Short recap: Itô isometry and Itô formula

**Itô isometry:** For any $\phi \in H^2[0,T]$ we have

$$E \left| \int_0^T \phi(t) dB_t \right|^2 = E \int_0^T |\phi(s)|^2 ds.$$  

**Itô formula ( = stochastic version of the fundamental theorem of calculus):** Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. Then

$$f(t, B_t) = f(0, 0) + \int_0^t \partial_t f(s, B_s) + \frac{1}{2} \partial_{xx} f(s, B_s) \, ds + \int_0^t \partial_x f(s, B_s) \, dB_s.$$  

**Exercise 16 (Itô integrals)**

Let $(B_t)_{t \geq 0}$ be an $\mathbb{R}$-valued Brownian motion.

a) For $t \geq 0$ determine the variances of the following integrals:

i) $\int_0^t |B_s|^2 \, dB_s$;

ii) $\int_0^t (B_s + s)^2 \, dB_s$.

b) For $t \geq 0$ calculate the following Itô integrals:

i) $\int_0^t B_s^2 \, dB_s$;

ii) $\int_0^t B_s^n \, dB_s$, $n \in \mathbb{N}$.

**Solution:**  

a) i) By Itô isometry we get

$$E \left| \int_0^t |B_s|^2 \, dB_s \right|^2 = E \int_0^t |B_s|^2 \, ds = \int_0^t E|B_s|^2 \, ds \overset{\text{Fub.}}{=} \frac{1}{2} \Gamma(1) t^2 = \frac{1}{2} \frac{4}{3} t^2.$$  

a) ii) Using Itô’s isometry we obtain

$$E \left| \int_0^t (B_s + s)^2 \, dB_s \right|^2 = E \int_0^t |B_s + s|^4 \, ds$$

$$\overset{\text{Fub.}}{=} \int_0^t \underbrace{E B_s^4}_{s^4} + 4s \underbrace{E B_s^3}_{s^3} + 6s^2 \underbrace{E B_s^2}_{s^2} + 4s^3 \underbrace{E B_s}_{s} + s^4 \, ds$$

$$= \int_0^t 3s^2 + 6s^3 + s^4 \, ds = t^3 + \frac{3}{2} t^4 + \frac{1}{5} t^5.$$  

b) By Itô formula we have

$$\int_0^t \partial_x f(s, B_s) \, dB_s = f(t, B_t) - f(0, 0) - \int_0^t \partial_t f(s, B_s) + \frac{1}{2} \partial_{xx} f(s, B_s) \, ds.$$  

In particular,

$$\int_0^t \frac{d}{ds} f(B_s) \, dB_s = f(B_t) - f(0) - \frac{1}{2} \int_0^t \frac{d^2}{ds^2} f(B_s) \, ds.$$  

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b) i) In this case we choose \( f(x) = \frac{1}{3} x^3 \). Then \( f'(x) = x^2 \) and \( f''(x) = 2x \). Now Itô’s formula implies
\[
\int_0^t B_s^2 \, dB_s = \frac{1}{3} B_t^3 - \frac{1}{2} \int_0^t 2B_s \, ds = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds.
\]

b) ii) Here we take \( f(x) = \frac{1}{n+1} x^{n+1} \). Then \( f'(x) = x^n \) and \( f''(x) = nx^{n-1} \). Then, by Itô’s formula
\[
\int_0^t B_s^n \, dB_s = \frac{1}{n+1} B_t^{n+1} - \frac{n}{2} \int_0^t B_s^{n-1} \, ds.
\]
In particular, for \( n = 1 \) we get
\[
\int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t.
\]

Exercise 16 b) shows very clearly, that the stochastic calculus does not share the same integration rules we know from real analysis. If we define \( h_n(t) = \frac{1}{n!} t^n \), then we get the well-known formula
\[
\int_0^t h_n(s) \, ds = h_{n+1}(t),
\]
but
\[
\int_0^t h_n(B_s) \, dB_s = h_{n+1}(B_t) - \frac{1}{2} \int_0^t h_{n-1}(B_s) \, ds.
\]
In the Itô calculus the substitutes for those polynomials \( h_n \) as above are the Hermite polynomials.

