GLOBAL BOUNDEDNESS OF THE FUNDAMENTAL SOLUTION OF PARABOLIC EQUATIONS WITH UNBOUNDED COEFFICIENTS

ESTHER BLEICH

Abstract. The purpose of this paper is to obtain an upper bound for the fundamental solution for parabolic Cauchy problem $\partial_t u = Au$, $u(x,0) = f(x)$, on $\mathbb{R}^N \times (0,\infty)$, where A is a second order elliptic partial differential operator with unbounded coefficients such that its potential and the potential of the formal adjoint operator A^* are bounded from below.

1. Introduction

Let A be a second order elliptic partial differential operator with real coefficients given by

$$A = \sum_{i,j=1}^{N} D_j\left(a_{ij} D_i\right) + \sum_{i=1}^{N} F_i D_i - H = A_0 + F \cdot D - H, \tag{1.1}$$

where $A_0 = \sum_{i,j=1}^{N} D_j\left(a_{ij}D_i\right)$ and $F = (F_i)_{i=1,\dots,N}$. We consider the parabolic Cauchy problem

$$\begin{cases} \partial_t u(x,t) = Au(x,t), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $f \in C_b(\mathbb{R}^N)$ for $N \in \mathbb{N}$ is given.

It is known that if $a_{ij}, D_j a_{ij}, F_i, H \in C^{\alpha}_{loc}(\mathbb{R}^N)$ for all $i,j \in \{1,...,N\}$ and some $\alpha \in (0,1)$ and if $\inf_{x\in\mathbb{R}^N} H(x) > -\infty$, then problem (1.2) has at least one solution $u \in C(\mathbb{R}^N \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^N \times (0,\infty))$ given by

$$u(x,t) = \int_{\mathbb{R}^N} p(x,y,t) f(y)dy, \qquad (x,t) \in \mathbb{R}^N \times [0,\infty),$$

where $p = p(x,y,t) > 0$, $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$, is the fundamental solution (see [1, Theorem 2.2.5]).

We assume the following conditions on the coefficients of A which will be kept without further mentioning.

Condition 1.1.

(i) $N \geq 3$.

(ii) $a_{ij} \in C^{2+\alpha}_{loc}(\mathbb{R}^N)$, $F_i \in C^{1+\alpha}_{loc}(\mathbb{R}^N)$, $H \in C^{\alpha}_{loc}(\mathbb{R}^N)$, $a_{ij} = a_{ji}$ *for all* $i,j = 1,...,N$ *and some* $\alpha \in (0,1)$.

(iii) $H(x) \geq H_0$ *and* $\operatorname{div} F(x) + H(x) \geq H_0^*$ *for each* $x \in \mathbb{R}^N$, *where* $H_0, H_0^* \leq 0$.

(iv) *There exists a constant* $\lambda > 0$ *such that*

$$\lambda\left|\xi\right|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \qquad \textit{for all } x,\xi \in \mathbb{R}^N. \tag{1.3}$$

Date: Januar 31, 2013.

2000 *Mathematics Subject Classification.* 35A08, 47D07, 60J35.

Key words and phrases. fundamental solution, transition probabilities, parabolic Cauchy problem, Markov semigroup.

Notice that the diffusion coefficients $a_{ij}{}_{,i,j=1,\dots,N}$, the drift $F=(F_i)_{i=1,\dots,N}$ and the potential H are not assumed to be bounded in $\mathbb{R}^N$.

1.1. **The main result.** We prove that under above conditions the fundamental solution p satisfies

$$p(x,y,t)\le C_{N,\lambda}e^{\gamma t}t^{-\frac{N}{2}},\qquad (x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,\infty), \tag{1.4}$$

for the constants

$$C_{N,\lambda}=\frac{2^{N-1}\Gamma\left(\frac{N+1}{2}\right)}{\pi^{\frac{N+1}{2}}(\lambda(N-2))^{\frac{N}{2}}} \tag{1.5}$$

and

$$\gamma=-\frac{3}{4}\left(H_0^*+H_0\right)\ge 0. \tag{1.6}$$

1.2. **Notation.** For $x\in\mathbb{R}^N$, $|x|$ denotes the Euclidean norm. The function spaces, $L^q(\Omega)$ spaces, $1\le q<\infty$, $\Omega\subseteq\mathbb{R}^N$ are always meant with respect to the Lebesgue measure and are endowed with the usual norm

$$\|\psi\|_{L^q(\Omega)}=\left(\int_\Omega|\psi(y)|^q\,dy\right)^{\frac{1}{q}}$$

For $0<\alpha<1$ we denote by $C^{k+\alpha}_{loc}(\Omega)$ the space of all functions u whose k^{th} derivatives are locally α-Hölder continuous. Furthermore, we denote by $C^{2+\alpha,1+\alpha/2}_{loc}(\Omega\times J)$, where $J\subset[0,\infty)$ is an interval, the space of all functions u such that $u,\partial_t u, D_i u$ and $D_{ij}u$ are locally α-Hölder continuous. $B(x,R)$ denotes the open ball of $\mathbb{R}^N$ of radius R and centre x. If $u:\mathbb{R}^N\times J\to\mathbb{R}$, where $J\subset[0,\infty)$ is an interval, we use the notations

$$\partial_t u=\frac{\partial u}{\partial t},\qquad D_i u=\frac{\partial u}{\partial x_i},\qquad D_{ij}u=D_iD_ju,\qquad Du=(D_1u,\dots,D_Nu)$$

and

$$|Du|^2=\sum_{i=1}^N|D_iu|^2.$$

We write $a(\xi,\nu)$ for $\sum_{i,j=1}^N a_{ij}(\cdot)\xi_i\nu_j$ and $\xi,\nu\in\mathbb{R}^N$. It then holds

$$|a(\xi,\nu)|^2\le a(\xi,\xi)\,a(\nu,\nu)\qquad\text{for all }\xi,\nu\in\mathbb{R}^N. \tag{1.7}$$

