ON FREDHOLM PROPERTIES OF OPERATOR PRODUCTS

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Abstract

Suppose that A and B are bounded linear operators on a Banach space such that AB is a Fredholm operator. In general, BA is not a Fredholm operator. In this note we show that BA is a generalised Fredholm operator in the sense of Caradus.

Let X be an infinite-dimensional complex Banach space and let $\mathscr{L}(X)$ denote the Banach algebra of all bounded linear operators on X. By $\mathscr{F}(X)$ we denote the ideal of all finite-dimensional operators in $\mathscr{L}(X)$, by $\hat{\mathscr{L}}$ we denote the quotient algebra $\mathscr{L}(X)/\mathscr{F}(X)$ and by $\hat{A}$ we denote the equivalence class of $A \in \mathscr{L}(X)$ in $\hat{\mathscr{L}}$, i.e. $\hat{A} = A + \mathscr{F}(X)$. Moreover, by $N(A)$ and $A(X)$ we denote the kernel and the range of A, respectively.

As usual, $A \in \mathscr{L}(X)$ is called a *Fredholm operator* if $\dim N(A)$ and $\operatorname{codim} A(X)$ are both finite. It is well known that

$$A \text{ is Fredholm} \Longleftrightarrow \hat{A} \text{ is invertible in } \hat{\mathscr{L}}$$

(see [3, satz 81.1]). If $A \in \mathscr{L}(X)$ is Fredholm then $A(X)$ is closed; furthermore, A is *relatively regular*, i.e. there exists a *pseudo-inverse* $S \in \mathscr{L}(X)$ such that $ASA = A$ (see [3, §74]).

Remarks. The term 'relatively regular' is also known as 'von Neumann regular'. A ring in which every element is relatively regular is called a von Neumann regular ring (see [2]).

The set of all Fredholm operators on X is denoted by $\Phi(X)$. In [1, chapter 3] Caradus shows the following:

Theorem 1. *Let $A \in \mathscr{L}(X)$ be a Fredholm operator and S any pseudo-inverse of A. Then $I - SA - AS \in \Phi(X)$.*

This suggests the following definition due to Caradus [1]:

Definition 1. $A \in \mathscr{L}(X)$ is called a *generalised Fredholm operator* if A is relatively regular and $I - SA - AS \in \Phi(X)$ for some pseudo-inverse S of A.

The set of all generalised Fredholm operators on X is denoted by $\Phi_g(X)$. In [4–10] we investigated the class $\Phi_g(X)$.

Theorem 2. (a) $\mathscr{F}(X) \subseteq \Phi_g(X)$.
(b) $\Phi(X) \subseteq \Phi_g(X)$.

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(c) *If* $A \in \Phi_g(X)$ *then there is* $\delta > 0$ *such that*

$$A - \lambda I \in \Phi_g(X) \quad \text{for } |\lambda| < \delta$$

and

$$A - \lambda I \in \Phi(X) \quad \text{for } 0 < |\lambda| < \delta.$$

(d) $A \in \Phi_g(X)$ *if and only if there exists* $S \in \mathscr{L}(X)$ *such that*

$$\hat{A}\hat{S}\hat{A} = \hat{S} \text{ and } \hat{I} - \hat{S}\hat{A} - \hat{A}\hat{S} \text{ is invertible in } \hat{\mathscr{L}}.$$

(e) $\Phi_g(X) + \mathscr{F}(X) \subseteq \Phi_g(X)$.

Proof. (a) is shown in [4, proposition 1.3], (b) follows from Theorem 1, (c) is due to Caradus [1], (d) is shown in [4, theorem 2.3] and (e) follows from (d). ■

Remarks. (i) Part (e) of Theorem 2 does not hold with $\mathscr{K}(X)$ instead of $\mathscr{F}(X)$, where $\mathscr{K}(X)$ denotes the ideal of all compact operators in $\mathscr{L}(X)$ (see [4, (1.5)]).

(ii) From Theorem 2(c) it follows that $\Phi_g(X) \subseteq \overline{\Phi(X)}$ (where the bar denotes closure).

Part (d) of Theorem 2 suggests the following notations, where $\mathscr{A}$ is any complex algebra with identity $e \neq 0$:

$$\mathscr{A}^{-1} = \{a \in \mathscr{A} : \ a \text{ is invertible in } \mathscr{A}\},$$
$$\mathscr{A}^{g} = \{a \in \mathscr{A} : \text{ there is } s \in \mathscr{A} \text{ such that } asa = a \text{ and } e - sa - as \in \mathscr{A}^{-1}\}\,.$$

With these notations we have

$$\mathscr{A} \in \Phi(X) \Longleftrightarrow \hat{A} \in \hat{\mathscr{L}}^{-1}, \tag{1}$$

and, by Theorem 2(d),

$$\mathscr{A} \in \Phi_g(X) \Longleftrightarrow \hat{A} \in \hat{\mathscr{L}}^{g}. \tag{2}$$

From (1) it is easy to see that the following is true:

Theorem 3. (a) *If* $A, B \in \Phi(X)$ *then* $AB \in \Phi(X)$.

