Short recap: Itô isometry and Itô formula

Itô isometry: For any $\phi \in \mathcal{H}^2[0,T]$ we have

$$\mathbb{E}\Big|\int_0^T \phi(t)\,\mathrm{d}B_t\Big|^2 = \mathbb{E}\int_0^T |\phi(s)|^2\,\mathrm{d}s.$$

Itô formula (= stochastic version of the fundamental theorem of calculus): Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. Then

$$f(t,B_t) = f(0,0) + \int_0^t \partial_t f(s,B_s) + \tfrac{1}{2}\partial_{xx} f(s,B_s)\,\mathrm{d}s + \int_0^t \partial_x f(s,B_s)\,\mathrm{d}B_s.$$

Exercise 16 (Itô integrals)

Let $(B_t)_{t\geq 0}$ be an $\mathbb{R}$-valued Brownian motion.

a) For $t \geq 0$ determine the variances of the following integrals:

   i) $\int_0^t |B_s|^{\frac{1}{2}}\,\mathrm{d}B_s$;

   ii) $\int_0^t (B_s+s)^2\,\mathrm{d}B_s$.

b) For $t \geq 0$ calculate the following Itô integrals:

   i) $\int_0^t B_s^2\,\mathrm{d}B_s$;

   ii) $\int_0^t B_s^n\,\mathrm{d}B_s, \quad n \in \mathbb{N}$.

Solution: a) i) By Itô isometry we get

$$\mathbb{E}\Big|\int_0^t |B_s|^{\frac{1}{2}}\,\mathrm{d}B_s\Big|^2 = \mathbb{E}\int_0^t |B_s|\,\mathrm{d}s = \int_0^t \mathbb{E}|B_s|\,\mathrm{d}s \overset{\gamma\sim N(0,1)}{=} \int_0^t s^{\frac{1}{2}}\mathbb{E}|\gamma|\,\mathrm{d}s$$
$$= \mathbb{E}|\gamma|\int_0^t s^{\frac{1}{2}}\,\mathrm{d}s \overset{\text{Ex. 8}}{=} \tfrac{1}{\sqrt{2\pi}}2\Gamma(1)\tfrac{2}{3}t^{\frac{3}{2}} = \tfrac{1}{\sqrt{2\pi}}\tfrac{4}{3}t^{\frac{3}{2}},$$

a) ii) Using Itô's isometry we obtain

$$\mathbb{E}\Big|\int_0^t (B_s+s)^2\,\mathrm{d}B_s\Big|^2 = \mathbb{E}\int_0^t |B_s+s|^4\,\mathrm{d}s$$
$$\overset{\text{Fub.}}{=} \int_0^t \underbrace{\mathbb{E}B_s^4}_{\overset{\text{Ex. 8}}{=}3s^2} + 4s\underbrace{\mathbb{E}B_s^3}_{=0} + 6s^2\underbrace{\mathbb{E}B_s^2}_{=s} + 4s^3\underbrace{\mathbb{E}B_s}_{=0} + s^4\,\mathrm{d}s$$
$$= \int_0^t 3s^2 + 6s^3 + s^4\,\mathrm{d}s = t^3 + \tfrac{3}{2}t^4 + \tfrac{1}{5}t^5.$$

b) By Itô formula we have

$$\int_0^t \partial_x f(s,B_s)\,\mathrm{d}B_s = f(t,B_t) - f(0,0) - \int_0^t \partial_t f(s,B_s) + \tfrac{1}{2}\partial_{xx} f(s,B_s)\,\mathrm{d}s.$$