We further set

$$|a|^2=\sum_{i,j=1}^N a_{ij}^2,\qquad |F|^2=\sum_{i=1}^N F_i^2.$$

Observe that

$$|a(\xi,\nu)|\le|a|\,|\xi|\,|\nu|\qquad\text{for all }\xi,\nu\in\mathbb{R}^N. \tag{1.8}$$

We further define a cut-off function η_n. Let $\eta\in C^2_c(\mathbb{R}^N)$ be such that $\eta(y)=1$ if $|y|\le 1$, $\eta(y)=0$ if $|y|\ge 3$, $0\le\eta\le 1$ and $|D\eta|\le 1$. For each $n\in\mathbb{N}$ we set $\eta_n(y):=\eta\left(\frac{y}{n}\right)$. Then $\eta_n|_{B(0,n)}=1$, $\eta_n|_{\mathbb{R}^N\setminus B(0,3n)}=0$ and $0\le\eta_n\le 1$.It follows that

$$|D\eta_n(y)|\le\frac{1}{n},\qquad\text{for all }y\in\mathbb{R}^N\text{ and }n\in\mathbb{N}. \tag{1.9}$$

If B is a differential operator, then we write $B(Dx)$ (or $B(Dy)$) instead of B to emphasize that we derive with respect to x (or y).

2. Preliminaries

2.1. Construction of p. We briefly recall the construction of a fundamental solution p. For more details we refer to [1, Chapter 2] and [7, Section 4] for the case $H = 0$. The idea is to consider the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_n(x,t) = A u_n(x,t), & x \in B(0,n), \, t > 0, \\ u_n(x,t) = 0, & x \in \partial B(0,n), \, t > 0, \\ u_n(x,0) = f(x), & x \in B(0,n), \end{cases} \tag{2.1}$$

in the ball $B(0,n)$ for a given $f \in C\left(\overline{B(0,n)}\right)$ and $n \in \mathbb{N}$. By classical results for parabolic Cauchy problems in bounded domains (e.g. [3, Chapter III, §4]) we know that the problem (2.1) admits a unique solution

$$u_n \in C(\overline{B(0,n)} \times [0,\infty)) \cap C^{2,1}(B(0,n) \times (0,\infty)).$$

Moreover, Condition 1.1 implies existence and uniqueness of a Green function

$$0 < p_n = p_n(x,y,t) \in C(B(0,n) \times B(0,n) \times (0,\infty))$$

such that for each fixed $x \in B(0,n)$ it holds

$$p_n(x,\cdot,\cdot) \in C^{2+\alpha,1+\alpha/2}_{loc}(B(0,n) \times (0,\infty))$$

and for each fixed $y \in B(0,n)$ it holds

$$p_n(\cdot,y,\cdot) \in C^{2+\alpha,1+\alpha/2}_{loc}(B(0,n) \times (0,\infty)).$$

Furthermore, for each fixed $y \in B(0,n)$ the function $p_n(\cdot,y,\cdot)$ satisfies

$$\partial_t p_n(x,y,t) = A(Dx) p_n(x,y,t)$$

with respect to $(x,t) \in B(0,n) \times (0,\infty)$ and for each fixed $x \in B(0,n)$ it holds

$$\partial_t p_n(x,y,t) = A^*(Dy) p_n(x,y,t)$$

with respect to $(y,t) \in B(0,n) \times (0,\infty)$, where

$$A^* = A_0 - F \cdot D - (\operatorname{div} F + H) \tag{2.2}$$

is the formal adjoint operator of A, such that

$$p_n^*(y,x,t) = p_n(x,y,t) \tag{2.3}$$

is the unique Green function for the problem

$$\begin{cases} \partial_t v_n(y,t) = A^* v_n(y,t), & y \in B(0,n), \, t > 0, \\ v_n(y,t) = 0, & y \in \partial B(0,n), \, t > 0, \\ v_n(y,0) = f(y), & y \in B(0,n), \end{cases} \tag{2.4}$$

The proof of these statements one can find in [3, Section III, §7]. For the solution u_n of Problem (2.1) we hence have

$$u_n(x,t) = \int_{B(0,n)} p_n(x,y,t) f(y) dy$$

and

$$\int_{B(0,n)} p_n(x,y,t) f(y) dy \to f(x) \qquad \text{as } t \to 0 \text{ for each } x \in B(0,n)$$

and for the solution v_n of Problem (2.4) we have

$$v_n(y,t) = \int_{B(0,n)} p_n(x,y,t) f(x) dx$$

and

$$\int_{B(0,n)} p_n(x,y,t) f(x) dx \to f(y) \qquad \text{as } t \to 0 \text{ for each } y \in B(0,n).$$

