \in L^2(\Omega\times[0,T])\colon f(t) \text{ is } \mathcal{F}_t - \text{measurable for all } t \in [0,T]\}.$$

Using again that adapted step processes are dense in $\mathcal{H}^2[0,T]$ (i.e. for any $\phi \in \mathcal{H}^2[0,T]$ we can find a sequence of step processes $(\phi_n)_n$ such that $\lim_{n\to\infty}\phi_n = \phi$ in $L^2(\Omega\times[0,T])$), we can define the Itô integral

$$\int_0^T \phi(t)\,\mathrm{d}B_t := \lim_{n\to\infty}\int_0^T \phi_n(t)\,\mathrm{d}B_t \qquad \text{in } L^2(\Omega).$$

Properties:

a) The Itô integral is linear and independent of the approximating sequence.

b) If ϕ is independent of Ω, then this integral coincides with the Wiener integral.

c) In general, the Itô integral is no longer Gaussian.

d) We have Itô's isometry, i.e.

$$\mathbb{E}\Big|\int_0^T \phi(t)\,\mathrm{d}B_t\Big|^2 = \mathbb{E}\int_0^T |\phi(t)|^2\,\mathrm{d}t, \qquad \phi \in \mathcal{H}^2[0,T],$$

in particular, $I\colon \mathcal{H}^2[0,T] \to L^2(\Omega)$ is continuous.

Exercise 15 (Stochastic version of Fubini's theorem)

Let $\phi\colon \Omega\times[0,T]\times[0,T] \to \mathbb{R}$ be measurable, $(\phi(s,t))_{s\in[0,T]}$ be adapted with respect to the Brownian filtration for all $t \in [0,T]$, and

$$\int_0^T \Big(\int_0^T \mathbb{E}|\phi(s,t)|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\,\mathrm{d}t < \infty.$$

Proof that:

a) $(\omega,s) \mapsto \phi(\omega,s,t) \in L^2(\Omega\times[0,T])$ for almost all $t\in[0,T]$ and $t \mapsto \int_0^T \phi(s,t)\,\mathrm{d}B_s \in L^1([0,T],L^2(\Omega))$;

b) $t \mapsto \phi(\omega,s,t) \in L^1[0,T]$ for almost all $(\omega,s)\in\Omega\times[0,T]$, $(\omega,s)\mapsto \int_0^T \phi(s,t)\,\mathrm{d}t \in L^2(\Omega\times[0,T])$, and $\big(\int_0^T \phi(s,t)\,\mathrm{d}t\big)_{s\in[0,T]}$ is adapted with respect to the Brownian filtration;

c) We have

$$\int_0^T\int_0^T \phi(s,t)\,\mathrm{d}B_s\,\mathrm{d}t = \int_0^T\int_0^T \phi(s,t)\,\mathrm{d}t\,\mathrm{d}B_s \quad \text{in } L^2(\Omega).$$

Solution: a) By assumption $\|\phi(\cdot,t)\|_{L^2(\Omega\times[0,T])} < \infty$ for almost all $t\in[0,T]$, and

$$\int_0^T \underbrace{\Big\|\int_0^T \phi(s,t)\,\mathrm{d}B_s\Big\|_{L^2(\Omega)}}_{\overset{\text{Itô}}{=}(\mathbb{E}\int_0^T|\phi(s,t)|^2\,\mathrm{d}s)^{\frac{1}{2}}}\,\mathrm{d}t = \int_0^T\Big(\int_0^T \mathbb{E}|\phi(s,t)|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\,\mathrm{d}t < \infty,$$

which implies that $\int_0^T \phi(s,\cdot)\,\mathrm{d}B_s \in L^1([0,T];L^2(\Omega))$.

b) By Minkowski's integral inequality we get

$$\begin{aligned}\Big\|\int_0^T \phi(\cdot,t)\,\mathrm{d}t\Big\|_{L^2(\Omega\times[0,T])} &\le \Big\|\int_0^T |\phi(\cdot,t)|\,\mathrm{d}t\Big\|_{L^2(\Omega\times[0,T])} \le \int_0^T \|\phi(\cdot,t)\|_{L^2(\Omega\times[0,T])}\,\mathrm{d}t\\ &= \int_0^T\Big(\int_0^T \mathbb{E}|\phi(s,t)|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\,\mathrm{d}t < \infty,\end{aligned}$$

which implies the first two statements of part b). Finally, since $\int_0^T \phi(s,t)\,\mathrm{d}t$ is a pointwise limit of $\mathcal{F}_s$-measurable random variables, it is $\mathcal{F}_s$-measurable as well.

c) By assumption and part b) the integral $\int_0^T \phi(\cdot,t)\,\mathrm{d}t$ exists as an $\mathcal{H}^2[0,T]$-valued Bochner integral. Using now the continuity of the linear operator $I\colon \mathcal{H}^2[0,T] \to L^2(\Omega)$, we get that $I\phi(t)$ is Bochner-integrable in $L^2(\Omega)$ by part a) with

$$\int_0^T \int_0^T \phi(s,t)\,\mathrm{d}B_s\,\mathrm{d}t = \int_0^T I(\phi(\cdot,t))\,\mathrm{d}t = I\Big(\int_0^T \phi(\cdot,t)\,\mathrm{d}t\Big) = \int_0^T \int_0^T \phi(s,t)\,\mathrm{d}t\,\mathrm{d}B_s.$$