In particular,

$$\int_0^t \tfrac{\mathrm{d}}{\mathrm{d}x} f(B_s)\,\mathrm{d}B_s = f(B_t) - f(0) - \tfrac{1}{2}\int_0^t \tfrac{\mathrm{d}^2}{\mathrm{d}x^2} f(B_s)\,\mathrm{d}s.$$

b) i) In this case we choose $f(x) = \frac{1}{3}x^3$. Then $f'(x) = x^2$ and $f''(x) = 2x$. Now Itô's formula implies

$$\int_0^t B_s^2 \,\mathrm{d}B_s = \tfrac{1}{3}B_t^3 - \tfrac{1}{2}\int_0^t 2B_s \,\mathrm{d}s = \tfrac{1}{3}B_t^3 - \int_0^t B_s \,\mathrm{d}s.$$

b) ii) Here we take $f(x) = \frac{1}{n+1}x^{n+1}$. Then $f'(x) = x^n$ and $f''(x) = nx^{n-1}$. Then, by Itô's formula

$$\int_0^t B_s^n \,\mathrm{d}B_s = \tfrac{1}{n+1}B_t^{n+1} - \tfrac{n}{2}\int_0^t B_s^{n-1} \,\mathrm{d}s.$$

In particular, for $n = 1$ we get $\int_0^t B_s \,\mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$.

Exercise 16 b) shows very clearly, that the stochastic calculus does not share the same integration rules we know from real analysis. If we define $h_n(t) = \frac{1}{n!}t^n$, then we get the well-known formula

$$\int_0^t h_n(s) \,\mathrm{d}s = h_{n+1}(t),$$

but

$$\int_0^t h_n(B_s) \,\mathrm{d}B_s = h_{n+1}(B_t) - \tfrac{1}{2}\int_0^t h_{n-1}(B_s) \,\mathrm{d}s.$$

In the Itô calculus the substitutes for those polynomials h_n as above are the Hermite polynomials.

Exercise 17 (Hermite polynomials)

For any $n \in \mathbb{N}_0$ we define the Hermite polynomial by

$$h_n(t,x) := \frac{(-t)^n}{n!} e^{\frac{x^2}{2t}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\frac{x^2}{2t}}, \quad t, x \in \mathbb{R}.$$

Show that:

$$\int_0^t h_n(s, B_s) \,\mathrm{d}B_s = h_{n+1}(t, B_t), \quad t \geq 0, n \in \mathbb{N}_0.$$

Solution: 1) Let $\lambda \in \mathbb{R}$ and $f(t,x) := \exp(\lambda x - \frac{1}{2}\lambda^2 t)$. Then

$$\begin{aligned}
\partial_t f(t,x) &= -\tfrac{\lambda^2}{2}\exp(\lambda x - \tfrac{1}{2}\lambda^2 t) = -\tfrac{\lambda^2}{2} f(t,x); \\
\partial_x f(t,x) &= \lambda f(t,x); \\
\partial_{xx} f(t,x) &= \lambda^2 f(t,x).
\end{aligned}$$

Now Itô's formula implies

$$\begin{aligned}
f(t, B_t) &= 1 + \int_0^t -\tfrac{\lambda^2}{2} f(s, B_s) + \tfrac{\lambda^2}{2} f(s, B_s) \,\mathrm{d}s + \int_0^t \lambda f(s, B_s) \,\mathrm{d}B_s \\
&= 1 + \lambda \int_0^t f(s, B_s) \,\mathrm{d}B_s.
\end{aligned}$$

2) Next we show that for $g_{t,x}(\lambda) := \exp(-\frac{(x-\lambda t)^2}{2t})$ we have $g_{t,x}^{(n)}(0) = (-t)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp(-\frac{x^2}{2t})$. It holds that

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp\Big(-\frac{x^2}{2t}\Big) &= \frac{\mathrm{d}^n}{\mathrm{d}x^n} \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k \frac{1}{(2t)^k} x^{2k} \\
&= \begin{cases} \displaystyle\sum_{k=\frac{n}{2}}^{\infty} \frac{1}{k!}(-1)^k \frac{1}{(2t)^k} \big(2k \cdot \ldots \cdot (2k-(n-1))\big) x^{2k-n}, & \text{if } n \text{ is even,} \\ \displaystyle\sum_{k=\frac{n+1}{2}}^{\infty} \frac{1}{k!}(-1)^k \frac{1}{(2t)^k} \big(2k \cdot \ldots \cdot (2k-(n-1))\big) x^{2k-n}, & \text{if } n \text{ is odd,} \end{cases} \\
&= \sum_{k=[\frac{n}{2}]}^{\infty} \frac{1}{k!}(-1)^k \frac{1}{(2t)^k} \frac{(2k)!}{(2k-n)!} x^{2k-n},
\end{aligned}$$