Using the classical maximum principle, one obtains that the sequence (p_n) is increasing with respect to $n \in \mathbb{N}$. So we extend each function p_n to $\mathbb{R}^N \times \mathbb{R}^N \times (0, +\infty)$ with value zero for $x, y \in \mathbb{R}^N \backslash B(0, n)$ and still denote by p_n the so obtained function. It then holds

$$p_n(x, y, t) \leq p_{n+1}(x, y, t) \tag{2.5}$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$ and $n \in \mathbb{N}$. One sets

$$p\,(x, y, t) = \lim_{n \to \infty} p_n\,(x, y, t)\,, \qquad \text{pointwise for } (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)\,. \tag{2.6}$$

2.2. **Properties of p.** We formulate the main properties of p in the following proposition. The proof one can find in [1, Chapter 2] and in [7] for the case $H = 0$ (see also [2]).

Proposition 2.1. *Under assumptions of Condition 1.1 the following statements hold.*

(i) $\int_{\mathbb{R}^N} p(x, y, t)dy \leq e^{-H_0 t}$ *for all* $(x, t) \in \mathbb{R}^N \times (0, \infty)$.

(ii) $0 < p(x, y, t + s) = \int_{\mathbb{R}^N} p(x, z, t)p(z, y, s)dz$ *for all* $x, y \in \mathbb{R}^N$ *and* $s, t > 0$.

(iii) *For each fixed* $y \in \mathbb{R}^N$ *it holds* $\partial_t p(x, y, t) = A(Dx)p(x, y, t)$ *for all* $(x, t) \in \mathbb{R}^N \times (0, \infty)$.

(iv) *For each fixed* $x \in \mathbb{R}^N$ *it holds* $\partial_t p(x, y, t) = A^*(Dy)p(x, y, t)$ *for all* $(y, t) \in \mathbb{R}^N \times (0, \infty)$.

(v) $u(x, t) = \int_{\mathbb{R}^N} p(x, y, t)f(y)dy$ *solves for each* $f \in C_b(\mathbb{R}^N)$ *problem (1.2),* $u \in C(\mathbb{R}^N \times [0, \infty)) \cap C^{2+\alpha, 1+\alpha/2}_{loc}(\mathbb{R}^N \times (0, \infty))$ *and it holds*

$$|u(x, t)| \leq e^{-H_0 t}\, \|f\|_\infty \qquad \text{for all } (x, t) \in \mathbb{R}^N \times [0, \infty).$$

(vi) $v(y, t) = \int_{\mathbb{R}^N} p(x, y, t)f(x)dx$ *solves for each* $f \in C_b(\mathbb{R}^N)$ *problem*

$$\begin{cases} \partial_t v(y, t) = A^* v(y, t), & y \in \mathbb{R}^N,\ t > 0, \\ v(y, 0) = f(y), & y \in \mathbb{R}^N, \end{cases} \tag{2.7}$$

$v \in C(\mathbb{R}^N \times [0, \infty)) \cap C^{2+\alpha, 1+\alpha/2}_{loc}(\mathbb{R}^N \times (0, \infty))$ *and it holds*

$$|v(y, t)| \leq e^{-H_0^* t}\, \|f\|_\infty \qquad \text{for all } (y, t) \in \mathbb{R}^N \times [0, \infty).$$

(vii) *For any bounded Borel function* $f \geq 0$ *with* $f \not\equiv 0$ *it holds*

$$\int_{\mathbb{R}^N} p(x, y, t)f(y)dy > 0 \qquad \text{for all } (x, t) \in \mathbb{R}^N \times (0, \infty)$$

and

$$\int_{\mathbb{R}^N} p(x, y, t)f(x)dx > 0 \qquad \text{for all } (y, t) \in \mathbb{R}^N \times (0, \infty)$$

*(*positivity*).*

The global boundedness of p was studied for example in [6], [4] for the case of bounded diffusion coefficients a_{ij}, $i, j = 1, ..., N$, and in [2] for the general case. It was assumed the existence of some Lyapunov function $1 \leq V \in C^2(\mathbb{R}^N)$, that is

$$\lim_{|x| \to \infty} V(x) = \infty \qquad \text{and} \qquad AV(x) \leq kV(x) \qquad \text{for all } x \in \mathbb{R}^N$$

and some constant $k > -H_0$. Moreover, the coefficients of A must growth not faster as $V^{\frac{1}{N+1}}$. We remark that the existence of a Lyapunov function yields the uniqueness of the bounded solution of Problem (1.2).

The current case allows the nonuniqueness of the bounded solution and arbitrary grow of the coefficients of A.

The similar result one can find in [5] under assumption of bounded diffusion coefficients. Therefore the technics from [5] are unsuitable in the current case.

3. Global boundedness of the fundamental solution

From classical theory we know that if the operator A has bounded coefficients, then it holds

$$p(x,y,t) \leq Ct^{-\frac{N}{2}} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$$

for some constant $C > 0$, depending on the supremum norm of coefficients of the operator A (see e. g. [3, Chapter I, (6.12)]).

