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Notes for

Ringvorlesung Wavephenomena WS21/22: Conserved energies for the one dimensional Gross-Pitaevskii equation [1]

[1] Comments are welcome to be sent to me by email.

Contents

Main references:
[FT] L.D. Faddeev and L.A. Takhtajan: Hamiltonian methods in the theory of solitons (translated from the 1986 Russian original by A.G. Reyman). Springer, Berlin, 2007.
[KL] H. Koch and X. Liao: Conserved energies for the one-dimensional Gross-Pitaevskii equation. Adv. Math.: 2021, 377, Paper No. 107467, 83.

[20.12.2021]

In this minicourse we are going to consider the one dimensional Gross-Pitaevskii equation

$$i\partial_t q + \partial_{xx} q = 2(|q|^2 - 1)q, \tag{GP}$$

under the following *boundary condition* at infinity:

$$|q(x)| \to 1 \text{ as } x \to \pm\infty. \tag{BC}$$

Here the factor 2 on the righthand side is introduced for the notational simplicity in this minicourse, which can be simply removed by the rescaling consideration

$$t \mapsto 2t, \quad x \mapsto \sqrt{2}x.$$

It is straightforward to check that if q solves (GP), then the function

$$\psi = e^{-2it} q$$

solves the one dimensional cubic nonlinear defocusing Schrödinger equation

$$i\partial_t \psi + \partial_{xx} \psi = 2|\psi|^2 \psi, \tag{NLS}$$

which satisfies also the boundary condition (BC). Hence (GP)-(BC) is sometimes also called the cubic nonlinear defocusing Schrödinger equation with nonzero boundary condition in some references. Notice that the trivial solution

$$q(t,x) = 1$$

for (GP)-(BC) corresponds to the constant-amplitude wave solution

$$\psi(t,x) = e^{-2it}$$

for (NLS)-(BC).
We are going to study the Cauchy problem of the one dimensional GP equation, under the initial data assumption

$$q|_{t=0} = q_0(x). \tag{IC}$$

We are going to

- show the local-in-time well-posedness of this Cauchy problem in the general energy space X^s, $s \geqslant 0$ in Section 1.
- renormalise the transmission coefficient associated to the Lax operator with the potentials in X^s in Section 2.
- establish a family of conserved energies to show the global-in-time well-posedness in X^s in Section 3.

1 Energy spaces X^s

1.1 Heuristics

1.1.1 Special solutions

It is obvious that the constant function

$$q(t,x) = 1 \tag{1.1}$$

is a trivial solution of the GP equation (GP) under the boundary condition (BC). Observe that the GP equation (GP) is invariant under the multiplication of $e^{i\theta} \in \mathbb{S}^1$, and the constant functions

$$q(t,x) = e^{i\theta}, \quad \theta \in \mathbb{R} \tag{1.2}$$

are also trivial solutions of (GP)-(BC).
Although the GP equation is defocusing nonlinear Schrödinger type equation, it persists nontrivial solitary solutions

- Black soliton:
$$q(t,x) = \tanh(x), \tag{1.3}$$
which is independent of time variable;

- Dark solitons:
$$q_c(t,x) = \sqrt{1-c^2}\tanh(\sqrt{1-c^2}(x-2ct)) + ic, \quad c \in (-1,1) \tag{1.4}$$
which are travelling waves with the speed $2c \in (-2,2)$.

Notice that if $c = -1$ or 1, then the "dark soliton" solutions reduce to the trivial solutions in (1.2), and the black soliton is simply q_c with $c = 0$. Whenever $c \neq 0$, the dark soliton solutions do not have zero points.
It is also interesting to notice that the black soliton solution (1.3) has the following asymptotic behaviors at infinity:

$$\tanh(x) \to \pm 1 \text{ as } x \to \pm\infty.$$

The dark soliton solutions (1.4) has the following asymptotic behaviors at infinity:

$$q_c \to \pm\sqrt{1-c^2} + ic \text{ as } x \to \pm\infty.$$

1.1.2 Conserved quantities

Hamiltonian/Energy The Gross-Pitaevskii equation (GP) can viewed as the Hamiltonian flow withe respect to the Hamiltonian flow

$$\mathcal{E}(q) = \int_{\mathbb{R}} \left((|q|^2 - 1)^2 + |\partial_x q|^2 \right) \mathrm{d}x \,, \tag{1.5}$$

and the symplectic form $\omega(u,v) = 2\mathrm{Im} \int_{\mathbb{R}} u\bar{v} \,\mathrm{d}x$. The expression above motivates us to consider the following finite-energy solutions:

$$q \in H^1_{\mathrm{loc}}(\mathbb{R}) \text{ such that } |q|^2 - 1,\ \partial_x q \in L^2(\mathbb{R}). \tag{1.6}$$

Obviously the above special solutions in (1.2), (1.3) and (1.4) are of finite energy:

$$\begin{aligned} &|q_c|^2 - 1 = -(1-c^2)\,\mathrm{sech}^2(\sqrt{1-c^2}(x-2ct)) = -\partial_x q_c \\ &\Rightarrow \mathcal{E}(q_c) = 2\sqrt{1-c^2}^3 \int_{\mathbb{R}} \mathrm{sech}^4(x)\,\mathrm{d}x \,. \end{aligned} \tag{1.7}$$

We also keep in mind here that the slow varying functions

$$e^{i\ln(1+x^2)} \tag{1.8}$$

are also of finite energy with $|e^{i\ln(1+x^2)}| = 1$ and $\partial_x e^{i\ln(1+x^2)} = \frac{2ix}{1+x^2} e^{i\ln(1+x^2)}$, but do not have limits as $x \to \pm\infty$. Furthermore, we can rescale it with the large parameter ε^{-1}

$$e^{i\ln(1+(\varepsilon x)^2)} \tag{1.9}$$

such that it varies even slower and hence has small energy.

Mass and Momentum Aside from the conserved energy $\mathcal{E}$, the mass

$$\mathcal{M}(q) = \int_{\mathbb{R}} (|q|^2 - 1)\,\mathrm{d}x \,, \tag{1.10}$$

and the momentum

$$\mathcal{P}(q) = \mathrm{Im} \int_{\mathbb{R}} q \partial_x \bar{q} \,\mathrm{d}x \tag{1.11}$$

are also conserved by the Gross-Pitaevskii flow. However, $\mathcal{M}$ and $\mathcal{P}$ may not be well-defined for finite-energy solutions in (1.6), even if we take more regularity assumptions: $|q|^2 - 1, \partial_x q \in H^N(\mathbb{R})$. We will see that the mass and momentum quantities are indeed "enemies" when we consider the dynamics of the finite-energy solutions, and we are going to establish the conserved energies "modulo" mass and momentum. Nevertheless, it is interesting to calculate

$$\mathcal{M}(q_c) = -2\sqrt{1-c^2},$$
$$\mathcal{P}(q_c) = 2c\sqrt{1-c^2},$$

which are continuous functions in $c \in [-1, 1]$.

Phase change We mention here another conserved quantity which can be defined as the change of phase functions on the real line if q never vanishes:

$$\Theta(q) = \operatorname{Im} \int_{\mathbb{R}} \frac{q}{|q|} \partial_x \frac{\bar{q}}{|\bar{q}|} \, \mathrm{d}x \,. \tag{1.12}$$

If we can write $q = Ae^{i\varphi}$ with regular φ for q with no zeros, then

$$\Theta(q) = -\lim_{x\to\infty} (\varphi(x) - \varphi(-x)),$$

and in particular

$$\Theta(q_c) = 2\arccos(c), \quad -1 < c < 1 \text{ and } c \neq 0. \tag{1.13}$$

Notice that

$$\begin{aligned}
&\mathcal{M}(q_{\pm 1}) = \mathcal{P}(q_{\pm 1}) = \mathcal{E}(q_{\pm 1}) = 0, \\
&\lim_{c\to 1} \Theta(q_c) = 0, \quad \lim_{c\to -1} \Theta(q_c) = 2\pi, \quad \lim_{c\to 0} \Theta(q_c) = \pi, \\
&\text{and } \lim_{c\to 1} \Theta(q_c) = \lim_{c\to -1} \Theta(q_c) = 0 \mod 2\pi,
\end{aligned}$$

and we have found a continuous (with respect to the metric (ds) below) function

$$Q = \{q_c \,|\, c \in [-1, 1]\} \ni q_c \mapsto 2\arccos(c) \in \mathbb{R}/2\pi\,\mathbb{Z} \,.$$

If in general q has zeros, then we have to regularise q into $\tilde{q}$ in the sense that $\tilde{q}$ never vanishes and $\tilde{q}$ approximates q asymptotically at infinity (while itself is not necessarily close to q), to define the regularised quantity $\tilde{\Theta}(q)$ in terms of $\Theta(\tilde{q})$. However, we can define $\tilde{\Theta}$ only up to modulo $2\pi\,\mathbb{Z}$, as different choices of the regularisation procedure lead to the same value modulo $2\pi\,\mathbb{Z}$. Hence $\tilde{\Theta}(q)$ can be conserved only up to $2\pi\,\mathbb{Z}$.

1.1.3 Energy spaces X^s

We are motivated by the formulation of the conserved energy (1.5) as well as the modulation invariance to consider the generalised finite-energy solution spaces

$$X^s = \{q \in H^s_{\text{loc}}(\mathbb{R};\mathbb{C}) \,:\, |q|^2 - 1,\ \partial_x q \in H^{s-1}(\mathbb{R})\}/\mathbb{S}^1, \quad s \geqslant 0. \tag{Xs}$$

Here $\mathbb{S}^1$ denotes the unit circle, and we do not distinguish the functions which differ by a multiplcative constant of modulus 1 in X^s. In particular,

$$X^1 = \{q \in H^s_{\text{loc}}(\mathbb{R};\mathbb{C}) \,:\, \mathcal{E}(q) < \infty\}/\mathbb{S}^1,$$

and

$$\mathcal{E}(q) = 0 \Leftrightarrow q = e^{i\theta},\ \theta \in [0, 2\pi),\ \text{i.e. } q = 1 \text{ in } X^1.$$

Obviously X^s is not a vector (or affine) space, as the boundary condition (BC) at infinity: $|q| \to 1$ for complex-valued functions are taken into account. Notice that

$$\{q_c \,|\, c \in [-1,1]\} \subsetneq X^s, \quad q_c + \mathcal{S} \subsetneq X^s \text{ and } \{q_c \,|\, c \in (-1,1)\} \cap 1 + \mathcal{S} = \varnothing.$$

Definition of E^s. We introduce the energy "norm" in X^s

$$E^s(q) = \left(\|D^{s-1}(|q|^2 - 1)\|^2_{L^2} + \|D^{s-1}(\partial_x q)\|^2_{L^2}\right)^{\frac{1}{2}}, \tag{Es}$$

where the operator $D = \langle \partial_x \rangle$ is defined by

$$\widehat{Df}(\xi) = \langle \xi \rangle \hat{f}(\xi), \tag{1.14}$$

where $\langle \xi \rangle = \sqrt{2^2 + \xi^2}$, and $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)\, \mathrm{d}x$ denotes the Fourier transform of the function $f(x)$ (the factor 2 appears due to the mismatch between the usual Fourier transform and the (inverse) scattering transform terminology). Then we can rewrite X^s as

$$X^s = \{q \in H^s_{\text{loc}}(\mathbb{R};\mathbb{C}) \,|\, E^s(q) < \infty\}.$$

In particular we have

$$(E^1(q))^2 = \mathcal{E}(q), \quad (E^0(q))^2 = \|D^{-1}(|q|^2 - 1)\|^2_{L^2} + \|D^{-1}(\partial_x q)\|^2_{L^2}.$$

Obviously the energy norm $E^1(q)$ is conserved by the Gross-Pitaevskii flow. We aim to show the conservation of the energy norm $E^s(q)$ for all $s \geqslant 0$, and hence the global-in-time wellposedness for all $s \geqslant 0$.