and with that

$$\begin{aligned}
g_{t,x}(\lambda) = \exp\Big(-\frac{(x-\lambda t)^2}{2t}\Big) &= \sum_{k=0}^{\infty} \frac{1}{k!}\Big(-\frac{(x-\lambda t)^2}{2t}\Big)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k \frac{1}{(2t)^k} \sum_{n=0}^{2k} \binom{2k}{n} (-1)^n \lambda^n t^n x^{2k-n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n (-t)^n \sum_{k=[\frac{n}{2}]}^{\infty} \frac{1}{k!}(-1)^k \frac{1}{(2t)^k} \frac{(2k)!}{(2k-n)!} x^{2k-n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n (-t)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp\Big(-\frac{x^2}{2t}\Big) \stackrel{!}{=} \sum_{n=0}^{\infty} \frac{g_{t,x}^{(n)}(0)}{n!} \lambda^n.
\end{aligned}$$

Comparing the coefficients we get the stated result above.

3) By part 2) we obtain for $\widetilde{f}_{t,x}(\lambda) := \exp(\lambda x - \frac{1}{2}\lambda^2 t)$ the equality $\widetilde{f}_{t,x}(\lambda) = g_{t,x}(\lambda) \exp(\frac{x^2}{2t})$, which implies

$$\widetilde{f}_{t,x}^{(n)}(0) = \exp\Big(\frac{x^2}{2t}\Big) g_{t,x}^{(n)}(0) = (-t)^n \exp\Big(\frac{x^2}{2t}\Big) \frac{\mathrm{d}^n}{\mathrm{d}x^n} \exp\Big(-\frac{x^2}{2t}\Big) = n!\, h_n(t,x).$$

Having this at hand we obtain

$$f(t,B_t) = \widetilde{f}_{t,B_t}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \widetilde{f}_{t,B_t}^{(n)}(0) = \sum_{n=0}^{\infty} \lambda^n h_n(t,B_t).$$

Now we use 1) to get

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda^n h_n(t,B_t) = f(t,B_t) = 1 + \lambda \int_0^t f(s,B_s)\,\mathrm{d}B_s &= 1 + \lambda \sum_{n=0}^{\infty} \int_0^t \lambda^n h_n(s,B_s)\,\mathrm{d}B_s \\
&= 1 + \sum_{n=1}^{\infty} \lambda^n \int_0^t h_{n-1}(s,B_s)\,\mathrm{d}B_s.
\end{aligned}$$

Comparing the coefficients we finally obtain

$$\int_0^t h_{n-1}(s,B_s)\,\mathrm{d}B_s = h_n(t,B_t), \quad n \in \mathbb{N}.$$

Exercise 18 (Kolmogorov)

Let $X\colon [0,1]\times\Omega \to \mathbb{R}$ be a stochastic process with almost surely continuous paths such that

$$\mathbb{E}\big(|X(t)-X(s)|^\beta\big) \le c\,|t-s|^{1+\alpha}$$

for $\alpha,\beta>0$, $c\ge 0$, and $s,t\in[0,1]$. Show that for all $\gamma\in(0,\frac{\alpha}{\beta})$ and almost all $\omega\in\Omega$ there is a constant $C=C(\omega)$ such that

$$|X(t,\omega)-X(s,\omega)| \le C|t-s|^\gamma$$

for $s,t\in[0,1]$. This implies that the path $t\mapsto X(t,\omega)$ is γ-Hölder continuous for almost all $\omega\in\Omega$.