We will approximate the operator A by operators

$$A^{(m)} = A_0^{(m)} + F^{(m)} \cdot D - H^{(m)}, \qquad m \in \mathbb{N}.$$

Therefor, for $m \in \mathbb{N}$ we set

$$a_{ij}^{(m)} = \eta_m a_{ij} + \lambda \left(1 - \eta_m\right) \delta_{ij},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$, a constant $\lambda > 0$ is given as in (1.3) and the cut-off function η_m is given as in Section 1.2. Furthermore, we set

$$A_0^{(m)} = \sum_{i,j=1}^{N} D_i(a_{ij}^{(m)} D_j), \qquad F_i^{(m)} = \eta_m F_i$$

and

$$H^{(m)} = \eta_m H - F \cdot D\eta_m + |F| \, |D\eta_m| \, .$$

We then obtain that the coefficients of $A^{(m)}$ are bounded and it holds

$$a^{(m)}(\cdot)(\xi,\xi) := \sum_{i,j=1}^{N} a_{ij}^{(m)}(\cdot)\xi_i\xi_j \geq \lambda \left|\xi\right|^2 . \tag{3.1}$$

Thus $A^{(m)}$ is elliptic. Moreover, we have

$$H^{(m)}(x) \geq \eta_m(x)H(x) \geq H_0 \tag{3.2}$$

and

$$\operatorname{div} F^{(m)}(x) + H^{(m)}(x) \geq \eta_m(x)(\operatorname{div} F(x) + H(x)) \geq H_0^*. \tag{3.3}$$

Let $p^{(m)} = p^{(m)}(x,y,t)$ be the fundamental solution for $A^{(m)}$. It then holds

$$\partial_t p^{(m)} = A_0^{(m)}(Dy)p^{(m)} - F^{(m)} \cdot Dp^{(m)} - (\operatorname{div} F^{(m)} + H^{(m)})p^{(m)} \tag{3.4}$$

with respect to $(y,t) \in \mathbb{R}^N \times (0,\infty)$ for each fixed $x \in \mathbb{R}^N$. In the next lemma we present some estimate of $L^2(\mathbb{R}^N)$ norm of $p^{(m)}$, $m \in \mathbb{N}$. The calculation method was presented by John Nash in [8] for the case $F = 0$, $H = 0$ and $a_{ij} \in C_b^1(\mathbb{R}^N)$, $i,j = 1,...,N$. In the proof a special case of Gagliardo–Nirenberg–Sobolev inequality (see [9]) will be used

$$S\left(\int_{\mathbb{R}^N} |u(x)|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}} \leq \int_{\mathbb{R}^N} \left|Du(x)\right|^2 dx, \tag{3.5}$$

where the constant S is given by

$$S = \frac{4^{\frac{N-1}{N}} \pi^{\frac{N+1}{N}} N(N-2)}{\Gamma\left(\frac{N+1}{2}\right)^{\frac{2}{N}}}. \tag{3.6}$$

Lemma 3.1. *For each $m \in \mathbb{N}$ it holds*

$$\int_{\mathbb{R}^N} p^{(m)}\left(x,y,t\right)^2 dy \leq Ce^{\gamma_1 t}t^{-\frac{N}{2}} \tag{3.7}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where

$$C = \frac{2^{\frac{N-2}{2}}\Gamma(\frac{N+1}{2})}{\pi^{\frac{N+1}{2}}\left(\lambda(N-2)\right)^{\frac{N}{2}}}$$

and

$$\gamma_1 = -H_0^* - 2H_0 \geq 0.$$

Proof. We fix arbitrary $x \in \mathbb{R}^N$ and $m \in \mathbb{N}$. For each $n \in \mathbb{N}$ we set

$$\zeta_n(x,t) = \int_{\mathbb{R}^N} \eta_n(y)^2 p^{(m)}(x,y,t)^2 dy, \qquad t \in (0,\infty). \tag{3.8}$$

Since (3.4) for any $t \in (0,\infty)$, it holds

$$\begin{aligned}\partial_t \zeta_n &= \int_{\mathbb{R}^N} 2\eta_n^2 p^{(m)} \partial_t p^{(m)} dy \\ &= \int_{\mathbb{R}^N} 2\eta_n^2 p^{(m)} A_0^{(m)}(Dy) p^{(m)} dy - \int_{\mathbb{R}^N} 2\eta_n^2 p^{(m)} F^{(m)} \cdot Dp^{(m)} dy \\ &- \int_{\mathbb{R}^N} 2\eta_n^2 (p^{(m)})^2 \left(\operatorname{div} F^{(m)} + H^{(m)}\right) dy.\end{aligned}$$