Definition of d^s. Obviously we may not simply take the "naturally induced" distance function by the energy norm

$$\Big(\|D^{s-1}(|p-q|^2-1)\|_{L^2}^2 + \|D^{s-1}(\partial_x p - \partial_x q)\|_{L^2}^2 \Big)^{\frac{1}{2}}$$

to measure the distance between p and q in X^s. Indeed, the above quantity is simply ∞ if we take e.g. $p=1$ and $q=\tanh(x)$.
The choice of the distance function

$$\Big(\|D^{s-1}(|p|^2-|q|^2)\|_{L^2}^2 + \|D^{s-1}(p'-q')\|_{L^2}^2 \Big)^{\frac{1}{2}} \tag{1.15}$$

is too "rigid" that the distance between two identical functions q and $e^{i\theta}q$, $\theta \notin 2\pi\mathbb{Z}$ is not zero if $q \neq 1$. Or notice that the distance between q and its perturbation $e^{i\theta}q$ (with e.g. $\theta = \ln(1+(\varepsilon x)^2)$) is not small in general (as $\|(1-e^{i\theta})q'\|_{L^2}$ is not small).
In order to measure effectively the distance between two phase functions at infinity (as $|q| \to 1$ at infinity, there is no freedom for the amplitude to deviate from 1), we define the distance function $d^s(\cdot,\cdot)$ in the space X^s:

$$d^s(p,q) = \Big(\int_{\mathbb{R}} \inf_{|\lambda(y)|=1} \big\| \operatorname{sech}(\cdot - y)(\lambda p - q) \big\|_{H^s(\mathbb{R})}^2 \,\mathrm{d}y \Big)^{\frac{1}{2}}, \tag{ds}$$

where $\operatorname{sech}(x) = \frac{2}{e^x+e^{-x}}$ (it is indeed distance function). Notice then

$$1 = e^{i\theta} \text{ in } X^s, \quad \forall \theta \in \mathbb{R},$$

and

$$d^s(1, e^{i\theta}) = 0.$$

Roughly speaking, at each fixed point $y \in \mathbb{R}$, we choose appropriately a mulplicative $\lambda(y) \in \mathbb{S}^1$ such that the modified distance $|\lambda(y)p - q|$ is measured. For example, if $p=1$ and $q=\tanh(x)$, then

$$\begin{aligned}
(d^0(p,q))^2 \leqslant & \int_{y<0} \big\| \operatorname{sech}(x-y)(\tanh(x)+1) \big\|_{L_x^2(\mathbb{R})}^2 \,\mathrm{d}y \\
& + \int_{y>0} \big\| \operatorname{sech}(x-y)(\tanh(x)-1) \big\|_{L_x^2(\mathbb{R})}^2 \,\mathrm{d}y \\
= & \int_{y<0} \int_{\mathbb{R}} \operatorname{sech}^2(x-y)(\tanh(x)+1)^2 \,\mathrm{d}x\,\mathrm{d}y \\
& + \int_{y>0} \int_{\mathbb{R}} \operatorname{sech}^2(x-y)(\tanh(x)-1)^2 \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

is finite.

1.2 Properties of X^s

1.2.1 Special cases

Case $s = 1$: $X^1 \subset L^\infty$. We take a smooth nonnegative cutoff function $\eta \in C_c^\infty(\mathbb{R})$ with $\eta \in [0,1]$, $\int_{\mathbb{R}} \eta = 1$, $\mathrm{Supp}\,(\eta) \in (-1,1)$ and estimate

$$\begin{aligned}
|(\eta * q)(x)| &= \Big|\int_{\mathbb{R}} \eta(x-y)q(y)\,\mathrm{d}y\Big| \leqslant \Big(\int_{\mathbb{R}} \eta(x-y)|q(y)|^2\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&= \Big(1 + \int_{\mathbb{R}} \eta(x-y)(|q(y)|^2-1)\,\mathrm{d}y\Big)^{\frac{1}{2}} \\
&\leqslant \Big(1 + \|\eta\|_{L^2}\||q|^2-1\|_{L^2}\Big)^{\frac{1}{2}} \leqslant 1 + \||q|^2-1\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

We then control the difference between $\eta * q$ and q in terms of q':

$$\begin{aligned}
|q(x) - (\eta * q)(x)| &= \Big|\int_{\mathbb{R}} \eta(x-y)(q(x)-q(y))\,\mathrm{d}y\Big| \\
&\leqslant \|q'\|_{L^2} \int_{\mathbb{R}} \eta(x-y)|x-y|^{\frac{1}{2}}\,\mathrm{d}y \;\leqslant \|q'\|_{L^2}.
\end{aligned}$$

To conclude, we have the following L^∞ estimate:

$$\|q\|_{L^\infty} \leqslant 1 + \|q'\|_{L^2} + \||q|^2-1\|_{L^2}^{\frac{1}{2}}.$$

In particular, we may use the energy norm $E^1(q)$ to control the distance $||q|^2-1|$:

$$\||q|^2-1\|_{H^1} \leqslant \||q|^2-1\|_{L^2} + 2\|\bar{q}q'\|_{L^2} \leqslant 4(1+\|q\|_{L^\infty})E^1(q).$$

If the energy

$$E^1(q) \leqslant \delta \tag{1.16}$$

is small enough, then $\||q|^2-1\|_{L^\infty}$ is small enough and the function q has no zero points on $\mathbb{R}$.

More generally, we can show

$$X^s \subset L^\infty(\mathbb{R}), \quad \forall s > \frac{1}{2},$$

which can be compared with the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $s > \frac{1}{2}$.

Case of small energy $E^1(q) < \delta$. If the energy of $q \in X^1$ is sufficiently small (1.16), that is, if we restrict ourself in the neighborhood of 1, then we may write

$$q = Ae^{i\varphi}.$$

We have equivalent descriptions for the energy norm E^1 and the distance function d^1 in terms of the amplitude $A = |q|$ and the phase velocity $v := \varphi' = \mathrm{Im}\,[\frac{q'}{q}]$.

Equivalent descriptions in terms of $A-1$ and φ' in the small energy setting. By the above, we have firstly straightforward the following equivalence

$$\|A-1\|_{H^1} \sim_{\|q\|_{L^\infty}} \|A^2-1\|_{H^1},$$

since

$$\begin{aligned}\|A-1\|_{H^1} &= \big\|((|q|^2-1)+1)^{\frac{1}{2}}-1\big\|_{H^1} \leqslant C(\|q\|_{L^\infty})\||q|^2-1\|_{H^1},\\ \|A^2-1\|_{H^1} &= \big\|((A-1)+1)^2-1\big\|_{H^1} \leqslant C(\|q\|_{L^\infty})\|A-1\|_{H^1}.\end{aligned}$$

As

$$q' = A'e^{i\varphi} + i\varphi' Ae^{i\varphi} \equiv (\frac{A'}{A} + i\varphi')q,$$

we have

$$\|\varphi'\|_{L^2} = \|\mathrm{Im}\,(\frac{q'}{q})\|_{L^2} \leqslant (\inf|q|)^{-1}\|q'\|_{L^2} \leqslant (1-C(\delta)\delta)^{-1}\|q'\|_{L^2}.$$

Correspondingly we have the following equivalence relationship:

$$E^1(q) \sim_\delta \|A-1\|_{H^1} + \|\varphi'\|_{L^2}. \tag{1.17}$$

The smallness condition (1.16) is equivalent to

$$\|A^2-1\|_{H^1} + \|\varphi'\|_{L^2} \text{ is small.}$$

Equivalence between $E^1(q)$ and $d^1(1,q)$ in the small energy setting.
We calculate the distance

$$\begin{aligned} d^1(1,q) &= \Big(\int_{\mathbb{R}} \inf_{|\lambda(y)|=1} \big\|\operatorname{sech}(\cdot - y)(\lambda - q)\big\|^2_{H^1(\mathbb{R})}\,\mathrm{d}y\Big)^{\frac{1}{2}}\\ &\leqslant \Big(\int_{\mathbb{R}} \big\|\operatorname{sech}(x-y)(e^{i\varphi(y)} - e^{i\varphi(x)})\big\|^2_{H^1_x(\mathbb{R})}\,\mathrm{d}y\Big)^{\frac{1}{2}}\\ &\quad + \Big(\int_{\mathbb{R}} \big\|\operatorname{sech}(x-y)(A(x)-1)e^{i\varphi(x)}\big\|^2_{H^1_x(\mathbb{R})}\,\mathrm{d}y\Big)^{\frac{1}{2}}\\ &\leqslant C\|\varphi'\|_{L^2} + C(1+\|\varphi'\|_{L^2})\|A-1\|_{H^1} \lesssim_\delta E^1(q).\end{aligned}$$

On the other side, as for any λ with $|\lambda| = 1$ it holds

$$A^2 - 1 = |\lambda q|^2 - 1 = \mathrm{Re}\,\big((\lambda q + 1)(\overline{\lambda q} - 1)\big),$$

we have

$$\|A^2 - 1\|_{H^1} \leqslant C(\|q\|_{L^\infty} + \|q'\|_{L^2}) d^1(1, q).$$

Furthermore,

$$\|\varphi'\|_{L^2} = \inf_{|\lambda|=1} \|(\lambda e^{i\varphi} - 1)'\|_{L^2} \leqslant \inf_{|\lambda|=1} \|(\lambda e^{i\varphi} - 1)\|_{H^1},$$

and

$$\|(A-1)e^{i\varphi}\|_{H^1} \leqslant C\|A - 1\|_{H^1}(1 + \|\varphi'\|_{L^2})$$

we have the equivalence relationship

$$d^1(1, q) \sim_\delta E^1(q). \tag{1.18}$$

Equivalence between $\|A - B\|_{H^1} + \|\varphi' - \psi'\|_{L^2}$ *and* $d^1(Ae^{i\varphi}, Be^{i\psi})$
As above we notice that for $q = Ae^{i\varphi}$ and $p = Be^{i\psi}$ with small energy $E^1(q), E^1(p) \leqslant \delta$, it holds

$$A^2 - B^2 = \mathrm{Re}\,\big((\lambda q + p)(\overline{\lambda q - p})\big),$$

which implies

$$\|A^2 - B^2\|_{H^1} \lesssim_\delta d^1(p, q).$$

Similarly, as

$$\|\varphi' - \psi'\|_{L^2} \leqslant C(1 + \|\psi'\|_{L^2}) \inf_{|\lambda|=1} \|(\lambda e^{i(\varphi-\psi)} - 1)\|_{H^1},$$

we have

$$\|\varphi' - \psi'\|_{L^2} \lesssim_\delta d^1(p, q).$$

And we finally have the following equivalence relationship

$$d^1(p, q) \sim_\delta \|A - B\|_{H^1} + \|\varphi' - \psi'\|_{L^2}. \tag{1.19}$$

However, noticing that

$$(Ae^{i\varphi} - Be^{i\psi})' = (A' - B')e^{i\varphi} + i(A - B)\varphi' e^{i\varphi} + iB(\varphi' - \psi')e^{i\varphi} + (B' + iB\psi')(e^{i\varphi} - e^{i\psi}),$$

we do not have in general

$$\||p| - |q|\|_{H^1} + \|p' - q'\|_{L^2} \sim_\delta \|A - B\|_{H^1} + \|\varphi' - \psi'\|_{L^2},$$

while only

$$\||p| - |q|\|_{H^1} + \|p' - q'\|_{L^2} \gtrsim_\delta \|A - B\|_{H^1} + \|\varphi' - \psi'\|_{L^2}.$$

Hence the “distance function” given in (1.15) is “rigider” than $d^s(p, q)$ given in (ds).

1.2.2 Regularisation

In order to show the analytical properties for X^s with small s (in particular for $s = 0$), it is often convenient to perform a regularisation procedure.
Let us fix an even mollifier $j \in C_c^\infty(\mathbb{R})$ with

$$j \in [0,1], \quad \mathrm{Supp}\,(j) \subset [-1,1], \quad j = 1 \text{ on } [-\frac{1}{3}, \frac{1}{3}] \text{ and } \int_{\mathbb{R}} j = 1.$$

For a large parameter $\tau > 0$ we rescale j as

$$j_\tau(x) = \tau j(\tau x).$$

Notice that $(j_\tau)_{\tau \geqslant 1}$ is a Dirac sequence, and for $q \in X^s$ we expect $d^s(j_\tau * q, q) \to 0$ as $\tau \to \infty$. In order to investigate the operator $j_\tau * : X^s \mapsto X^s$ (and in particular its dependence on the parameter τ), we introduce the rescaled derivative operator

$$D_\tau := (\tau^2 - \partial_x^2)^{\frac{1}{2}}, \text{ that is, } \widehat{(D_\tau f)}(\xi) = (\tau^2 + \xi^2)^{\frac{1}{2}} \hat{f}(\xi). \tag{1.20}$$

Correspondingly we introduce the rescaled energy norm

$$E_\tau^s(q) = \left(\||q|^2 - 1\|^2_{H_\tau^{s-1}} + \|q'\|^2_{H_\tau^{s-1}} \right)^{\frac{1}{2}}, \tag{1.21}$$

where the rescaled Sobolev norm H_τ^s is defined to be

$$\|f\|_{H_\tau^s} = \|D_\tau^s f\|_{L^2}.$$

The energy norm $E^s(q)$ defined in (Es) corresponds to the parameter $\tau = 2$:

$$E^s(q) = E_2^s(q),$$

and $D = D_2$, $H^s = H_2^s$.
Observe the homogeneity relationship

$$(\frac{\tau}{2})^{-(\frac{1}{2}+s)} \|j_{\tau/2}\|_{H_\tau^s} = \|j\|_{H^s}$$

which roughly implies that if we localize j at the frequency τ, then its rescaled H_τ^s-norm is of size $\tau^{s+\frac{1}{2}}$. We are wondering if we localize q at the frequency τ: $r = q|_{\text{frequency } \tau}$, then whether the "nonlinear" norm of r can also "observe" this scaling property

$$\text{"}\frac{1}{\tau}(\||r|^2 - 1\|_{L^2} + \|r'\|_{L^2}) \lesssim E_\tau^0(q)?\text{"}$$

to some extended sense.
We have the following lemma (whose proof is omitted here).