Exercise 17  (Hermite polynomials)
For any \( n \in \mathbb{N}_0 \) we define the Hermite polynomial by
\[
h_n(t, x) := \frac{(-t)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad t, x \in \mathbb{R}.
\]
Show that:
\[
\int_0^t h_n(s, B_s) \, dB_s = h_{n+1}(t, B_t), \quad t \geq 0, n \in \mathbb{N}_0.
\]

Solution: 1) Let \( \lambda \in \mathbb{R} \) and \( f(t, x) := \exp(\lambda x - \frac{1}{2} \lambda^2 t) \). Then
\[
\partial_t f(t, x) = -\frac{\lambda^2}{2} \exp(\lambda x - \frac{1}{2} \lambda^2 t) = -\frac{\lambda^2}{2} f(t, x);
\]
\[
\partial_x f(t, x) = \lambda f(t, x);
\]
\[
\partial_{xx} f(t, x) = \lambda^2 f(t, x).
\]
Now Itô’s formula implies
\[
f(t, B_t) = 1 + \int_0^t -\frac{\lambda^2}{2} f(s, B_s) + \frac{\lambda^2}{2} f(s, B_s) \, ds + \int_0^t \lambda f(s, B_s) \, dB_s
\]
\[
= 1 + \lambda \int_0^t f(s, B_s) \, dB_s.
\]
2) Next we show that for \( g_{t,x}(\lambda) := \exp(-\frac{(x-\lambda t)^2}{2t}) \) we have \( g^{(n)}_{t,x}(0) = (-t)^n \frac{d^n}{dx^n} \exp(-\frac{x^2}{2t}) \).

It holds that
\[
\frac{d^n}{dx^n} \exp(-\frac{x^2}{2t}) = \frac{d^n}{dx^n} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{1}{(2t)^k} x^{2k} = \begin{cases} 
\sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{1}{(2t)^k} (2k \cdot \ldots \cdot (2k - (n-1))) x^{2k-n}, & \text{if } n \text{ is even,} \\
\sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{\infty} \frac{1}{k!} (-1)^k \frac{1}{(2t)^k} (2k \cdot \ldots \cdot (2k - (n-1))) x^{2k-n}, & \text{if } n \text{ is odd,}
\end{cases}
\]

and with that
\[
g_{t,x}(\lambda) = \exp(-\frac{(x-\lambda t)^2}{2t}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{(x-\lambda t)^2}{2t}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{1}{(2t)^k} (2k \cdot \ldots \cdot (2k - (n-1))) x^{2k-n} n!
\]

Comparing the coefficients we get the stated result above.

3) By part 2) we obtain for \( \tilde{f}_{t,x}(\lambda) := \exp(\lambda x - \frac{1}{2} \lambda^2 t) \) the equality \( \tilde{f}_{t,x}(\lambda) = g_{t,x}(\lambda) \exp(\frac{x^2}{2t}) \), which implies
\[
\tilde{f}^{(n)}_{t,x}(0) = \exp(\frac{x^2}{2t}) g^{(n)}_{t,x}(0) = (-t)^n \frac{d^n}{dx^n} \exp(-\frac{x^2}{2t}) = n! h_n(t, x).
\]

Having this at hand we obtain
\[
f(t, B_t) = \tilde{f}_{t,B_t}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \tilde{f}^{(n)}_{t,B_t}(0) = \sum_{n=0}^{\infty} \lambda^n h_n(t, B_t).
\]

Now we use 1) to get
\[
\sum_{n=0}^{\infty} \lambda^n h_n(t, B_t) = f(t, B_t) = 1 + \lambda \int_0^t f(s, B_s) dB_s = 1 + \lambda \sum_{n=0}^{\infty} \int_0^t \lambda^n h_n(s, B_s) dB_s = 1 + \sum_{n=1}^{\infty} \int_0^t h_{n-1}(s, B_s) dB_s.
\]

Comparing the coefficients we finally obtain
\[
\int_0^t h_{n-1}(s, B_s) dB_s = h_n(t, B_t), \quad n \in \mathbb{N}.
\]
Exercise 18  (Kolmogorov)