Integration by parts yields

$$\begin{aligned}-\partial_t \zeta_n &= \int_{\mathbb{R}^N} 2\eta_n^2 a^{(m)}(Dp^{(m)}, Dp^{(m)}) dy \\ &+ \int_{\mathbb{R}^N} 4\eta_n p^{(m)} a^{(m)}(D\eta_n, Dp^{(m)}) dy \\ &- \int_{\mathbb{R}^N} 2\eta_n (p^{(m)})^2 \eta_m F \cdot D\eta_n dy \\ &+ \int_{\mathbb{R}^N} \eta_n^2 (p^{(m)})^2 \eta_m (\operatorname{div} F + H) dy + \int_{\mathbb{R}^N} \eta_n^2 (p^{(m)})^2 \eta_m H dy \\ &+ \int_{\mathbb{R}^N} \eta_n^2 (p^{(m)})^2 \left(2\,|F|\,|D\eta_m| - F \cdot D\eta_m\right) dy\end{aligned} \tag{3.9}$$

Moreover, it holds

$$\begin{aligned}\int_{\mathbb{R}^N} 4\eta_n p^{(m)} a^{(m)}(D\eta_n, Dp^{(m)}) dy &= \int_{\mathbb{R}^N} 2a^{(m)}(D(\eta_n p^{(m)}), D(\eta_n p^{(m)})) dy \\ &- \int_{\mathbb{R}^N} 2(p^{(m)})^2 a^{(m)}(D\eta_n, D\eta_n) dy \\ &- \int_{\mathbb{R}^N} 2\eta_n^2 a^{(m)}(Dp^{(m)}, Dp^{(m)}) dy.\end{aligned}$$

Applying this identity to (3.9), we obtain

$$\begin{aligned}-\partial_t \zeta_n &= \int_{\mathbb{R}^N} 2a^{(m)}(D(\eta_n p^{(m)}), D(\eta_n p^{(m)})) dy \\ &- \int_{\mathbb{R}^N} 2(p^{(m)})^2 a^{(m)}(D\eta_n, D\eta_n) dy \\ &- \int_{\mathbb{R}^N} 2\eta_n (p^{(m)})^2 \eta_m F \cdot D\eta_n dy \\ &+ \int_{\mathbb{R}^N} \eta_n^2 (p^{(m)})^2 \eta_m (\operatorname{div} F + H) dy + \int_{\mathbb{R}^N} \eta_n^2 (p^{(m)})^2 \eta_m H dy \\ &+ \int_{\mathbb{R}^N} \eta_n^2 (p^{(m)})^2 \left(2\,|F|\,|D\eta_m| - F \cdot D\eta_m\right) dy.\end{aligned} \tag{3.10}$$

We fix an arbitrary $t \in (0,\infty)$. We estimate the terms of (3.10). Using (3.1), Proposition 2.1 and (1.9), we obtain

$$\int_{\mathbb{R}^N} 2a^{(m)}(D(\eta_n p^{(m)}), D(\eta_n p^{(m)})) dy \geq \int_{\mathbb{R}^N} 2\lambda \left| D(\eta_n p^{(m)}) \right|^2 dy,$$

$$-\int_{\mathbb{R}^N} 2(p^{(m)})^2 a^{(m)}(D\eta_n, D\eta_n)dy \geq -\int_{\mathbb{R}^N} \frac{2}{n^2}(p^{(m)})^2 \left|a^{(m)}\right| dy$$
$$\geq -\frac{2}{n}\left\|p^{(m)}(x,\cdot,t)\right\|_\infty \left\|\left|a^{(m)}\right|\right\|_\infty e^{-H_0 t},$$

$$-\int_{\mathbb{R}^N} 2\eta_n(p^{(m)})^2\eta_m F\cdot D\eta_n dy \geq -\int_{\mathbb{R}^N} \frac{2}{n}(p^{(m)})^2\eta_m |F|\, dy$$
$$\geq -\frac{2}{n}\left\|p^{(m)}(x,\cdot,t)\right\|_\infty \left\|\left|F^{(m)}\right|\right\|_\infty e^{-H_0 t},$$

$$\int_{\mathbb{R}^N} \eta_n^2(p^{(m)})^2\eta_m \left(\operatorname{div} F + H\right) dy + \int_{\mathbb{R}^N} \eta_n^2(p^{(m)})^2\eta_m H dy$$
$$\geq \left(H_0^* + H_0\right)\zeta_n$$

and
$$\int_{\mathbb{R}^N} \eta_n^2(p^{(m)})^2 \left(2\,|F|\,|D\eta_m| - F\cdot D\eta_m\right) dy \geq 0.$$

We set
$$\theta = H_0^* + H_0 \leq 0$$

Hence, from (3.10) it follows
$$-\partial_t\zeta_n \geq \int_{\mathbb{R}^N} 2\lambda \left|D(\eta_n p^{(m)})\right|^2 dy + \theta\zeta_n - \omega_n, \tag{3.11}$$

where
$$\omega_n = \omega_n(x,t) = \frac{2}{n}e^{-H_0 t}\left\|p^{(m)}(x,\cdot,t)\right\|_\infty \left(\left\|\left|a^{(m)}\right|\right\|_\infty + \left\|\left|F^{(m)}\right|\right\|_\infty\right).$$