Lemma 1.1. *Let $q \in X^s$, $s \geqslant 0$. Let $\tau \geqslant 2$. Let*

$$r = j_\tau * q.$$

Then $r \in X^\infty = \cap_{\sigma \geqslant 0} X^\sigma$, and $|r|^2 - 1, r', q - r, r$ can be estimated in terms of $E^0_\tau(q)$ as follows (with an absolute constant C):

$$\begin{aligned}
&\|\frac{r'}{\tau}\|_{L^2} \leqslant C\|D_\tau^{-1} q'\|_{L^2}, \\
&\|q - r\|_{L^2} \leqslant C\|D_\tau^{-1} q'\|_{L^2}, \\
&\|\frac{r^{(k)}}{\tau}\|_{L^\infty} \leqslant C\tau^k \left(1 + \frac{1}{\sqrt{\tau}} E^0_\tau(q)\right), \quad \forall k \in \mathbb{N}, \\
&\||r|^2 - 1\|_{L^2} \leqslant C(\tau + \|r\|_{L^\infty}) E^0_\tau(q),
\end{aligned} \tag{1.22}$$

and hence

$$\left\|\frac{r'}{\tau}\right\|_{L^2} + \left\|q - r\right\|_{L^2} + \left\|\frac{|r|^2 - 1}{\tau}\right\|_{L^2} \leqslant C\left(1 + \frac{1}{\sqrt{\tau}} E^0_\tau(q)\right) E^0_\tau(q). \tag{1.23}$$

1.2.3 Relations between E^s and d^s

As seen in Subsection 1.2.1 that there is some (nonlinear) relationship between the energy norm $E^1(q)$ and the distance function $d^1(1, q)$ in the small energy setting, we would like to establish the relationship between E^s and d^s for general $s \geqslant 0$. The regularisation procedure in Subsection 1.2.2 will help in the consideration of low regularity cases.
We have the following relationship between E^s and d^s (whose proof can be found in [Section 6, KL]).

Proposition 1.1 (E^s and d^s)**.** *There is an absolute constant c such that the following hold true:*

(i) *If $q \in X^s$, then*

$$d^s(1, q) \leqslant cE^s(q). \tag{1.24}$$

(ii) *If $p, q \in X^s$, then*

$$\||q|^2 - |p|^2\|_{H^{s-1}} \leqslant c\big(1 + \|(|q|^2 - 1, |p|^2 - 1)\|^{\frac{1}{2}}_{H^{s-1}} + \|(q', p')\|_{H^{s-1}}\big) d^s(p, q). \tag{1.25}$$

and

$$\|q'\|_{H^{s-1}} \leqslant \|p'\|_{H^{s-1}} + cd^s(p, q). \tag{1.26}$$

In particular if $p \in X^s$ and $q \in H^s_{loc}$ so that $d^s(p,q) < \infty$, then $q \in X^s$ and

$$E^s(q) \leqslant E^s(p) + c(1 + \|p'\|_{H^{s-1}} + \||p|^2 - 1\|^{\frac{1}{2}}_{H^{s-1}}) d^s(p,q) + c(d^s(p,q))^2. \tag{1.27}$$

Correspondingly, there exists a constant C depending on $\Lambda > 0$ such that for any $q, p \in X^s$ with $E^s(q), E^s(p) \leqslant \Lambda$,

$$|E^s(q) - E^s(p)| \leqslant c d^s(q,p).$$

1.2.4 Properties of (X^s, d^s)

We now state the important analytic and topological properties of the metric space (X^s, d^s) below (interested readers are referred to Section 6 in [KL] for the proofs).

Theorem 1.1. *The metric space (X^s, d^s), $s \geqslant 0$ has the following properties*

- *The space $(X^s, d^s(\cdot,\cdot))$ is a complete metric space.*
- *The subset $\{q \,|\, q - 1 \in C_0^\infty(\mathbb{R})\}$ is dense in X^s.*
- *Any set $\{q \in H^s_{\text{loc}}(\mathbb{R}) : E^s(q) < C\}$ is contained in some ball $B^s_r(1)$ with r depending on C.*
- *Any closed ball $\overline{B^s_r(q)}$ in X^s, $s > 0$ is weakly sequentially compact.*
- *The subset*

  $$Q = \{q_c : -1 \leqslant c \leqslant 1\} \subset X^s$$

  is a strong deformation retract of X^s, which means that there is a continuous map (called deformation)

  $$\Xi : [0,1] \times X^s \to X^s$$

  so that

  1. $\Xi(0,q) = q, \quad \forall q \in X^s$,
  2. $\Xi(t,q_c) = q_c, \quad \forall t \in [0,1], \forall c \in [-1,1]$,
  3. $\Xi(1,q) \subset Q, \quad \forall q \in X^s$.

- *There is an analytic structure on X^s. More precisely, Fix a function $\eta \in C_0^\infty([-1,1])$ with $\eta = 1$ on $[-1/2, 1/2]$. Let $q \in X^s$. There exist r and L depending only on $\||q|^2 - 1\|_{H^{s-1}}, \|\partial_x q\|_{H^{s-1}}$ such that the map*

  $$B^s_r(q) \ni p \mapsto ((a_n)_n, \mathfrak{b}) \in l^2_d \times \tilde{H}^s, \text{ with}$$

$$\|(a_n)_n\|_{l^2_d} = \big(\sum_n |a_n - a_{n-1}|^2\big)^{\frac{1}{2}},$$
$$\tilde{H}^s = \{\mathfrak{b} \in H^s \,|\, \langle \eta((x-4Ln)/L)\mathfrak{b}, \eta((x-4Ln)/L)q\rangle_{H^s} \in \mathbb{R}, \quad \forall n \in \mathbb{Z}\}$$

is a biLipschitz map to its image. If $d^s(q, q_1) < r$ then the coordinate change in the intersection is an analytic diffeomorphism with uniformly bounded derivatives.

In particular, for any smooth partition of unity

$$\sum_n \rho((x-4Ln)/L) = 1 \text{ with } \rho = 1 \text{ on } [-1,1] = \operatorname{Supp}(\eta),$$

we can define the real-valued function

$$\theta(x) = \sum a_n \rho((x-4Ln)/L)$$

to parametrize $B^s_r(q)$ as

$$(\theta, \mathfrak{b}) \to e^{i\theta}(q + \mathfrak{b}).$$

1.3 Local-in-time well-posedness in X^s

Theorem 1.2 (LWP in X^s)**.** *Let $s \geqslant 0$. The Gross-Pitaevskii equation* (GP) *is locally-in-time well-posed in the metric space (X^s, d^s) in the following sense: For any initial data $q_0 \in X^s$, there exists a positive time $\bar{t} \in (0, \infty)$ and a unique local-in-time solution $q \in \mathcal{C}((-\bar{t}, \bar{t}); X^s)$ of* (GP) *and for any $t \in (0, \bar{t})$, the Gross-Pitaevskii flow map $X^s \ni q_0 \mapsto q \in \mathcal{C}([-t, t]; X^s)$ is continuous.*
Here the solution $q \in \mathcal{C}((-\bar{t}, \bar{t}); X^s)$ is defined in terms of the representatives in (X^s, d^s) as follows. There is $\tilde{q} : (-\bar{t}, \bar{t}) \to H^s_{loc}(\mathbb{R})$ which satisfies that

$$(-\bar{t}, \bar{t}) \ni t \to \tilde{q}(t) - \tilde{q}(0) \in L^2(\mathbb{R}), \tag{1.28}$$

is weakly continuous and

$$\|\tilde{q}(\cdot) - \tilde{q}_{0,\varepsilon}\|_{L^4([a,b]\times\mathbb{R})} < \infty, \tag{1.29}$$

for some regularized initial data $\tilde{q}_{0,\varepsilon}$ of $\tilde{q}(0)$ and for all time intervals $[a, b] \subset (-\bar{t}, \bar{t})$ with $0 \in [a, b]$, such that the equation (GP) *holds in the distributional sense on $(-\bar{t}, \bar{t}) \times \mathbb{R}$ and $\tilde{q}(t)$ projects to $q(t)$.*

Sketch of the proof. We first consider the case $s = 0$. We will always choose a representative in the equivalence class in X^0. Let $r = j * q_0 \in X^\infty$ be the regularised initial data. Then if $q = q(t)$ solves the Cauchy problem of (GP) with the initial data q_0, then

$$b := q - r$$

satisfies the following nonlinear Schrödinger-type equation

$$i\partial_t b + \partial_{xx} b = g(b), \quad b|_{t=0} = b_0 = q_0 - r \in L^2, \tag{1.30}$$

where

$$g(b) = 2|b|^2 b + 4r|b|^2 + 2\overline{r}b^2 + (4|r|^2 - 2)b + 2r^2\overline{b} + 2r(|r|^2 - 1) - r''.$$

Vice versa: If b satisfies this equation (1.30) then q satisfies (GP) with the initial data q_0.
The standard Strichartz estimates for the Schrödinger group $S(t) = e^{it\partial_{xx}}$ implies the a priori estimate

$$\|b\|_{t_0} := \|b(t)\|_{L^\infty([-t_0,t_0];L^2_x)} + \|b\|_{L^8([-t_0,t_0];L^4(\mathbb{R}_x))} + \|b\|_{L^6([-t_0,t_0]\times\mathbb{R})} \leqslant C_0 E^0(q_0)$$

if the time $t_0 > 0$ is chosen small enough. The fixed point argument gives a unique solution $b \in C([-t_0, t_0]; L^2)$ for (1.30). The function

$$\begin{aligned} q = b + r \in C([-t_0, t_0]; L^2 + X^0) = C([-t_0, t_0]; X^0) \\ \text{with } \|q(t) - r\|_{L^\infty([-t_0,t_0];L^2)\cap L^6([-t_0,t_0]\times\mathbb{R})} \leqslant C_0 E^0(q_0). \end{aligned}$$

solves (GP) in the sense of (1.28)-(1.29). If there are two solutions $q_1, q_2 \in C((-t_0, t_0); X^0)$ of (GP) in the sense of (1.28)-(1.29) with the same initial data in X^0, we choose the same initial representative q_0. By virtue of the energy inequality on any compact time interval I including 0, their difference

$$b_{12} = q_1 - q_2 \in L^\infty(I; L^2) \cap L^4(I \times \mathbb{R}) \text{ with } b_{12}(0, x) = 0$$

should vanish identically on I.
If $s \in (0, 1)$, we make use of the characterisation of the H^s-norm:

$$\|f\|_{\dot{H}^s} = \left\| \frac{\|f(x-y) - f(x)\|_{L^2_x}}{|y|^s} \right\|_{L^2(\mathbb{R}; \frac{dy}{|y|})}, \quad \|f\|_{H^s} = \|f\|_{L^2} + \|f\|_{\dot{H}^s}.$$

We apply the previous construction to the finite differences, and integrate the estimates for fixed y according to the norm above. It follows from this construction that the existence time is the same for all $s \in [0, 1)$. For $s \geqslant 1$ it follows similarly.

Finally as

$$d^s(q(t), p(t)) \leqslant d^s(r_q, r_p) + C\|b_q(t) - b_p(t)\|_{H^s},$$

the continuity of the flow $q_0 \mapsto q(t)$ follows from the continuity of the map $j_* : X^s \ni q_0 \to r_q \in X^s$ and the Lipschitz continuity of the flow $b(0) \mapsto b(t)$ in H^s. □

[20.12.2021]

[10.01.2022]

2 Transmission coefficient

2.1 The general framework

We recall briefly the general framework of the (inverse) scattering transform of the completely integrable system (GP) (without rigorous proofs), under the classical assumption that the potential $q \in 1 + \mathcal{S}$ (this assumption can be relaxed).