Let $X: [0,1] \times \Omega \to \mathbb{R}$ be a stochastic process with almost surely continuous paths such that

$$\mathbb{E}(|X(t) - X(s)|^\beta) \leq c|t - s|^{1+\alpha}$$

for $\alpha, \beta > 0$, $c \geq 0$, and $s, t \in [0,1]$. Show that for all $\gamma \in (0, \frac{\alpha}{\beta})$ and almost all $\omega \in \Omega$ there is a constant $C = C(\omega)$ such that

$$|X(t, \omega) - X(s, \omega)| \leq C|t - s|^\gamma$$

for $s, t \in [0,1]$. This implies that the path $t \mapsto X(t, \omega)$ is $\gamma$-Hölder continuous for almost all $\omega \in \Omega$.

Hints for the proof:

1) For $0 < \gamma < \frac{\alpha}{\beta}$ and $n \in \mathbb{N}$ we consider the set

$$A_n := \{|X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})| > \frac{1}{2^n} \text{ for some } i \in \{0, \ldots, 2^n - 1\}\}.$$

Now estimate $\mathbb{P}(A_n)$ appropriately.

2) Show that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, and apply the Lemma of Borel-Cantelli to get

$$|X(\frac{i+1}{2^n}, \omega) - X(\frac{i}{2^n}, \omega)| \leq K(\omega)\frac{1}{2^n}$$

for some $K(\omega) > 0$, all $i \in \{0, \ldots, 2^n - 1\}$, and almost all $\omega \in \Omega$.

3) Using 2), prove that

$$|X(t_1, \omega) - X(t_2, \omega)| \leq C(\omega)|t_1 - t_2|^\gamma$$

for all dyadic rationals $t_1, t_2 \in [0,1]$ and a constant $C = C(\omega) > 0$.

4) Finally, use that $X$ has almost surely continuous paths and 3) to conclude the proof.

Solution: 1) For $n \in \mathbb{N}$ we define

$$A_n := \{|X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})| > \frac{1}{2^n} \text{ for some } i \in \{0, \ldots, 2^n - 1\}\}.$$

Then by Markov’s inequality

$$\mathbb{P}(A_n) \leq \sum_{i=0}^{2^n-1} \mathbb{P}\left(|X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})| > \frac{1}{2^n}\right) \leq \sum_{i=0}^{2^n-1} \left(\frac{1}{2^n}\right)^{-\beta} \mathbb{E}\left|X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})\right|^\beta$$

$$\leq c \sum_{i=0}^{2^n-1} 2^{n\gamma-\beta-n(1+\alpha)} = c 2^{n(\gamma\beta-\alpha)}.$$

2) Using that $\gamma\beta - \alpha < 0$ by assumption, we obtain $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Therefore, the Lemma of Borel-Cantelli implies that $\mathbb{P}(\limsup_{n \to \infty} A_n) = 0$, which means that for almost all $\omega \in \Omega$ there is a number $m(\omega) \in \mathbb{N}$ such that

$$|X(\frac{i+1}{2^n}, \omega) - X(\frac{i}{2^n}, \omega)| \leq \frac{1}{2^n} \text{ for all } i = 0, \ldots, 2^n - 1$$

if $n \geq m(\omega)$. However, this implies that for those $\omega \in \Omega$ we get

$$|X(\frac{i+1}{2^n}, \omega) - X(\frac{i}{2^n}, \omega)| \leq K(\omega)\frac{1}{2^n} \text{ for all } i = 0, \ldots, 2^n - 1 \text{ and all } n \in \mathbb{N} \quad (*)$$