Moreover, $0 \leq \omega_n(x,t) \to 0$ as $n\to\infty$ for any $(x,t)\in\mathbb{R}^N\times(0,\infty)$. Furthermore, the Gagliardo–Nirenberg–Sobolev inequality (3.5) implies
$$\int_{\mathbb{R}^N} \left|D\left(\eta_n p^{(m)}\right)\right|^2 dy \geq S\left(\int_{\mathbb{R}^N}\left(\eta_n p^{(m)}\right)^{\frac{2N}{N-2}} dy\right)^{\frac{N-2}{N}} \tag{3.12}$$

for the Sobolev constant $S = S(N)$ given in (3.6). Since
$$0 < \int_{\mathbb{R}^N} \eta_1 p^{(m)} dy \leq \int_{\mathbb{R}^N} \eta_n p^{(m)} dy \leq \int_{\mathbb{R}^N} p^{(m)} dy \leq e^{-H_0 t},$$

it holds
$$0 < e^{H_0 t} \leq \frac{1}{\int_{\mathbb{R}^N}\eta_n p^{(m)} dy} < \infty.$$

For $r > 1$, this fact leads to
$$\left(\int_{\mathbb{R}^N}\left(\eta_n p^{(m)}\right)^{\frac{2N}{N-2}} dy\right)^{\frac{1}{r}}$$
$$= \left(\int_{\mathbb{R}^N}\left(\left(\eta_n(p^{(m)})\right)^{\frac{2N}{(N-2)r}}\right)^r dy\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N}\left(\left(\eta_n p^{(m)}\right)^{\frac{r-1}{r}}\right)^{\frac{r}{r-1}} dy\right)^{\frac{r-1}{r}}$$
$$\cdot\left(\frac{1}{\int_{\mathbb{R}^N}\eta_n p^{(m)} dy}\right)^{\frac{r-1}{r}}$$
$$\geq \left\|\left(\eta_n p^{(m)}\right)^{\frac{2N}{(N-2)r}}\right\|_r \left\|\left(\eta_n p^{(m)}\right)^{\frac{r-1}{r}}\right\|_{\frac{r}{r-1}} e^{H_0\frac{r-1}{r}t}.$$

Hölder's inequality then yields

$$\left(\int_{\mathbb{R}^N}\left(\eta_n p^{(m)}\right)^{\frac{2N}{N-2}}dy\right)^{\frac{1}{r}} \geq \left\|\left(\eta_n p^{(m)}\right)^{\frac{2N}{(N-2)r}+\frac{r-1}{r}}\right\|_1 e^{H_0\frac{r-1}{r}t}. \tag{3.13}$$

Choosing $r=\frac{N+2}{N-2}$ in (3.13), we infer

$$\left(\int_{\mathbb{R}^N}\left(\eta_n p^{(m)}\right)^{\frac{2N}{N-2}}dy\right)^{\frac{N-2}{N+2}} \geq \left\|\eta_n^2 (p^{(m)})^2\right\|_1 e^{\frac{4H_0}{N+2}t} = \zeta_n e^{\frac{4H_0}{N+2}t}$$

and hence

$$\left(\int_{\mathbb{R}^N}\left(\eta_n p^{(m)}\right)^{\frac{2N}{N-2}}dy\right)^{\frac{N-2}{N}} \geq \zeta_n^{1+\frac{2}{N}} e^{\frac{4H_0}{N}t}.$$

We combine the above inequality with (3.12) and arrive at

$$\int_{\mathbb{R}^N}\left|D\left(\eta_n p^{(m)}\right)\right|^2 dy \geq S\zeta_n^{1+\frac{2}{N}} e^{\frac{4H_0}{N}t}. \tag{3.14}$$

It then follows from (3.11)

$$-\partial_t \zeta_n \geq 2\lambda S\zeta_n^{1+\frac{2}{N}} e^{\frac{4H_0}{N}t} + \theta\zeta_n - \omega_n$$

and hence

$$-\partial_t\left(e^{\theta t}\zeta_n\right) \geq 2\lambda S\zeta_n^{1+\frac{2}{N}} e^{\theta t} e^{\frac{4H_0}{N}t} - e^{\theta t}\omega_n.$$

We remark that for $n\in\mathbb{N}$ it holds

$$0<\delta=\delta(x,t) := \int_{\mathbb{R}^N} p^{(m)}(x,y,t)\cdot\eta_1(y)^2 p^{(m)}(x,y,t)dy \leq \zeta_n(x,t) < \infty \tag{3.15}$$

Taking into account (3.15), we conclude

$$\partial_t\left((e^{\theta t}\zeta_n)^{-\frac{2}{N}}\right) \geq \frac{4\lambda S}{N} e^{-\frac{2\theta}{N}t} e^{\frac{4H_0}{N}t} - \frac{2}{N}\delta^{-1-\frac{2}{N}} e^{-\frac{2\theta}{N}t}\omega_n. \tag{3.16}$$