2.1.1 Lax pair formulation

According to the seminar paper by Zakharov-Shabat in 1973, $q(t, x)$ solves the equation (GP) if and only if there holds the operator evolution equation

$$L_t = PL - LP,$$

where

$$L = \begin{pmatrix} i\partial_x & -iq \\ i\overline{q} & -i\partial_x \end{pmatrix}, \tag{2.31}$$

$$P = i\begin{pmatrix} 2\partial_x^2 - (|q|^2 - 1) & -q\partial_x - \partial_x q \\ \bar{q}\partial_x + \partial_x\bar{q} & -2\partial_x^2 + (|q|^2 - 1) \end{pmatrix}.$$

that is, the two operators (L, P) form the so-called Lax-pair, which *formally* implies the invariance of the spectra of L by time evolution. Indeed, let the skewadjoint operator P generate a unitary family of evolution operators $U(t', t)$, then

$$L(t) = U^*(t', t)L(t')U(t', t)$$

and $L(t)$ and $L(t')$ are similar.

The inverse scattering transform relates the evolution of the Gross-Pitaevskii flow to the study of the spectral and scattering property of the Lax operator

L. And the equation (GP) is completely integrable by means of the inverse scattering method. In the classical framework where $q-1$ is Schwartz function (see Faddeev-Takhtajan), the self-adjoint operator $L : H^1(\mathbb{R};\mathbb{C}^2) \mapsto L^2(\mathbb{R};\mathbb{C}^2)$ has essential spectrum $(-\infty,-1] \cup [1,\infty)$ and at most countably many simple real eigenvalues $\{\lambda_m\}$ on $(-1,1)$. We can define the reflection coefficient $R(\lambda)$ on the continuous spectrum $\{\lambda \in (-\infty,-1] \cup [1,\infty)\}$, as well as the transition coefficient $\{\gamma_m\}$ on the discrete part of the spectrum $\{\lambda_m\}$. It turns out that if $q(t,x)$ solves the Gross-Pitaevskii equation, then the spectrum of $L = L(q(t,x))$ is invariant in time, while the time evolution of the scattering data $\{R(t,\lambda),\gamma_m(t)\}$ of $L = L(q(t,x))$ is rather simple by GP flow:

$$\partial_t R(t,\lambda) = 4i\lambda z R(t,\lambda), \quad \partial_t \gamma_m(t) = 4i\lambda_m z_m \gamma_m(t),$$

where $z^2 = \lambda^2 - 1$ and $\operatorname{sgn}(z) = \operatorname{sgn}(\lambda)$ if $\lambda \in (-\infty,-1] \cup [1,\infty)$ while $z_m = i\sqrt{1-\lambda_m^2}$. (See the following context for more explanations.) The inverse scattering transform recovers the potential $q(t,x)$ from the time-dependent scattering data $\{R(t,\lambda),\gamma_m(t)\}$.

The direct/inverse scattering transforms give an algorithmic way to solve (GP). This idea can be compared with the resolution of the linear Schrödinger equation via Fourier and inverse Fourier transform:

$$\begin{aligned} & i\partial_t u + u_{xx} = 0, \quad u|_{t=0} = u_0, \\ & \Rightarrow i\partial_t \hat{u}(\xi) - \xi^2 \hat{u}(\xi) = 0, \quad \hat{u}|_{t=0} = \widehat{u_0}(\xi), \\ & \Rightarrow \hat{u}(t,\xi) = e^{-i\xi^2 t}\widehat{u_0}(\xi) \Rightarrow u(t,x) = \mathcal{F}_x^{-1}(\hat{u}). \end{aligned} \tag{2.32}$$

The direct and inverse scattering transform play the same role of Fourier and inverse Fourier transform here:

$$\begin{array}{ccc} q_0(x) & \text{- - - - - - - - - -} & q(t,x) \\ \text{direct scattering transform} \downarrow & & \uparrow \text{inverse scattering transform} \\ \{R(0,\lambda),\gamma_m(0)\} & \xrightarrow[\cdot e^{4i\lambda z t}]{} & \{R(t,\lambda),\gamma_m(t)\} \end{array}$$

However, the inverse scattering transform step is rather involved and it is hard to say that this machinery can work easier than other methods. Nevertheless it offers an algorithm to solve (NLS) and we can derive much information from the formulation itself, e.g. asymptotic behaviors of the solutions.

2.1.2 Direct scattering problem

Equivalently as the Lax-pair reformulation of the equation (GP), the GP equation can be viewed as the compatibility condition for the following two overdetermined systems of ODEs

$$\begin{aligned} u_x &= \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u, \\ u_t &= i \begin{pmatrix} -2\lambda^2 - (|q|^2 - 1) & -2i\lambda q + \partial_x q \\ -2i\lambda \overline{q} - \partial_x \overline{q} & 2\lambda^2 + (|q|^2 - 1) \end{pmatrix} u, \end{aligned} \tag{2.33}$$

where $u : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2$ is the unknown vector-valued function and $\lambda \in \mathbb{C}$ can be viewed as parameter. The first system in (2.33) can be written in the form of a spectral problem $Lu = \lambda u$ of the Lax operator.

We study the spectral problem of the Lax operator L, or equivalently, the scattering problem $(2.33)_1$:

$$u_x = \begin{pmatrix} -i\lambda & q \\ \bar{q} & i\lambda \end{pmatrix} u, \tag{2.34}$$

in the classical framework

$$q \in 1 + \mathcal{S}.$$

We observe that if $q = 1$, then the constant matrix

$$\begin{pmatrix} -i\lambda & 1 \\ 1 & i\lambda \end{pmatrix}$$

has the eigenvalues

$$\pm iz, \text{ with } \lambda^2 - z^2 = 1.$$

This situation is different from the case $q = 0$, where the two eigenvalues are simply $\pm i\lambda$ themselves.

A Riemann surface This leads us to the introduction of the Riemann surface

$$\{(\lambda, z) \in \mathbb{C}^2 \mid z^2 = \lambda^2 - 1\}.$$

Notice that here $z(\lambda) = (\lambda^2 - 1)^{\frac{1}{2}}$ is a double-valued function of λ, and is defined on a two-sheet Riemann surface with cuts $(-\infty, -1]$ and $[1, \infty)$: On the upper sheet of the surface we have

$$\mathrm{Im}\,(z) > 0$$

and on the lower sheet of the surface we have

$$\mathrm{Im}\,(z) < 0.$$

This variable z is considered to belonging to the complex plane by gluing the upper and lower half complex planes along the full real axis.
We typically choose the upper sheet $\mathcal{R}$ of this Riemann surface:

$$\begin{aligned}&\mathcal{R} = \{(\lambda, z) \,|\, \lambda \in \mathcal{V}, \quad z = z(\lambda) = \sqrt{\lambda^2 - 1} \in \mathcal{U}\}, \\ &\text{with } \mathcal{V} := \mathbb{C}\backslash I,\ I := (-\infty, -1] \cup [1, +\infty),\ \mathcal{U} := \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\},\end{aligned} \tag{2.35}$$

We can define a conformal mapping

$$\mathcal{R} \ni (\lambda, z) \mapsto \zeta = \zeta(\lambda) = \lambda + z \in \mathcal{U},$$

which maps the cuts $\lambda \in I$ to the real axis $\zeta \in \mathbb{R}$ and the neighborhood of ∞ in the λ-complex plane for $\mathrm{Im}\,\lambda > 0$ (resp. < 0) to the neighborhood of ∞ (resp. 0) in ζ-upper complex plane. The inverse mapping is given by $\lambda = \frac{1}{2}(\zeta + \frac{1}{\zeta})$ and $z = \frac{1}{2}(\zeta - \frac{1}{\zeta})$.

Analysis on the cuts We fix the real

$$z = \xi \in \mathbb{R}\backslash\{0\},$$

and $\lambda = \pm\sqrt{\xi^2 + 1} \in (-\infty, -1) \cup (1, \infty) \subset \mathbb{R}$. The constant matrix $\begin{pmatrix} -i\lambda & 1 \\ 1 & i\lambda \end{pmatrix}$ has two eigenvalues

$$\pm i\xi,$$

with the associated eigenspaces generated by the following vectors respectively:

$$\begin{pmatrix} 1 \\ i(\lambda + \xi) \end{pmatrix} \text{ resp. } \begin{pmatrix} 1 \\ i(\lambda - \xi) \end{pmatrix} = i(\lambda - \xi) \begin{pmatrix} \overline{i(\lambda + \xi)} \\ 1 \end{pmatrix}$$

Let u_l be the (left) Jost solution of the Lax equation (2.34) with $q \in 1 + \mathcal{S}$ satisfying the following boundary condition at $-\infty$:

$$u_l(x) = e^{-i\xi x} \begin{pmatrix} 1 \\ i(\lambda - \xi) \end{pmatrix} + o(1) \text{ as } x \to -\infty.$$

Then u_l will take the following asymptotic behavior at $+\infty$:

$$u_l(x) = e^{-i\xi x} T^{-1} \begin{pmatrix} 1 \\ i(\lambda - \xi) \end{pmatrix} + e^{i\xi x} R T^{-1} \begin{pmatrix} 1 \\ i(\lambda + \xi) \end{pmatrix} + o(1) \text{ as } x \to +\infty.$$

Here R, T^{-1} are two complex numbers which are independent of x-variable, and are called (right) reflection coefficient and transmission coefficient respectively. Obviously

$$R = R(\lambda; q), \quad T^{-1} = T^{-1}(\lambda; q).$$

Analog we can consider the (right) Jost solution u_r to be the solution of the ODE (2.34) with $q \in 1 + \mathcal{S}$ satisfying the boundary condition at $+\infty$:

$$u_r(x) = e^{i\xi x} \begin{pmatrix} 1 \\ i(\lambda + \xi) \end{pmatrix} + o(1) \text{ as } x \to +\infty.$$

Then u_r takes the following asymptotic behaviour at $-\infty$:

$$u_r(x) = e^{-i\xi x} L T^{-1} \begin{pmatrix} 1 \\ i(\lambda - \xi) \end{pmatrix} + e^{i\xi x} T^{-1} \begin{pmatrix} 1 \\ i(\lambda + \xi) \end{pmatrix} + o(1) \text{ as } x \to -\infty.$$

The complex number L denotes the left reflection coefficient. As the matrix in (2.34) is traceless, $\det(u_l, u_r)$ is a constant, which ensures that the transmission coefficient T^{-1} is indeed the same for the left resp. right Jost solutions.
Observe also that in this case $(\lambda, z) \in \mathbb{R}^2$, if $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$ solves (2.34), then $\begin{pmatrix} \overline{u^2} \\ \overline{u^1} \end{pmatrix}$ also solves (2.34) and the quantity $|u^1|^2 - |u^2|^2$ is a constant. Hence we have the following relationships between T and L, R in this case:

$$L\overline{T} = -(\lambda + \xi)^2 T \overline{R}, \quad |T|^2 = 1 - (\lambda + \xi)^2 |R|^2 = 1 - (\lambda - \xi)^2 |L|^2 \leqslant 1.$$

This means that

$$T^{-1}(\lambda) \neq 0 \text{ for } \lambda \in I. \tag{2.36}$$

In the trivial case $q = 1$, the left/right Jost solutions are simply

$$u_l(x) = e^{-i\xi x} \begin{pmatrix} 1 \\ i(\lambda - \xi) \end{pmatrix}, \quad u_r(x) = e^{i\xi x} \begin{pmatrix} 1 \\ i(\lambda + \xi) \end{pmatrix}.$$

Thus in this case

$$T^{-1}(\lambda) = T(\lambda) = 1, \quad R(\lambda) = L(\lambda) = 0.$$

Analysis on the upper Riemann sheet We fix a point on the upper Riemann surface sheet

$$(\lambda, z) \in \mathcal{R} \text{ with } \lambda \notin I \text{ and } \operatorname{Im} z > 0.$$

Exactly as before, the constant matrix $\begin{pmatrix} -i\lambda & 1 \\ 1 & i\lambda \end{pmatrix}$ has two eigenvalues

$$\pm iz,$$

with the associated eigenspaces generated by the following vectors respectively:

$$\begin{pmatrix} 1 \\ i(\lambda+z) \end{pmatrix} \text{ resp. } \begin{pmatrix} 1 \\ i(\lambda-z) \end{pmatrix} = i(\lambda-z)\begin{pmatrix} \overline{i(\lambda+z)} \\ 1 \end{pmatrix}$$