Hints for the proof:

1) For $0<\gamma<\frac{\alpha}{\beta}$ and $n\in\mathbb{N}$ we consider the set

$$A_n := \big\{|X(\tfrac{i+1}{2^n})-X(\tfrac{i}{2^n})| > \tfrac{1}{2^{n\gamma}} \text{ for some } i\in\{0,\dots,2^n-1\}\big\}.$$

Now estimate $\mathbb{P}(A_n)$ appropriately.

2) Show that $\sum_{n=1}^\infty \mathbb{P}(A_n)<\infty$, and apply the Lemma of Borel-Cantelli to get

$$|X(\tfrac{i+1}{2^n},\omega)-X(\tfrac{i}{2^n},\omega)| \le K(\omega)\tfrac{1}{2^{n\gamma}}$$

for some $K(\omega)>0$, all $i\in\{0,\dots,2^n-1\}$, and almost all $\omega\in\Omega$.

3) Using 2), prove that

$$|X(t_1,\omega)-X(t_2,\omega)| \le C(\omega)|t_1-t_2|^\gamma$$

for all dyadic rationals $t_1,t_2\in[0,1]$ and a constant $C=C(\omega)>0$.

4) Finally, use that X has almost surely continuous paths and 3) to conclude the proof.

Solution: 1) For $n\in\mathbb{N}$ we define

$$A_n := \big\{|X(\tfrac{i+1}{2^n})-X(\tfrac{i}{2^n})| > \tfrac{1}{2^{n\gamma}} \text{ for some } i\in\{0,\dots,2^n-1\}\big\}.$$

Then by Markov's inequality

$$\begin{aligned}\mathbb{P}(A_n) &\le \sum_{i=0}^{2^n-1}\mathbb{P}\big(|X(\tfrac{i+1}{2^n})-X(\tfrac{i}{2^n})| > \tfrac{1}{2^{n\gamma}}\big) \le \sum_{i=0}^{2^n-1}\big(\tfrac{1}{2^{n\gamma}}\big)^{-\beta}\mathbb{E}\big|X(\tfrac{i+1}{2^n})-X(\tfrac{i}{2^n})\big|^\beta\\ &\le c\sum_{i=0}^{2^n-1}2^{n\gamma\beta-n(1+\alpha)} = c\,2^{n(\gamma\beta-\alpha)}.\end{aligned}$$

2) Using that $\gamma\beta-\alpha<0$ by assumption, we obtain $\sum_{n=1}^\infty\mathbb{P}(A_n)<\infty$. Therefore, the Lemma of Borel-Cantelli implies that $\mathbb{P}(\limsup_{n\to\infty}A_n)=0$, which means that for almost all $\omega\in\Omega$ there is a number $m(\omega)\in\mathbb{N}$ such that

$$|X(\tfrac{i+1}{2^n},\omega)-X(\tfrac{i}{2^n},\omega)| \le \tfrac{1}{2^{n\gamma}} \quad \text{for all } i=0,\dots,2^n-1$$

if $n\ge m(\omega)$. However, this implies that for those $\omega\in\Omega$ we get

$$|X(\tfrac{i+1}{2^n},\omega)-X(\tfrac{i}{2^n},\omega)| \le K(\omega)\tfrac{1}{2^{n\gamma}} \quad \text{for all } i=0,\dots,2^n-1 \text{ and all } n\in\mathbb{N} \qquad (*)$$

if we choose $K(\omega)$ large enough.