(b) *If* $A, B \in \mathscr{L}(X)$ *and* $AB, BA \in \Phi(X)$ *then* $A, B \in \Phi(X)$.

The following example shows that if AB is Fredholm, neither A nor B need to be Fredholm, and so, by Theorem 3(b), BA is not Fredholm in general.

Example 1. Take $X = l^2$ with the usual orthonormal basis $(u_k)_{k=1}^{\infty}$. Define $A, B \in \mathscr{L}(X)$ by

$$Au_{2k} = u_k, \ Au_{2k+1} = 0 \quad \text{for all } k \in \mathbb{N}$$

and

$$Bu_k = u_{2k} \quad \text{for all } k \in \mathbb{N}.$$

Then $AB = I$, hence AB is Fredholm, but neither A nor B is Fredholm. Thus $BA \notin \Phi(X)$.

Now we are in a position to state the main results of this paper:

Theorem 4. *If* $A, B \in \mathscr{L}(X)$ *and* $AB \in \Phi(X)$ *then* $BA \in \Phi_g(X)$.

Theorem 5. *Let A, $B \in \mathscr{L}(X)$.*

(a) *If $A^n \in \Phi_g(X)$ for some integer $n \geq 1$ then $A^{n+1} \in \Phi_g(X)$.*

(b) *If $(AB)^n \in \Phi_g(X)$ for some integer $n \geq 1$ then $(BA)^{n+1} \in \Phi_g(X)$.*

The following examples show that if $T \in \mathscr{L}(X)$ and $T^2 \in \Phi_g(X)$ then, in general, it does not follow that $T \in \Phi_g(X)$, and if $AB \in \Phi_g(X)$ then, in general, BA does not belong to the class $\Phi_g(X)$.

Example 2. Take $X = l^2$.

If $(u_k)_{k=1}^{\infty}$ is the usual orthonormal basis in l^2, define $T \in \mathscr{L}(X)$ by

$$Tu_k = \frac{1}{k^2} u_{2k} \quad \text{for } k \text{ odd}$$

and

$$Tu_k = 0 \quad \text{for } k \text{ even.}$$

Then T is compact but not finite-dimensional so that $T(X)$ is not closed; hence $T \notin \Phi_g(X)$. Obviously $T^2 = 0 \in \Phi_g(X)$.

Example 3. Take $X = l^2$.

Define A, $B \in \mathscr{L}(X)$ by

$$A(x_1, x_2, x_3, \dots) = (x_1, x_1, x_3, \tfrac{1}{2}x_3, x_5, \tfrac{1}{3}x_5, \dots)$$

and

$$B(x_1, x_2, x_3, \dots) = (0, x_2, 0, x_4, 0, x_6, \dots).$$

Then

$$BA(x_1, x_2, x_3, \dots) = (0, x_1, 0, \tfrac{1}{2}x_3, 0, \tfrac{1}{3}x_5, \dots),$$

and thus BA is compact, but $BA \notin \mathscr{F}(X)$. Therefore $BA \notin \Phi_g(X)$, since BA is not relatively regular. Since $AB = 0$, we have $AB \in \Phi_g(X)$. Observe that A and B are projections; thus A, $B \in \Phi_g(X)$, but the product BA is not generalised Fredholm. The reason is as follows: $BA - AB \notin \mathscr{F}(X)$ (see [5, theorem 4.5]).

For the proofs of Theorems 4 and 5 we need some preparations, where $\mathscr{A}$ is any complex algebra with identity $e \neq 0$.

Proposition 1. *If $a \in \mathscr{A}^g$ then there exists $t \in \mathscr{A}$ such that $ata = a$, $at = ta$ and $tat = t$. Furthermore, if $r \in \mathscr{A}$ and $ra = ar$ then $rt = tr$.*

PROOF. [4, proposition 3.9] and [10, proposition 2]. ■

Proposition 2. *If a, $b \in \mathscr{A}$ and $ab \in \mathscr{A}^{-1}$ then $ba \in \mathscr{A}^g$.*