We take the left Jost solution u_l of the Lax equation (2.34) with $q \in 1+\mathcal{S}$ and taking the following asymptotic behavior at $\pm\infty$:

$$\begin{aligned} u_l(x) &= e^{-izx}\begin{pmatrix} 1 \\ i(\lambda-z) \end{pmatrix} + o(1)e^{\operatorname{Im}(z)x} \text{ as } x \to -\infty, \\ u_l(x) &= e^{-izx}T^{-1}\begin{pmatrix} 1 \\ i(\lambda-z) \end{pmatrix} + o(1)e^{\operatorname{Im}(z)x} \text{ as } x \to +\infty. \end{aligned} \tag{2.37}$$

Here the transmission coefficient

$$T^{-1} = T^{-1}(\lambda)$$

is a holomorphic function on $\mathcal{R}$ and

$$\lim_{|\lambda|\to\infty} T^{-1}(\lambda) = 1.$$

Notice that if $(\lambda, z) \in \mathcal{R}$, then $(\bar\lambda, -\bar z) \in \mathcal{R}$. If u_l is the left Jost solution associated to $(\lambda, z) \in \mathcal{R}$, then

$$u_l^* := i(\bar\lambda + \bar z)\begin{pmatrix} \overline{u_l^2} \\ \overline{u_l^1} \end{pmatrix}$$

is the left Jost solution associated to $(\bar\lambda, -\bar z) \in \mathcal{R}$ with the following asymptotic bahaviour at $\pm\infty$:

$$\begin{aligned} u_l^*(x) &= e^{-i(-\bar z)x}\begin{pmatrix} 1 \\ i(\bar\lambda-(-\bar z)) \end{pmatrix} + o(1)e^{\operatorname{Im}(-\bar z)x} \text{ as } x \to -\infty, \\ u_l^*(x) &= e^{-i(-\bar z)x}\bar{T}^{-1}(\lambda)\begin{pmatrix} 1 \\ i(\bar\lambda-(-\bar z)) \end{pmatrix} + o(1)e^{\operatorname{Im}(-\bar z)x} \text{ as } x \to +\infty. \end{aligned}$$

This implies the following symmetry

$$\bar{T}^{-1}(\lambda) = T^{-1}(\bar{\lambda}), \quad \forall(\lambda, z) \in \mathcal{R}. \tag{2.38}$$

In particular

$$\lim_{\lambda \to a+i0} |T^{-1}(\lambda)| = \lim_{\lambda \to a-i0} |T^{-1}(\lambda)|, \quad \forall a \in (-\infty, -1) \cup (1, \infty).$$

Hence the superharmonic function $-\ln|T^{-1}(\lambda)|$ in continuous on the cut lines.

The zeros of T^{-1} correspond to the eigenvalues of the spectral problem (2.34): Indeed, by virtue of (2.36), the zeros of $T^{-1}(\lambda)$ can only be located away from the cut lines. We consider the upper Riemann sheet such that $e^{\pm izx}$ decays exponentially fast as $x \to \pm\infty$. If $T^{-1}(\lambda) = 0$, then u_l and u_r are linearly dependent:

$$u_l(x) = \gamma u_r(x),$$

such that λ, u_l are the eigenvalue and the corresponding eigenvector of the self-adjoint Lax operator L. In particular there are only eigenvalues $\{\lambda_m\} \subset (-1, 1)$ and we define the transmission coefficient γ_m for these discrete spectrum. One can further show that they are simple by calculating explicitly $(T^{-1})'(\lambda_m) \neq 0$.

More generally [DPVV], if $q - 1 \in L^1(\mathbb{R})$, then the transmission coefficient $T^{-1}(\lambda)$ is continuous in $\overline{\mathcal{R}}\backslash\{\pm1\}$ and analytic in $\lambda \in \mathcal{R}$, while its complex conjugate $\overline{T^{-1}}(\lambda)$ is continuous in the closure of the lower Riemann sheet except the two points $\{\pm1\}$ and analytic in the lower Riemann sheet. The functions $R(\lambda), \bar{R}(\lambda)$ are continuous in the cut lines $I\backslash\{\pm1\}$, but in general cannot be continuous away from I.

If $q - 1 \in L^{1,2}(\mathbb{R})$ with sufficiently decay at infinity, then $z(\lambda)T^{-1}(\lambda)$ is continuous in $\overline{\mathcal{R}}$ and analytic in $\mathcal{R}$, while $z\overline{T^{-1}}(\lambda)$ is continuous on the closure of the lower Riemann sheet and analytic in the lower Riemann sheet. The functions $zR(\lambda), z\bar{R}(\lambda)$ are continuous on the whole real line $\lambda \in \mathbb{R}$.

If $q - 1 \in L^{1,4}(\mathbb{R})$, then there are only finite number of zeros for $T^{-1}(\lambda)$ in $(-1, 1)$. Nevertheless for generic q we have $R(\pm1) = \mp1$ (recalling $|R(\lambda)| < 1$ for $\lambda \in (-\infty, -1) \cup (1, \infty)$).

2.1.3 Inverse scattering transform

We now take into account the time variable. The time-dependent left Jost solution $e^{-2i\lambda zt}u_l$ should solve the compatible ODE systems (2.33) if $q = q(t, x)$ solves the Gross-Pitaevskii equation (GP). In particular, we infer from the evolutionary equation $(2.33)_2$ that

$$\partial_t(T^{-1}) = 0, \quad \partial_t R = 4iz\lambda R. \tag{2.39}$$

This means that the transmission coefficient

$$T^{-1}(\lambda; q(t,x)) = T^{-1}(\lambda; q_0(x))$$

is independent of time, and in particular, the zeros of $T^{-1}(\lambda)$ (i.e. the eigenvalues of the Lax operator) are independent of time. The reflection coefficient evolves simply as

$$R(\lambda; q(t,x)) = e^{4i\lambda zt} R(\lambda; q_0(x)).$$

In the reflectionless case $R = 0$, if the Lax operator has only finitely many simple eigenvalues (under e.g. the assumption $q - 1 \in \mathcal{S}$), then the corresponding potential is indeed the multisoliton solution of the Gross-Pitaevskii equation. In particular, the Lax operator with the dark soliton solution $q_c(t,x)$ in (1.4) as the potential persists a simple eigenvalue $\lambda = -c$ and $R(\lambda; q_c) = 0$.
Theoretically the solution $q(t,x)$ of (GP) can be recovered by the reconstruction formula

$$q(t,x) = \lim_{\zeta \to \infty} \zeta m_{21}(\zeta, t, x),$$

where the matrix $m(\zeta, t, x)$ is characterized by some Riemann-Hilbert problem, which is omitted here.

2.2 Renormalised transmission coefficient

We are going to define the renormalised transmission coefficient $T_{\mathrm{c}}^{-1}(\lambda; q)$ for the potential q in our (general) energy space X^s, in the upper Riemann sheet $\mathcal{R}$.
By the regularisation procedure in Subsection 1.2.2, we can deal with the general case X^s with $s \geqslant 0$. Here for simplicity we are going to consider the case $q \in X^s$, $s > \frac{1}{2}$, and we are going to define $T_{\mathrm{c}}^{-1}(\lambda; q)$ as the original transmission coefficient T^{-1} modulo the mass and momentum quantities. The renormalised transmission coefficient T_{c}^{-1} is well-defined in X^s and is invariant by GP-flow. We are going to use T_{c}^{-1} to define a family of conserved energies, which imply the global-in-time well-posedness of the GP equation in next section.

Formulation of the renormalised Lax equation We first would like to renormalise the Lax equation (2.34) with the boundary condition $(2.37)_1$ into the following form

$$w_x = \begin{pmatrix} 0 & 0 \\ 0 & 2iz \end{pmatrix} w + \begin{pmatrix} 0 & q_2 \\ q_3 & q_4 \end{pmatrix} w, \quad \lim_{x \to -\infty} w(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{ODE}$$

where the "potentials" depend on

$$|q|^2 - 1, \quad q'$$

linearly, up to the coefficients $q, \bar{q}, \zeta$ or $\frac{1}{|q|^2-\zeta^2}$ (recalling $\zeta = \lambda + z \in \mathcal{U}$). More precisely, we observe that the transformed Jost solution

$$v = \begin{pmatrix} -i\zeta & q \\ \bar{q} & i\zeta \end{pmatrix} u, \text{ such that } u = \frac{1}{|q|^2 - \zeta^2} \begin{pmatrix} -i\zeta & q \\ \bar{q} & i\zeta \end{pmatrix} v,$$

solves

$$v_x = \begin{pmatrix} -iz & 0 \\ 0 & iz \end{pmatrix} v + \begin{pmatrix} -q_1 & q_2 \\ q_3 & q_4 - q_1 \end{pmatrix} v,$$

where

$$\begin{aligned} q_1 &= \frac{i\zeta(|q|^2 - 1) - \bar{q}q'}{|q|^2 - \zeta^2}, \\ q_2 &= \frac{i\zeta q' + (|q|^2 - 1)q}{|q|^2 - \zeta^2}, \\ q_3 &= \frac{-i\zeta\bar{q}' + (|q|^2 - 1)\bar{q}}{|q|^2 - \zeta^2}, \\ q_4 &= \frac{2i\zeta(|q|^2 - 1) + q\bar{q}' - \bar{q}q'}{|q|^2 - \zeta^2}. \end{aligned} \tag{$*$}$$

We want to remove the upper left entries of the two matrices: Let

$$w = -\frac{1}{2iz} e^{izx + \int_{-\infty}^{x} q_1 \,\mathrm{d}m} v = -\frac{1}{2iz} e^{izx + \int_{-\infty}^{x} q_1 \,\mathrm{d}m} \begin{pmatrix} -i\zeta & q \\ \bar{q} & i\zeta \end{pmatrix} u, \tag{2.40}$$

then it satisfies the renormalised (ODE) above. Furthermore, it has the following asymptotic behaviour at $+\infty$:

$$w(\lambda, x, t) = \begin{pmatrix} e^{\int_{-\infty}^{\infty} q_1 \,\mathrm{d}m} T^{-1}(\lambda) \\ 0 \end{pmatrix} + o(1) \text{ as } x \to +\infty. \tag{2.41}$$

Formal resolution of the renormalised Lax equation Equivalently, the renormalized Jost solution w satisfies the following integral equation

$$w(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 & q_2(x_1) \\ e^{2iz(x-x_1) + \int_{x_1}^{x} q_4 \,\mathrm{d}m} q_3(x_1) & 0 \end{pmatrix} w(x_1)\, dx_1.$$

We can formally solve (2.34) iteratively as follows

$$
\begin{aligned}
& w=\sum_{n=0}^{\infty} w_n, \quad w_0=\begin{pmatrix}1\\0\end{pmatrix}, \\
& w_n(x)=\int_{-\infty}^{x}\begin{pmatrix}0 & q_2(x_1)\\ e^{\varphi(x)-\varphi(x_1)}q_3(x_1) & 0\end{pmatrix} w_{n-1}(x_1)\,\mathrm{d}x_1, \\
& \text{with } \varphi(x)=2izx+\int_0^x q_4(x_1)\,\mathrm{d}x_1 .
\end{aligned} \tag{2.42}
$$

Thus the asymptotic behaviour of the first component w^1 at infinity in (2.41) reads as

$$
\begin{aligned}
& e^{\int_{-\infty}^{\infty} q_1\,\mathrm{d}m} T^{-1}(\lambda)=\lim_{x\to\infty} w^1(x)=\sum_{n=0}^{\infty}\lim_{x\to\infty} w_n^1(x) \\
& =1+\int_{x_1<y_1} e^{\varphi(y_1)-\varphi(x_1)} q_3(x_1) q_2(y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\
& \quad +\cdots+\int_{x_1<y_1<\cdots<x_j<y_j}\prod_{n=1}^{j} e^{\varphi(y_n)-\varphi(x_n)} q_3(x_n) q_2(y_n)\,\mathrm{d}x\,\mathrm{d}y+\cdots \\
& =: 1+\sum_{j=1}^{\infty} \wedge^j .
\end{aligned}
$$

More precisely, we have the following asymptotic expansion of $T^{-1}(\lambda)$ and $\ln T^{-1}(\lambda)$.

Proposition 2.1. *Let $q-1\in\mathcal{S}$. Let $(\lambda,z)\in\mathcal{R}$ the upper Riemann sheet, and $\zeta=\lambda+z\in\mathcal{U}$ the upper complex plane.*
Then the transmission coefficient $T^{-1}(\lambda)$ defined in the upper Riemann sheet expands asymptotically as follows:

$$
e^{\int_{-\infty}^{\infty} q_1\,\mathrm{d}x} T^{-1}(\lambda)=1+\sum_{j=1}^{\infty} T_{2j}(\lambda), \quad T_{2j}=\wedge^j, \tag{2.43}
$$

and its logarithm expands asymptotically as

$$
\int_{-\infty}^{\infty} q_1\,\mathrm{d}x+\ln T^{-1}(\lambda)=T_2+\sum_{j=2}^{\infty}\tilde{T}_{2j}, \tag{2.44}
$$

where $\tilde{T}_{2j}$ is linear combination of <u>*connected*</u> *symbols $\wedge_{2j}$ of degree $2j$.*

Notice here that T_2 is "quadratic" while $\tilde{T}_{2j}$ is homogeneous of order $2j$ in $|q|^2-1, q'$, up to the coefficients $q,\bar{q},\zeta,\frac{1}{|q|^2-\zeta^2}$, and $e^{\varphi(x)}$, $e^{-\varphi(x)}$.