Let further $t_0>0$ be such that $2t_0<t$. We define $\tau\in C^\infty(\mathbb{R})$ by $0\leq\tau\leq 1$, $\tau(s)=0$ for $0\leq s\leq t_0$, $\tau(s)=1$ for $s\geq 2t_0$ and $\tau'\geq 0$. We multiply (3.16) by τ and get

$$\begin{aligned}\partial_t\left(\tau(t)(e^{\theta t}\zeta_n(x,t))^{-\frac{2}{N}}\right) \geq{}& \frac{4\lambda S}{N}\tau(t) e^{-\frac{2\theta}{N}t} e^{\frac{4H_0}{N}t} - \frac{2}{N}\tau(t)\delta^{-1-\frac{2}{N}}(x,t) e^{-\frac{2\theta}{N}t}\omega_n(x,t)\\ &+\tau'(t)(e^{\theta t}\zeta_n(x,t))^{-\frac{2}{N}},\end{aligned} \tag{3.17}$$

where the last term on the right side is nonnegative. We set

$$\nu_n(x,t)=\delta(x,t)^{-1-\frac{2}{N}} e^{-\frac{4H_0}{N}t}\omega_n(x,t).$$

From (3.17) we conclude

$$\partial_t\left(\tau(e^{\theta t}\zeta_n)^{-\frac{2}{N}}\right) \geq \frac{2}{N}\tau e^{-\frac{2\theta}{N}t} e^{\min\{0,H_0\}\frac{4}{N}t}\left(2\lambda S-\nu_n\right).$$

Since $v_n(x,t)\to 0$ as $n\to\infty$ for any $(x,t)\in\mathbb{R}^N\times(0,\infty)$, we can chose $n_0\in\mathbb{N}$ such that $2\lambda S-\nu_n\geq\lambda S$ for each $n\geq n_0$. For such n we obtain

$$\partial_t\left(\tau(e^{\theta t}\zeta_n)^{-\frac{2}{N}}\right) \geq \frac{2\lambda S}{N}\tau e^{-\frac{2\theta}{N}t} e^{\frac{4H_0}{N}t}.$$

Integration from t_0 to t yields

$$\begin{aligned}(e^{\theta t}\zeta_n(x,t))^{-\frac{2}{N}} &\geq \frac{2\lambda S}{N}\int_{t_0}^t \tau(s) e^{-\frac{2\theta}{N}s} e^{\frac{4H_0}{N}s}ds \geq \frac{2\lambda S}{N}\int_{2t_0}^t e^{-\frac{2\theta}{N}s} e^{\frac{4H_0}{N}s}ds\\ &\geq \frac{2\lambda S}{N} e^{\frac{2H_0}{N}t}(t-2t_0).\end{aligned} \tag{3.18}$$

For $n \geq n_0$ from (3.18) we deduce

$$\zeta_n(x,t)^{-1} \geq \left(\frac{2\lambda S}{N}\right)^{\frac{N}{2}} e^{(H_0^*+2H_0)t}\,(t-2t_0)^{\frac{N}{2}}\,. \tag{3.19}$$

Since $\zeta_n(x,t) > 0$ for any $(x,t) \in \mathbb{R}^N \times (0,\infty)$, we obtain from (3.19)

$$\zeta_n \leq \left(\frac{N}{2\lambda S}\right)^{\frac{N}{2}} e^{\gamma_1 t}(t-2t_0)^{-\frac{N}{2}},$$

where $\gamma_1 = -H_0^* - 2H_0 \geq 0$.

Letting $n \to \infty$, Fatou's lemma implies

$$\int_{\mathbb{R}^N} p^{(m)}(x,y,t)^2\,dy \leq \left(\frac{N}{2\lambda S}\right)^{\frac{N}{2}} e^{\gamma_1 t}\,(t-2t_0)^{-\frac{N}{2}}$$

Since $t_0 > 0$ can be arbitrary close to 0 and $(x,t) \in \mathbb{R}^N \times (0,\infty)$ are arbitrary, we deduce

$$\int_{\mathbb{R}^N} p^{(m)}(x,y,t)^2\,dy \leq \left(\frac{N}{2\lambda S}\right)^{\frac{N}{2}} e^{\gamma_1 t} t^{-\frac{N}{2}}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Using (3.6), we then observe

$$\int_{\mathbb{R}^N} p^{(m)}(x,y,t)^2\,dy \leq Ce^{\gamma_1 t} t^{-\frac{N}{2}}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and each $m \in \mathbb{N}$. □

The next step is to show that estimate (3.7) is true for p instead of $p^{(m)}$. Therefore, we recall the construction of $p^{(m)}$. For fixed $m \in \mathbb{N}$ we consider the parabolic Cauchy problem

$$\begin{cases} \partial_t u_n(x,t) = A^{(m)} u_n(x,t), & x \in B(0,n),\, t > 0, \\ u_n(x,t) = 0, & x \in \partial B(0,n),\, t > 0, \\ u_n(x,0) = f(x), & x \in B(0,n), \end{cases} \tag{3.20}$$

for $f \in C(\overline{B(0,n)})$ and $n \in \mathbb{N}$. We denote by $p_n^{(m)}$ the Green function for the problem (3.20). We remark that from (2.5) and (2.6) it follows that

$$p_n^{(m)}(x,y,t) \leq p^{(m)}(x,y,t), \qquad (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty),$$

for each $n \in \mathbb{N}$. Note that we consider extended $p_n^{(m)}$ on $\mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$ with $p_n^{(m)}(x,y,t) = 0$ for $x, y \in \mathbb{R}^N \setminus B(0,n)$ as in Section 2.1. Since $A^{(m)} = A$ on $B(0,m)$, we deduce that $p_m^{(m)} = p_m$, where p_m is the Green function for the problem

$$\begin{cases} \partial_t u(x,t) = Au(x,t), & x \in B(0,m),\, t > 0, \\ u(x,t) = 0, & x \in \partial B(0,m),\, t > 0, \\ u(x,0) = f(x), & x \in B(0,m). \end{cases}$$