3) Let Ω_0 be set of full measure where $(*)$ holds. Let $t_1, t_2 \in [0,1]$, $t_2 > t_1$, be dyadic rationals. Then choose $n \in \mathbb{N}$ such that

$$2^{-n} \le t_2 - t_1 < 2^{-n+1}.$$

Then there are $i, j \in \{0, \dots, 2^n\}$ such that

$$t_1 \le \frac{i}{2^n} \le \frac{j}{2^n} \le t_2,$$

which implies that

$$\frac{j-i}{2^n} \le t_2 - t_1 < 2^{-n+1} = \frac{2}{2^n}.$$

Thus, $j - i \in \{0, 1\}$, i.e. $j = i$ or $j = i + 1$. Hence, by $(*)$ we get for $\omega \in \Omega_0$

$$\left|X(\tfrac{i}{2^n}, \omega) - X(\tfrac{j}{2^n}, \omega)\right| \le K(\omega)\left|\tfrac{i-j}{2^n}\right|^\gamma \le K(\omega)|t_2 - t_1|^\gamma.$$

Moreover, by the dyadic nature there may be numbers $n < p_1 < p_2 < \ldots < p_k$ and $n < q_1 < q_2 < \ldots < q_\ell$ such that

$$t_1 = \frac{i}{2^n} - \sum_{r=1}^{k} \frac{1}{2^{p_r}} \quad \text{and} \quad t_2 = \frac{j}{2^n} + \sum_{r=1}^{\ell} \frac{1}{2^{q_r}}.$$

Again by $(*)$ we obtain for $\omega \in \Omega_0$

$$\left|X\left(\tfrac{i}{2^n} - \tfrac{1}{2^{p_1}} - \ldots - \tfrac{1}{2^{p_r}}, \omega\right) - X\left(\tfrac{i}{2^n} - \tfrac{1}{2^{p_1}} - \ldots - \tfrac{1}{2^{p_{r-1}}}, \omega\right)\right| \le K(\omega)\tfrac{1}{2^{p_r\gamma}}$$

for any $r \in \{1, \dots, k\}$, which implies

$$\left|X(t_1, \omega) - X(\tfrac{i}{2^n}, \omega)\right| \le K(\omega)\sum_{r=1}^{k} 2^{-p_r\gamma} \le \frac{K(\omega)}{2^{n\gamma}}\sum_{r=1}^{\infty} 2^{-r\gamma} = \frac{K'(\omega)}{2^{n\gamma}} \le K'(\omega)|t_2 - t_1|^\gamma,$$

where we used that $p_r > n + r$ by assumption. Similarly, we get

$$\left|X(t_2, \omega) - X(\tfrac{j}{2^n}, \omega)\right| \le K'(\omega)|t_2 - t_1|^\gamma.$$

With that we obtain by triangle inequality

$$\begin{aligned}\left|X(t_1, \omega) - X(t_2, \omega)\right| &\le \left|X(t_1, \omega) - X(\tfrac{i}{2^n}, \omega)\right| + \left|X(\tfrac{i}{2^n}, \omega) - X(\tfrac{j}{2^n}, \omega)\right| + \left|X(\tfrac{j}{2^n}, \omega) - X(t_2, \omega)\right| \\ &\le \big(2K'(\omega) + K(\omega)\big)|t_2 - t_1|^\gamma = C(\omega)|t_2 - t_1|^\gamma.\end{aligned}$$

4) Let Ω_1 be the set where X is continuous. Then the result follows immediately for all $\omega \in \Omega_0 \cap \Omega_1$ by continuity of X. However, since $\mathbb{P}(\Omega_0 \cap \Omega_1) = 1$ this already concludes the proof.

Remarks:

a) For any Brownian motion B Exercise 8 implies

$$\mathbb{E}|B_t - B_s|^{2k} = C_{2k}|t - s|^k \quad \forall\, k \in \mathbb{N},$$

so Exercise 18 implies that almost surely every Brownian motion has γ-Hölder continuous paths for $\gamma < \frac{1}{2}$.

b) We only needed the continuity assumption in the last step. If X is not continuous, then this step can be modified to construct a *version* $\widetilde{X}$ of X (i.e. $\mathbb{P}(X(t) = \widetilde{X}(t)) = 1$ for all $t \in [0,1]$) such that almost surely $\widetilde{X}$ has γ-Hölder continuous sample paths for $\gamma < \frac{\alpha}{\beta}$. This result is usually known as the Theorem of Kolmogorov-Chentsov.