Renormalised transmission coefficient For $q-1 \in \mathcal{S}$, we calculate explicitly

$$
\begin{aligned}
\int_{-\infty}^{\infty} q_1 \,\mathrm{d}x &= -\frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\mathcal{P} - \Phi, \\
\Phi := -\frac{i}{2z}\int_{\mathbb{R}} \frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x &+ \frac{1}{2z\zeta}\int_{\mathbb{R}} \frac{\bar{q}q'(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x\,,
\end{aligned}
\tag{2.45}
$$

where $\mathcal{M} = \int_{\mathbb{R}}(|q|^2-1)\,\mathrm{d}x$, $\mathcal{P} = \operatorname{Im}\int_{\mathbb{R}} q\bar{q}'\,\mathrm{d}x$ are the mass and momentum in (1.10) and (1.11), which are in general not well-defined for $q \in X^s$. The quantity Φ, however, is "quadratic" in $|q|^2-1, q'$ and is well-defined for $q \in X^s$, $s > \frac{1}{2}$.
The asymptotic expansion (2.44) then reads as

$$
\ln T^{-1}(\lambda) - \frac{i}{2z}\mathcal{M} - \frac{i}{2z\zeta}\mathcal{P} = \Phi + T_2 + \sum_{j=2}^{\infty}\tilde{T}_{2j},
$$

or equivalently,

$$
e^{-\frac{i}{2z}\mathcal{M}-\frac{i}{2z\zeta}\mathcal{P}}T^{-1}(\lambda) = e^{\Phi}(1+\sum_{j=1}^{\infty}T_{2j}) = e^{\Phi}\lim_{x\to\infty} w^1(x).
$$

This motivates us to define the renormalised transmission coefficient T_{c}^{-1} as $e^{-\frac{i}{2z}\mathcal{M}-\frac{i}{2z\zeta}\mathcal{P}}T^{-1}(\lambda)$ if $q \in 1+\mathcal{S}$, and more generally

Theorem 2.1. *Let $q \in X^s$, $s > \frac{1}{2}$. Let $(\lambda, z) \in \mathcal{R}$, then the renormalised Lax equation* (ODE) *has a unique solution $w \in L^\infty(\mathbb{R};\mathbb{C}^2)$ with $w^1 - 1 \in U^2$. We define the renormalised transmission coefficient as*

$$
T_{\mathrm{c}}^{-1}(\lambda) = e^{\Phi}\lim_{x\to\infty} w^1(x).
$$

Then

- *$T_{\mathrm{c}}^{-1}(\lambda; q)$ is a well-defined holomorphic function in $(\lambda, z) \in \mathcal{R}$ and depends analytically on $q \in X^s$ with respect to the analytic structure given in Theorem 1.1;*
- *When $q-1 \in \mathcal{S}$, the relation $T_{\mathrm{c}}^{-1}(\lambda) = e^{-i\mathcal{M}(2z)^{-1}-i\mathcal{P}(2z\zeta)^{-1}}T^{-1}(\lambda)$ and the properties $|T_{\mathrm{c}}^{-1}(\lambda)| \geqslant 1$ if $\lambda \in (-\infty,-1]\cup[1,\infty)$, $T_{\mathrm{c}}^{-1} \to 1$ as $|\lambda| \to \infty$, all hold true;*

- $T_{\rm c}^{-1}(\lambda)$ *has the following formal asymptotic expansion*

$$T_{\rm c}^{-1}(\lambda) = e^{\Phi(\lambda)}\Big(1 + \sum_{j=1}^{\infty} T_{2j}(\lambda)\Big), \quad T_{2j} = \cap^j, \tag{2.46}$$

and its logarithm expands asymptotically as

$$\ln T_{\rm c}^{-1}(\lambda) = \Phi(\lambda) + T_2(\lambda) + \sum_{j=2}^{\infty} \tilde{T}_{2j}(\lambda), \tag{2.47}$$

where

$$\Phi(\lambda) := -\frac{i}{2z}\int_{\mathbb{R}} \frac{(|q|^2-1)^2}{|q|^2-\zeta^2}\,\mathrm{d}x + \frac{1}{2z\zeta}\int_{\mathbb{R}} \frac{\bar{q}q'(|q|^2-1)}{(|q|^2-\zeta^2)}\,\mathrm{d}x\,, \tag{2.48}$$

and $\tilde{T}_{2j}$ is linear combination of <u>*connected*</u> *symbols $\cap_{2j}$ of degree $2j$.*

- *Let $q(t,x) \in \mathcal{C}(I; X^s)$ be a solution of the Gross-Pitaevskii equation, then $T_{\rm c}^{-1}(\lambda; q(t))$ is conserved on the existence time interval I.*

- $\mathrm{Re}\, T_{\rm c}^{-1}(\lambda) = \mathrm{Re}\, T_{\rm c}^{-1}(\bar{\lambda})$ *if $(\lambda, z) = (i\sigma, \pm i\frac{\tau}{2}) \in \mathcal{R}$, $\tau \geqslant 2$, $\sigma = \sqrt{\frac{\tau^2}{4} - 1}$.*

- $T_{\rm c}^{-1}$ *has at most countably many simple zeros $\{\lambda_m\} \subset (-1, 1)$.*

We can define a superharmonic function $G(z)$ on the upper half plane $\mathcal{U}$ as follows

$$G(z) := \frac{1}{2}\sum_{\pm} \mathrm{Re}\Big(4z^2 \ln T_{\rm c}^{-1}\big(\pm\sqrt{z^2+1}\big)\Big), \quad \mathrm{Im}\, z > 0, \quad \mathrm{Im}\sqrt{z^2+1} \geqslant 0, \tag{2.49}$$

such that $G \geqslant 0$ on the upper half plane $\mathcal{U}$ and $-\Delta G \geqslant 0$ is a nonnegative measure on the upper half plane $\mathcal{U}$ as follows

$$\nu_G(z) = -\Delta_z G(z) = -\pi \sum_m (2z)^2 \delta_{z=z_m} \geqslant 0,\ z_m = i\sqrt{1-\lambda_m^2} \in i(0,1], \tag{2.50}$$

where $\{\lambda_m\}$ are the simple zeros of $T_{\rm c}^{-1}(\lambda)$.

Remark 2.1. *We can define another renormalised transmission coefficient $\mathcal{T}^{-1}$ for all $q \in X^s$, $s \geqslant 0$ (instead of $s > \frac{1}{2}$ here), which is the original T^{-1} modulo the conserved quantity Θ (which is only well-defined up to modulo $2\pi\,\mathbb{Z}$) and the conserved mass $\mathcal{M}$. Indeed we can take the transformation $u \mapsto \begin{pmatrix} -i\zeta & r \\ \bar{r} & i\zeta \end{pmatrix} u$, with r the regularisation of q for $q \in X^s$, $s \geqslant 0$.*

When evaluated on the imaginary axis (see (3.74) *below) however, the real parts of both the renormalised transmission coefficients* T_{c}^{-1} *and* $\mathcal{T}^{-1}$ *coincide. Hence* $G(i\frac{\tau}{2})$ *and thus the conserved energies defined below* (3.76) *remain the same.*

Ideas of the proof: The resolution of the ODE (ODE) relies on the estimates of the operator

$$(Sf)(t) = \int_{x<y<t} e^{\varphi(y)-\varphi(x)} q_2(y)(q_3 f)(x)\, \mathrm{d}x\ \mathrm{d}y\,.$$

Under the smallness condition

$$\|(|q|^2-1, q')\|_{l^\infty_\tau DU^2} \leqslant \frac{1}{2C_0}, \tag{2.51}$$

we have

$$\|S\|_{V^2\mapsto U^2} \leqslant C\frac{2(|\operatorname{Re} z| + |\operatorname{Im} z|)}{\operatorname{Im} z}\|(|q|^2-1, q')\|^2_{l^2_\tau DU^2}, \tag{2.52}$$

and hence

$$\|w_{2j}\|_{U^2} = \|S^j(1)\|_{U^2} \leqslant \big(C\frac{2(|\operatorname{Re} z| + |\operatorname{Im} z|)}{\operatorname{Im} z}\big)^j \|(|q|^2-1, q')\|^{2j}_{l^2_\tau DU^2}.$$

We have similar estimates for w_{2j+1}. Finally for any fixed $(\lambda, z) \in \mathcal{R}$, we can take two real numbers a, b such that $\|(|q|^2-1, q')|_{(-\infty,a]\cup[b,\infty)}\|_{l^2_\tau DU^2}$ is small enough. We then solve (ODE) firstly on $(-\infty, a]$, then on the finite interval $[a, b]$, and finally on $[b, \infty)$.

[10.01.2022]

[17.01.2022]

3 Conserved energies

3.1 Trace formular

3.1.1 Harmonic functions on the upper half-plane

Recall that if H is a nonnegative harmonic function in the upper half complex plane, which is bounded on the positive imaginary axis. Then it has a trace on the real line, which is a locally finite nonnegative measure μ. Furthermore, one can represent H in terms of its trace via Poisson kernel

$$H(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-\xi|^2} d\mu(\xi). \tag{3.53}$$

In particular, if $z = i\tau$, $\tau > 0$, then we have

$$H(i\tau) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\tau}{\xi^2+\tau^2} d\mu(\xi),$$

and we have the following identity which relates the integral with the weight $(\xi^2+1)^{-1}$ on the real axis and the harmonic function evaluated at the purely imaginary point i:

$$\frac{1}{\pi}\int_{\mathbb{R}} (\xi^2+1)^{-1} d\mu(\xi) = H(i). \tag{3.54}$$

Let $s \in (-1, 0)$, then the integral with the weight $(\xi^2+1)^s$ on the real axis

$$\frac{1}{\pi}\int_{\mathbb{R}} (\xi^2+1)^s d\mu(\xi) \tag{3.55}$$

can be reformulated as the following integral of the harmonic function on the imaginary axis

$$-\frac{2\sin(\pi s)}{\pi}\int_1^\infty (\tau^2-1)^s H(i\tau) d\tau \tag{3.56}$$

by Cauchy's formular, if either (3.55) or (3.56) is finite.
If the measure $(1+|\xi|^2)^N\mu$, $N \geqslant 0$ is finite, then we can expand H at $i\infty$ as follows:

$$H(i\tau) = \sum_{l=0}^{N} (-1)^l H_{2l}\, \tau^{-(2l+1)} + H_{>2N}(i\tau),$$

where

$$\begin{aligned} H_{2l} &= \frac{1}{\pi}\int_{\mathbb{R}} \xi^{2l} d\mu(\xi), \\ H_{>2N}(i\tau) &= (-1)^{N+1}\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^{2N+2}}{\xi^2+\tau^2} d\mu(\xi)\tau^{-(2N+1)} = o(\tau^{-(2N+1)}). \end{aligned}$$

If $s \geqslant 0$, then we have to remove the parts involving H_{2l}, $0 \leqslant l \leqslant [s]$ in $H(i\tau)$, and we have the following more general trace formular

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{R}} (\xi^2+1)^s d\mu(\xi) &= \sum_{l=0}^{[s]} \binom{s}{l} H_{2l} \\ &- \frac{2\sin(\pi s)}{\pi}\int_1^\infty (\tau^2-1)^s \Big(H(i\tau) - \sum_{l=0}^{[s]} (-1)^l H_{2l}\tau^{-(2l+1)} \Big) d\tau. \end{aligned} \tag{3.57}$$

We observe in particular

- If $s=-1$, then (3.57) holds in the sense

$$\frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+1)^{-1}d\mu(\xi)=\lim_{s\to(-1)_+}-\frac{2\sin(\pi s)}{\pi}\int_1^\infty(\tau^2-1)^sH(i\tau)d\tau=H(i);$$

- If $s=N\geqslant 0$ is an integer, then (3.57) reduces to

$$\frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+1)^Nd\mu(\xi)=\sum_{l=0}^{N}\binom{N}{l}H_{2l}=\frac{1}{\pi}\int_{\mathbb{R}}\binom{N}{l}\xi^{2l}d\mu(\xi).$$