if we choose $K(\omega)$ large enough.
3) Let \( \Omega_0 \) be set of full measure where (*) holds. Let \( t_1, t_2 \in [0, 1] \), \( t_2 > t_1 \), be dyadic rationals. Then choose \( n \in \mathbb{N} \) such that
\[
2^{-n} \leq t_2 - t_1 < 2^{-n+1}.
\]
Then there are \( i, j \in \{0, \ldots, 2^n\} \) such that
\[
t_1 \leq \frac{i}{2^n} \leq \frac{j}{2^n} \leq t_2,
\]
which implies that
\[
\frac{j - i}{2^n} \leq t_2 - t_1 < 2^{-n+1} = \frac{2}{2^n}.
\]
Thus, \( j - i \in \{0, 1\} \), i.e. \( j = i \) or \( j = i + 1 \). Hence, by (*) we get for \( \omega \in \Omega_0 \)
\[
|X(\frac{i}{2^n}, \omega) - X(\frac{j}{2^n}, \omega)| \leq K(\omega)|\frac{j - i}{2^n}|^\gamma \leq K(\omega)|t_2 - t_1|^\gamma.
\]
Moreover, by the dyadic nature there may be numbers \( n < p_1 < p_2 < \ldots < p_k \) and \( n < q_1 < q_2 < \ldots < q_\ell \) such that
\[
t_1 = \frac{i}{2^n} - \sum_{r=1}^{k} \frac{1}{2^{p_r}} \quad \text{and} \quad t_2 = \frac{j}{2^n} + \sum_{r=1}^{\ell} \frac{1}{2^{q_r}}.
\]
Again by (*) we obtain for \( \omega \in \Omega_0 \)
\[
|X(\frac{i}{2^n} - \frac{1}{2^{p_1}} - \ldots - \frac{1}{2^{p_k}}, \omega) - X(\frac{j}{2^n} - \frac{1}{2^{q_1}} - \ldots - \frac{1}{2^{q_\ell}}, \omega)| \leq K(\omega)\frac{1}{2^{q_\ell}}
\]
for any \( r \in \{1, \ldots, k\} \), which implies
\[
|X(t_1, \omega) - X(\frac{i}{2^n}, \omega)| \leq K(\omega)\sum_{r=1}^{k} 2^{-p_r} r \leq \frac{K(\omega)}{2^{n\gamma}} \sum_{r=1}^{\infty} 2^{-r} r = \frac{K'(\omega)}{2^{n\gamma}} \leq K'(\omega)|t_2 - t_1|^\gamma,
\]
where we used that \( p_r > n + r \) by assumption. Similarly, we get
\[
|X(t_2, \omega) - X(\frac{j}{2^n}, \omega)| \leq K'(\omega)|t_2 - t_1|^\gamma.
\]
With that we obtain by triangle inequality
\[
|X(t_1, \omega) - X(t_2, \omega)| \leq |X(t_1, \omega) - X(\frac{i}{2^n}, \omega)| + |X(\frac{i}{2^n}, \omega) - X(\frac{j}{2^n}, \omega)| + |X(\frac{j}{2^n}, \omega) - X(t_2, \omega)|
\leq (2K'(\omega) + K(\omega))|t_2 - t_1|^\gamma = C(\omega)|t_2 - t_1|^\gamma.
\]
4) Let \( \Omega_1 \) be the set where \( X \) is continuous. Then the result follows immediately for all \( \omega \in \Omega_0 \cap \Omega_1 \) by continuity of \( X \). However, since \( \mathbb{P}(\Omega_0 \cap \Omega_1) = 1 \) this already concludes the proof.

Remarks:

a) For any Brownian motion \( B \) Exercise 8 implies
\[
\mathbb{E}|B_t - B_s|^{2k} = C_{2k}|t - s|^k \quad \forall k \in \mathbb{N},
\]
so Exercise 18 implies that almost surely every Brownian motion has \( \gamma \)-Hölder continuous paths for \( \gamma < \frac{1}{2} \).

b) We only needed the continuity assumption in the last step. If \( X \) is not continuous, then this step can be modified to construct a version \( \tilde{X} \) of \( X \) (i.e. \( \mathbb{P}(X(t) = \tilde{X}(t)) = 1 \) for all \( t \in [0, 1] \)) such that almost surely \( \tilde{X} \) has \( \gamma \)-Hölder continuous sample paths for \( \gamma < \frac{1}{2} \). This result is usually known as the Theorem of Kolmogorov-Chentsov.