So we obtain

$$p_m(x,y,t) \leq p^{(m)}(x,y,t), \qquad (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty),$$

for each $m \in \mathbb{N}$. Thus Lemma 3.1 yields

$$\int_{\mathbb{R}^N} p_m(x,y,t)^2\,dy \leq Ce^{\gamma_1 t} t^{-\frac{N}{2}}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$ and $m \in \mathbb{N}$, where constants C and γ_1 are given as in Lemma 3.1. Using (2.6) and Fatou's lemma we conclude that

$$\int_{\mathbb{R}^N} p(x,y,t)^2\,dy \leq Ce^{\gamma_1 t} t^{-\frac{N}{2}}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Applying this estimate to the adjoint problem (2.7), we obtain

$$\int_{\mathbb{R}^N} p\left(x,y,t\right)^2 dx \le Ce^{\gamma_2 t} t^{-\frac{N}{2}}$$

for all $(y,t) \in \mathbb{R}^N \times (0,\infty)$, where

$$\gamma_2 = -2H_0^* - H_0 \ge 0. \tag{3.21}$$

We formulate this result in the following corollary.

Corollary 3.2. *Under assumptions of condition 1.1 it holds*

$$\int_{\mathbb{R}^N} p\left(x,y,t\right)^2 dy \le Ce^{\gamma_1 t} t^{-\frac{N}{2}} \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty)$$

and

$$\int_{\mathbb{R}^N} p\left(x,y,t\right)^2 dx \le Ce^{\gamma_2 t} t^{-\frac{N}{2}} \qquad \text{for all } (y,t) \in \mathbb{R}^N \times (0,\infty),$$

where γ_1 and C are given as in Lemma 3.1 and γ_2 is given as in (3.21).

We can now show a global boundedness of $p(\cdot,\cdot,t)$ on $\mathbb{R}^N \times \mathbb{R}^N$ for each $t \in (0,\infty)$ using Proposition 2.1 (ii) (the Chapman–Kolmogorov equation).

Theorem 3.3. *Under assumptions of condition 1.1 it holds*

$$p(x,y,t) \le C_{N,\lambda} e^{\gamma t} t^{-\frac{N}{2}} \tag{3.22}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$, where

$$C_{N,\lambda} = \frac{2^{N-1}\Gamma\left(\frac{N+1}{2}\right)}{\pi^{\frac{N+1}{2}}\left(\lambda(N-2)\right)^{\frac{N}{2}}}$$

and

$$\gamma = -\frac{3}{4}(H_0^* + H_0) \ge 0.$$

Proof. Using Hölder's inequality and Corollary 3.2, we obtain

$$\begin{aligned} p(x,y,t) &= \int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right) p\left(z,y,\frac{t}{2}\right) dz \\ &\le \left(\int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} p\left(z,y,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \\ &\le C_{N,\lambda} e^{\gamma t} t^{-\frac{N}{2}}. \end{aligned}$$

□

Example 3.4. *It is well known that if $A = \sum_{i=1}^N D_{ii}$, then*

$$p(x,y,t) = \frac{1}{2^N \pi^{\frac{N}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) t^{-\frac{N}{2}} \qquad \text{for } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

In this case we have $H_0 = H_0^* = 0$ and $\lambda = 1$. One sees easyly that p satisfies inequality (3.22).

References

[1] M. Bertoldi and L. Lorenzi, *Analytical Methods for Markov Semigroups,* Chapman & Hall /CRC (2007).

[2] E. Bleich, *Global Properties of Kernels of Transition Semigroups,* Dissertation (2010).

[3] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Inc., Englewood Cliffs, N.J. (1964).

[4] K. Laidoune, G. Metafune, D. Pallara and A. Rhandi, *Global Properties of Transition Kernels Associated with Second Order Elliptic Operators,* to appear in: J. Escher et al (Eds.): "Parabolic Problems: Herbert Amann Festschrift", Birkhäuser.

[5] G. Metafune, El Maati Ouhabaz, D. Pallara, *Long time behavior of heat kernels of operators with unbounded drift terms*, J. Math. Anal. Appl. v. 377 (2011), pp. 170-179

[6] G. Metafune, D. Pallara and A. Rhandi, *Global Properties of Transition Probabilities of Singular Diffusions,* Theory Probab. Appl., 54 (2010), No. 1, pp. 68–96.

[7] G. Metafune, D. Pallara and M. Wacker, *Feller Semigroups on* $\mathbb{R}^N$, Semigroup Forum 65 (2002), pp. 159–205.

[8] J. Nash, *Continuity of Solutions of Parabolic and Elliptic Equations,* American Journal of Mathematics, Vol. 80, No. 4 (Oct., 1958), pp. 931-954.

[9] G. Talenti, *Best Constant in Sobolev Inequality,* Ann. Mat. Pura Appl. 110 (1976), pp. 353-372.

Karlsruhe Institute of Technology, Department of Mathematics, Institute for Analysis, 76128 Karlsruhe, Germany

E-mail address: esther.bleich@partner.kit.edu