We can further introduce a large parameter $\tau_0>0$ in (3.57) to arrive at

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+{\tau_0}^2)^sd\mu(\xi)&=\sum_{l=0}^{[s]}\binom{s}{l}{\tau_0}^{2(s-l)}H_{2l}\\&-\frac{2\sin(\pi s)}{\pi}\int_{\tau_0}^\infty(\tau^2-{\tau_0}^2)^s\Big(H(i\tau)-\sum_{l=0}^{[s]}(-1)^lH_{2l}\tau^{-(2l+1)}\Big)d\tau,\end{aligned}\tag{3.58}$$

and in particular

- If $s=-1$, then

$$\frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+{\tau_0}^2)^{-1}d\mu(\xi)=\frac{1}{\tau_0}H(i\tau_0);\tag{3.59}$$

- If $s=N\geqslant 0$ is an integer, then (3.57) reduces to

$$\frac{1}{\pi}\int_{\mathbb{R}}(\xi^2+{\tau_0}^2)^Nd\mu(\xi)=\sum_{l=0}^{N}\binom{N}{l}{\tau_0}^{2(N-l)}H_{2l}=\frac{1}{\pi}\int_{\mathbb{R}}\binom{N}{l}{\tau_0}^{2(N-l)}\xi^{2l}d\mu(\xi).\tag{3.60}$$

An intuitive example If we take the measure

$$d\mu(\xi)=\pi|\hat{f}(\xi)|^2d\xi$$

on the real axis, where $\hat{f}(\xi)$ is the Fourier transform of the function $f\in H^{-1}(\mathbb{R})$, then the function H given in (3.53):

$$H(z)=\int_{\mathbb{R}}\frac{\operatorname{Im}z}{(\xi-\operatorname{Re}z)^2+(\operatorname{Im}z)^2}|\hat{f}(\xi)|^2\,d\xi$$

is a well-defined nonnegative harmonic function in the upper half-plane, which is bounded on the positive imaginary axis.
We now apply the trace formula (3.58):

(i) We can relate the (rescaled) $\|f\|_{H^{-1}}$-norm with the harmonic function evaluated at the imaginary axis:

$$H(i\tau) = \int_{\mathbb{R}} \frac{\tau}{\xi^2 + \tau^2} |\hat{f}(\xi)|^2 d\xi = \tau \|f\|^2_{H^{-1}_\tau}, \tag{3.61}$$

which is indeed the trace formular (3.59) with $s = -1$ and $\tau_0 = \tau$.

This shows roughly that the well-definedness of the harmonic function H on the real axis *resp.* on the imaginary axis requires $f \in L^2(\mathbb{R})$ *resp.* $f \in H^{-1}(\mathbb{R})$.

(ii) For $s \in (-1, 0)$, the rescaled $\|f\|_{H^s_{\tau_0}}$-norm is formulated in terms of $\|f\|_{H^{-1}_\tau}$ as follows:

$$\begin{aligned} \|f\|^2_{H^s_{\tau_0}} &= \int_{\mathbb{R}} (\xi^2 + {\tau_0}^2)^s |\hat{f}(\xi)|^2 d\xi \\ &= -\frac{2\sin(\pi s)}{\pi} \int_{\tau_0}^{\infty} (\tau^2 - {\tau_0}^2)^s \tau \|f\|^2_{H^{-1}_\tau} d\tau. \end{aligned} \tag{3.62}$$

(iii) For $f \in H^N(\mathbb{R})$ we expand H at $i\infty$ as follows:

$$H(i\tau) = \sum_{l=0}^{N} (-1)^l H_{2l} \tau^{-(2l+1)} + H_{>2N}(i\tau), \tag{3.63}$$

where

$$\begin{aligned} H_{2l} &= \int_{\mathbb{R}} \xi^{2l} |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{\dot{H}^l}, \\ H_{>2N}(i\tau) &= (-1)^{N+1} \int_{\mathbb{R}} \frac{\xi^{2N+2}}{\xi^2 + \tau^2} |\hat{f}(\xi)|^2 d\xi \tau^{-(2N+1)} = o(\tau^{-(2N+1)}). \end{aligned}$$

In other words, we can expand $\|f\|^2_{H^{-1}_\tau}$ as $\tau \to \infty$ as:

$$\|f\|^2_{H^{-1}_\tau} = \sum_{l=0}^{N} (-1)^l \|f^{(l)}\|^2_{L^2} \tau^{-(2l+2)} + (-1)^{N+1} \|f^{(N+1)}\|^2_{H^{-1}_\tau} \tau^{-(2N+2)}.$$

We then have the following trace formula

$$\|f\|^2_{H^N_{\tau_0}} = \int_{\mathbb{R}} (\xi^2 + {\tau_0}^2)^N |\hat{f}(\xi)|^2 d\xi = \sum_{l=0}^{N} \binom{N}{l} {\tau_0}^{2(N-l)} \|f\|^2_{\dot{H}^l},$$

and in particular

$$\begin{aligned} \|f\|^2_{L^2} &= \|f\|^2_{L^2}, \\ \|f\|^2_{H^1_{\tau_0}} &= {\tau_0}^2 \|f\|^2_{L^2} + \|f'\|^2_{L^2}. \end{aligned}$$

(iv) For $f \in H^s(\mathbb{R})$, $s \geqslant 0$, then we have

$$\|f\|^2_{H^s_{\tau_0}} = \int_{\mathbb{R}} (\xi^2 + {\tau_0}^2)^s |\hat{f}(\xi)|^2 d\xi = \sum_{l=0}^{[s]} \binom{s}{l} {\tau_0}^{2(s-l)} \|f^{(l)}\|^2_{L^2}$$
$$- \frac{2\sin(\pi s)}{\pi} \int_{\tau_0}^{\infty} (\tau^2 - {\tau_0}^2)^s (-1)^{[s]+1} \|f^{([s]+1)}\|^2_{H^{-1}_\tau} \tau^{-(2[s]+1)} d\tau,$$

and in particular for $s \in (0,1)$

$$\|f\|^2_{H^s_{\tau_0}} = {\tau_0}^{2s} \|f\|^2_{L^2} + \frac{2\sin(\pi s)}{\pi} \int_{\tau_0}^{\infty} (\tau^2 - {\tau_0}^2)^s \tau^{-1} \|f'\|^2_{H^{-1}_\tau} d\tau.$$

3.1.2 Superharmonic functions on the upper half-plane

We have similar trace formula for the nonnegative superharmonic functions in the upper half-plane.

Lemma 3.1 (Koch-Tataru)**.** *Let G be a nonnegative superharmonic function in the upper half-plane which is bounded on the positive imaginary axis. Then it has a trace on the real line, which is a Radon measure μ, and $-\Delta G$ is a Radon measure ν in the upper half-plane.*
The followings hold true:

- *Representation of G through μ, ν.*
  The function G can be represented in terms of the Poisson kernel and the fundamental solution of the Laplace equation as follows:

$$G(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-\xi|^2} d\mu(\xi) + \frac{1}{2\pi} \int_{\mathcal{U}} \ln\Big|\frac{z-\bar{\zeta}}{z-\zeta}\Big| d\nu(\zeta). \tag{3.64}$$

  In particular, we have

$$G(i\tau) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\tau}{\xi^2+\tau^2} d\mu(\xi) + \frac{1}{\pi} \int_{\mathcal{U}} \operatorname{Im}\Big[\int_0^{\zeta} \frac{\tau}{w^2+\tau^2} dw\Big] d\nu(\zeta). \tag{3.65}$$

- *Expansion of G at $+i\infty$.*
  If two measures $(1+|\xi|^2)^N \mu$, $\operatorname{Im}\zeta (1+|\zeta|^2)^N \nu$ are finite, then we have the following precise expansion of G at $+i\infty$:

$$G(i\tau) = \sum_{l=0}^{N} (-1)^l G_{2l} \tau^{-2l-1} + G_{>2N}(i\tau), \tag{3.66}$$

where

$$G_{2l} = \frac{1}{\pi}\int_{\mathbb{R}} \xi^{2l} d\mu(\xi) + \frac{1}{\pi}\int_{\mathcal{U}} \frac{1}{2l+1} \operatorname{Im}\zeta^{2l+1} d\nu(\zeta),$$

$$G_{>2N} = (-1)^{N+1}\Big(\frac{1}{\pi}\int_{\mathbb{R}} \frac{\xi^{2N+2}}{\xi^2+\tau^2} d\mu(\xi) + \frac{1}{\pi}\int_{\mathcal{U}} \operatorname{Im}\Big[\int_0^\zeta \frac{w^{2(N+1)}}{w^2+\tau^2} dw\Big] d\nu(\zeta)\Big)\tau^{-2N-1}$$
$$= o(\tau^{-(2N+1)}).$$

- *Trace formula of G.*

$$\frac{1}{\pi}\int_{\mathbb{R}} (\xi^2+{\tau_0}^2)^s d\mu(\xi) + \frac{1}{\pi}\int_{\mathcal{U}} \operatorname{Im}\Big[\int_0^\zeta (w^2+{\tau_0}^2)^s dw\Big] d\nu(\zeta)$$
$$= -\frac{2\sin(\pi s)}{\pi}\int_{\tau_0}^\infty (\tau^2-{\tau_0}^2)^s\Big(G(i\tau) - \sum_{l=0}^{[s]}(-1)^l G_{2l}\tau^{-2l-1}\Big)d\tau \quad (3.67)$$
$$+ \sum_{l=0}^{[s]}\binom{s}{l} {\tau_0}^{2(s-l)} G_{2l},$$

whenever either side is finite. It is indeed (3.65) *if* $s = -1$.

3.2 Conserved energies

3.2.1 Rescaled energy norms

Recall the definition of the rescaled energy norm in (1.21):

$$(E^s_\tau(q))^2 = \||q|^2-1\|^2_{H^{s-1}_\tau} + \|q'\|^2_{H^{s-1}_\tau} = \int_{\mathbb{R}} (\xi^2+\tau^2)^{s-1}\big(|\widehat{|q|^2-1}|^2 + |\widehat{q'}|^2\big)(\xi)\, \mathrm{d}\xi\,.$$

For notational simplicity, we introduce the vector-valued function

$$\mathbf{q} = (|q|^2-1, q'), \quad (3.68)$$

such that we write

$$(E^s_\tau(q))^2 = \|\,\mathbf{q}\,\|^2_{H^{s-1}_\tau} = \int_{\mathbb{R}} (\xi^2+\tau^2)^{s-1}|\widehat{\mathbf{q}}|^2(\xi)\, \mathrm{d}\xi\,. \quad (3.69)$$

Trace formular for the rescaled energy norms If $s \in (0,1)$, then the trace formula (3.62) reads as

$$(E^s_{\tau_0}(q))^2 = \|\,\mathbf{q}\,\|^2_{H^{s-1}_{\tau_0}}$$
$$= -\frac{2\sin(\pi(s-1))}{\pi}\int_{\tau_0}^\infty (\tau^2-{\tau_0}^2)^{s-1}\tau(E^0_\tau(q))^2 d\tau. \quad (3.70)$$

If $s = N \geqslant 1$ is an integer, then

$$(E^N_{\tau_0}(q))^2 = \| \mathbf{q} \|^2_{H^{N-1}_{\tau_0}} = \sum_{l=0}^{N-1} \binom{N-1}{l} {\tau_0}^{2(N-l)} \| \mathbf{q} \|^2_{\dot{H}^l}. \tag{3.71}$$

For general $s \geqslant 1$, we have

$$\begin{aligned}(E^s_{\tau_0}(q))^2 = \| \mathbf{q} \|^2_{H^{s-1}_{\tau_0}} &= \sum_{l=0}^{[s-1]} \binom{s-1}{l} {\tau_0}^{2(s-1-l)} \| \mathbf{q}^{(l)} \|^2_{L^2} \\ -\frac{2\sin(\pi(s-1))}{\pi} &\int_{\tau_0}^{\infty} (\tau^2 - {\tau_0}^2)^{s-1} (-1)^{[s]} \tau^{-(2[s]-1)} \| \mathbf{q}^{([s])} \|^2_{H^{-1}_\tau} d\tau.\end{aligned} \tag{3.72}$$

3.2.2 Conserved energies and global-in-time wellposedness

Reformulation of H^{-1}_τ-norm in x-variable Recall the definitions of the unitary Fourier transform and inverse Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \,\mathrm{d}x\,, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi.$$

We first observe the following description of the H^{-1}_τ-norm:

$$\begin{aligned}&\mathrm{Re} \int_{x<y} e^{-\tau(y-x)} \bar{f}(x) f(y) \,\mathrm{d}x \;\mathrm{d}y \\ &= \frac{1}{2\pi} \mathrm{Re} \int_{\mathbb{R}^3} \frac{1}{\tau - i\xi} e^{iy\eta - iy\xi} \hat{f}(\eta) \overline{\hat{f}(\xi)} \,\mathrm{d}y \; d\xi \, d\eta \\ &= \mathrm{Re} \int_{\mathbb{R}} \frac{1}{\tau - i\xi} \hat{f}(\xi) \overline{\hat{f}(\xi)} \, d\xi = \int_{\mathbb{R}} \frac{\tau}{\tau^2 + \xi^2} |\hat{f}(\xi)|^2 \, d\xi \\ &= \tau \|f\|^2_{H^{-1}_\tau}.\end{aligned} \tag{3.73}$$

Asymptotic expansion of $\ln\, T_{\mathrm{c}}^{-1}$ **on the imaginary axis** We restrict ourselves in this subsection to the "imaginary axis" of the upper Riemann sheet $\mathcal{R}$:

$$(\lambda, z) = (i\sigma, i\tau/2), \text{ with } \tau \geqslant 2 \text{ and } \sigma = \sqrt{\frac{\tau^2}{4} - 1} \geqslant 0. \tag{3.74}$$

Recall the asymptotic expansion of $\ln\, T_{\mathrm{c}}^{-1}$ in (2.47):

$$\ln\, T_{\mathrm{c}}^{-1}(\lambda) = \Phi(\lambda) + T_2(\lambda) + \sum_{j=2}^{\infty} \tilde{T}_{2j}(\lambda).$$

We identify the quadratic term and the cubic (or higher order) term with respect to $\mathbf{q}$ in $\Phi + T_2$ as $\tilde{T}_2$ and $\tilde{T}_3$ respectively:

$$\Phi(\lambda) + T_2(\lambda) = \tilde{T}_2(\lambda) + \tilde{T}_3(\lambda).$$

We calculate the real part of $\tilde{T}_2$ on the imaginary axis as

$$\mathrm{Re}\,\tilde{T}_2(i\sigma) = -\frac{1}{\tau^2}\int_{x<y} e^{-\tau(y-x)}\bar{\mathbf{q}}(x)\cdot\mathbf{q}(y)\,\mathrm{d}x\ \mathrm{d}y\,.$$

We define the harmonic function

$$H(z) = \frac{1}{2}\sum_{\pm}\mathrm{Re}\left(4z^2\tilde{T}_2\big(\pm\sqrt{z^2+1}\big)\right)$$

on the upper half-plane, such that

$$H(i\frac{\tau}{2}) = \int_{x<y} e^{-\tau(y-x)}\bar{\mathbf{q}}(x)\cdot\mathbf{q}(y)\,\mathrm{d}x\ \mathrm{d}y\ = \tau\|\,\mathbf{q}\,\|^2_{H_\tau^{-1}}.$$

In particular, if $\mathbf{q}\in H^N$, $N\geqslant 0$, the asymptotic expansion of $H(i\frac{\tau}{2})$ reads as in (3.63):

$$H(i\frac{\tau}{2}) = \sum_{l=0}^{N}(-1)^l H_{2l}\tau^{-(2l+1)} + H_{>2N}(i\frac{\tau}{2}),$$
$$\text{where } H_{2l} = \|\,\mathbf{q}^{(l)}\,\|^2_{L^2},\quad H_{>2N}(i\frac{\tau}{2}) = (-1)^{N+1}\tau^{-(2N+1)}\|\,\mathbf{q}^{(N+1)}\,\|^2_{H_\tau^{-1}}.$$

Thus the above trace formular (3.70), (3.71) and (3.72) read respectively as

$$\begin{aligned}
(E_\tau^0(q))^2 &= \frac{1}{\tau}H(i\frac{\tau}{2}),\\
(E_{\tau_0}^s(q))^2 &= -\frac{2\sin(\pi(s-1))}{\pi}\int_{\tau_0}^{\infty}(\tau^2-{\tau_0}^2)^{s-1}H(i\frac{\tau}{2})d\tau,\quad s\in(0,1),\\
(E_{\tau_0}^s(q))^2 &= \sum_{l=0}^{[s-1]}\binom{s-1}{l}{\tau_0}^{2(s-1-l)}H_{2l}\\
&\quad-\frac{2\sin(\pi(s-1))}{\pi}\int_{\tau_0}^{\infty}(\tau^2-{\tau_0}^2)^{s-1}H_{>2N}(i\frac{\tau}{2})d\tau,\quad s\geqslant 1.
\end{aligned}\tag{3.75}$$

Conserved energies Recall the superharmonic function defined in the upper half-plane in (2.49):

$$G(z) = \frac{1}{2}\sum_{\pm}\mathrm{Re}\left(4z^2\ln\,T_{\mathrm{c}}^{-1}\big(\pm\sqrt{z^2+1}\big)\right).$$

Similarly as above, if $\mathbf{q} \in H^N$, $N \geqslant 0$, we can expand $G(i\frac{\tau}{2})$ as

$$G(i\frac{\tau}{2}) = \sum_{l=0}^{N}(-1)^l G_{2l}\tau^{-(2l+1)} + G_{>2N}(i\frac{\tau}{2}).$$

However, G_{2l}, $G_{>2N}$ are much more complicated, as the phase function $\varphi(x) = -\tau x + \int_0^x q_4$ involves the integral of $\mathbf{q}$. We have for example

$$\int_{x<y} e^{\varphi(y)-\varphi(x)} g(y)h(x)\,\mathrm{d}x\ \mathrm{d}y\ = \sum_{\ell=1}^{N} b_\ell + b_{\geqslant k+1},$$

where

$$b_\ell = \frac{1}{\tau^\ell}\int_{\mathbb{R}}(D_+^{m_\ell} g)(D_-^{n_\ell} h)\,\mathrm{d}y\ = \int_{\mathbb{R}}\Big((\frac{D_+}{\tau})^{m_\ell}\frac{g}{\tau}\Big)\Big((\frac{D_-}{\tau})^{n_\ell} h\Big)\,\mathrm{d}y\,,\ m_\ell + n_\ell + 1 = \ell,$$

$$\begin{aligned} b_{\geqslant k+1}(z) &= \frac{1}{\tau^k}\int_{x<y} e^{\varphi(y)-\varphi(x)}(D_+^m g)(y)(D_-^n h)(x)\,\mathrm{d}x\ \mathrm{d}y \\ &= \int_{x<y} e^{\varphi(y)-\varphi(x)}\Big((\frac{D_+}{\tau})^m g\Big)(y)\Big((\frac{D_-}{\tau})^n h\Big)(x)\,\mathrm{d}x\ \mathrm{d}y\,,\ m+n=k. \end{aligned}$$

Here $D_\pm(g) := (q_4 \pm \partial_x)g$.
Motivated by the reformulation of the rescaled energy norms in terms of the harmonic function (3.75), we define our conserved energies in terms of the *time invariant* superharmonic function G as follows:

$$\begin{aligned} (\mathcal{E}_\tau^0(q))^2 &= \frac{1}{\tau}G(i\frac{\tau}{2}), \\ (\mathcal{E}_{\tau_0}^s(q))^2 &= -\frac{2\sin(\pi(s-1))}{\pi}\int_{\tau_0}^{\infty}(\tau^2 - {\tau_0}^2)^{s-1}G(i\frac{\tau}{2})d\tau, \quad s \in (0,1), \\ (\mathcal{E}_{\tau_0}^s(q))^2 &= \sum_{l=0}^{[s-1]}\binom{s-1}{l}{\tau_0}^{2(s-1-l)}G_{2l} \\ &\quad - \frac{2\sin(\pi(s-1))}{\pi}\int_{\tau_0}^{\infty}(\tau^2 - {\tau_0}^2)^{s-1}G_{>2N}(i\frac{\tau}{2})d\tau, \quad s \geqslant 1. \end{aligned} \tag{3.76}$$

As the difference $G(i\frac{\tau}{2}) - H(i\frac{\tau}{2})$ involves only cubic or higher order terms (with respect to $\mathbf{q}$), we expect that the difference is comparably smaller than the quadratic term $H(i\frac{\tau}{2})$ under some smallness condition. More precisely, we have

Theorem 3.1 (Conserved energies & Global-in-time Wellposedness)**.** *Let $s \geqslant 0$. Then the Gross-Pitaevskii equation* (GP) *is globally-in-time well-posed in the metric space (X^s, d^s) in the sense in Theorem 1.2. Furthermore,* *there exist a constant $C \geqslant 2$ and a family of analytic energy functionals $(\mathcal{E}^s_\tau)_{\tau\geqslant 2} : X^s \mapsto [0,\infty)$, such that*

- $\mathcal{E}^s_\tau(q)$ *is equivalent to* $(E^s_\tau(q))^2$ *in the following sense: If* $q \in X^s$ *with* $\frac{1}{\sqrt{\tau}}E^0_\tau(q) < \frac{1}{2C}$*, then*

$$|\mathcal{E}^s_\tau(q) - (E^s_\tau(q))^2| \leqslant C\big(\frac{1}{\sqrt{\tau}}E^0_\tau(q)\big)(E^s_\tau(q))^2, \quad s \geqslant 0. \tag{3.77}$$

- $\mathcal{E}^s_\tau(\cdot)$, $\tau \geqslant 2$ *is conserved by Gross-Pitaevskii flow* (GP)*.*

Correspondingly, for any initial data $q_0 \in X^s$*, there exists* $\tau_0 \geqslant C$ *depending only on* $E^0(q_0)$ *such that the unique solution* $q \in \mathcal{C}(\mathbb{R}; X^s)$ *of the Gross-Pitaevskii equation* (GP) *satisfies the following energy conservation law:*

$$E^s_{\tau_0}(q(t)) \leqslant 2E^s_{\tau_0}(q_0), \quad \forall t \in \mathbb{R}\,. \tag{3.78}$$

Idea of the proof. If $s = 0$, then

$$G(i\frac{\tau}{2}) - H(i\frac{\tau}{2}) = -\tau^2 \mathrm{Re}\Big(\big(\tilde{T}_3 + \sum_{j=2}^{\infty}\tilde{T}_{2j}\big)\big(i\sqrt{\frac{\tau^2}{4}-1}\big)\Big), \quad \tau \geqslant 2.$$

By the regularisation procedure $q \mapsto r = j_\tau * q$, we have the following newly defined

$$\begin{aligned}
q_1 &= \frac{i\zeta(|r|^2-1) + i\zeta\big[r(\bar{q}-\bar{r}) + \bar{r}(q-r)\big] - \bar{r}r'}{|r|^2-\zeta^2},\\
q_2 &= \frac{r(|r|^2-1) + i\zeta r' + r^2(\bar{q}-\bar{r}) + \zeta^2(q-r)}{|r|^2-\zeta^2},\\
q_3 &= \frac{\bar{r}(|r|^2-1) - i\zeta\bar{r}' + \bar{r}^2(q-r) + \zeta^2(\bar{q}-\bar{r})}{|r|^2-\zeta^2},\\
q_4 &= \frac{2i\zeta(|r|^2-1) + 2i\zeta\big[r(\bar{q}-\bar{r}) + \bar{r}(q-r)\big] + r\bar{r}' - \bar{r}r'}{|r|^2-\zeta^2}.
\end{aligned}$$

By virtue of $\|f\|_{l^2_\tau DU^2} \lesssim \frac{1}{\sqrt{\tau}}\|f\|_{L^2}$ and Lemma 1.1, we improve the estimate for the operator S: $(Sf)(t) = \int_{x<y<t} e^{\varphi(y)-\varphi(x)} q_2(y)(q_3 f)(x)\,\mathrm{d}x\,\mathrm{d}y$ on the imaginary axis from (2.52) to

$$\|S\|_{V^2\mapsto U^2} \leqslant C(\frac{1}{\sqrt{\tau}}E^0_\tau(q))^2$$

under the smallness condition $\frac{1}{\sqrt{\tau}} E^0_\tau(q) \leqslant \frac{1}{2C}$. Finally under the smallness condition we have

$$\Big|\frac{1}{\tau}\Big(G(i\frac{\tau}{2}) - H(i\frac{\tau}{2})\Big)\Big| \leqslant C(\frac{1}{\sqrt{\tau}} E^0_\tau(q))(E^0_\tau(q))^2.$$

The equivalence relation (3.77) for the general case $s > 0$ follows similarly. By virtue of the equivalence relationship and the conservation of the energies $\mathcal{E}^s(q)$, we have the conservation of the rescaled energy norms (3.78) for large enough parameter τ_0. This together with the local-in-time well-posedness result in Theorem 1.2 implies the global-in-time well-posedness result.