PROOF. Put $c = (ab)^{-1}$, $s = bc^2a$ and $p = s(ba)$.
Then

$$(ba)s = (ba)bc^2a = b(abc)ca = bca,$$

$$s(ba) = bc^2a(ba) = bc(cab)a = bca,$$

and

$$p^2 = (s(ba))^2 = bc(abc)a = bca = s(ba) = p.$$

Hence

$$p = p^2 = s(ba) = (ba)s,$$

$$e - s(ba) - (ba)s = e - 2p \tag{3}$$

and

$$(e-2p)^2 = e - 4p + 4p^2 = e. \tag{4}$$

Furthermore,

$$(ba)s(ba) = ba(bca) = b(abc)a = ba. \tag{5}$$

From (3), (4) and (5) it follows that $ba \in \mathscr{A}^g$. ■

Proof of Theorem 4. By (1), $\widehat{AB} = \hat{A}\hat{B} \in \hat{\mathscr{L}}^{-1}$. Proposition 2 gives $BA = BA \in \mathscr{L}^g$. Thus $BA \in \Phi_g(X)$ by (2). ■

Proposition 3. *Let $a \in \mathscr{A}$ and suppose that $a^n \in \mathscr{A}^g$ for some integer $n \geq 1$. Then $a^{n+1} \in \mathscr{A}^g$.*

Proof. From Proposition 1 we know that there exists $t \in \mathscr{A}$ such that $a^n t a^n = a^n$, $a^n t = t a^n$, $t a^n t = t$, and $t a^{n-1} = a^{n-1} t$. Put $s = a^{n-1} t^2$ and $p = a^{n+1} s$. Then

$$\begin{aligned} a^{n+1} s a^{n+1} &= a^{n+1} a^{n-1} t^2 a^{n+1} = a^n (a^n t a^n) t a = a^n a^n t a = (a^n t a^n) a = a^{n+1}, \\ s a^{n+1} &= a^{n-1} t^2 a^n a = a^{n-1} t a^n t a = a^{n-1} t a = t a^n, \\ a^{n+1} s &= a^{n+1} a^{n-1} t^2 = a(t a^n t) a^{n-1} = a t a^{n-1} = a^n t. \end{aligned} \tag{6}$$

Thus $p = a^{n+1} s = s a^{n+1}$, and from

$$p^2 = (a^{n+1} s a^{n+1}) s = a^{n+1} s = p$$

we get

$$(e - s a^{n+1} - s a^{n+1})^2 = (e - 2p)^2 = e. \tag{7}$$

Now use (6) and (7) to see that $a^{n+1} \in \mathscr{A}^g$. ■

Proposition 4. *Let $a, b \in \mathscr{A}$ and suppose that $(ab)^n \in \mathscr{A}^g$ for some integer $n \geq 1$. Then $(ba)^{n+1} \in \mathscr{A}^g$.*

Proof. By Proposition 1, there exists $t \in \mathscr{A}$ such that

$$\begin{aligned} &(ab)^n t (ab)^n = (ab)^n, \quad t(ab)^n t = t, \\ &t(ab)^n = (ab)^n t \quad \text{and} \quad t(ab)^{n-1} = (ab)^{n-1} t. \end{aligned}$$

Put $s = b(ab)^{n-1} t^3 (ab)^{n-1} a$ and $p = (ba)^{n+1} s$.

Similar computations as in the proof of Proposition 3 show that

$$p^2 = p = bt(ab)^{n-1}a = b(ab)^{n-1}ta = s(ba)^{n+1},$$
$$(ba)^{n+1}s(ba)^{n+1} = (ba)^{n+1}$$

and

$$(e - s(ba)^{n+1} - (ba)^{n+1}s)^2 = (e - 2p)^2 = e.$$

Thus $(ba)^{n+1} \in \mathscr{A}^g$. ■

PROOF OF THEOREM 5. (a) Follows from (2) and Proposition 3. From (2) and Proposition 4 we see that (b) holds. ■

We close this paper with some corollaries.

Corollary 1. *Let* $A, B \in \mathscr{L}(X)$ *and* $AB \in \Phi(X)$. *Then*:
(a) $BA \in \overline{\Phi(X)}$;
(b) *there exists some* $S \in \Phi(X)$ *such that* $(BA)S(BA) = BA$;
(c) *if* $BA \notin \Phi(X)$, BA *is in the boundary of* $\Phi(X)$.

PROOF. Theorem 4 and [4, theorem 4.11]. ■

Corollary 2. *Let* $A, B \in \mathscr{L}(X)$ *and* $AB \in \Phi(X)$. *The following assertions are equivalent*:
(a) $BA \in \Phi(X)$;
(b) $\dim N(BA) < \infty$;
(c) $\operatorname{codim}(BA)(X) < \infty$;
(d) $BA + K \in \Phi_g(X)$ *for each* $K \in \mathscr{K}(X)$.

PROOF. Theorem 4, [5, theorem 3.19] and [5, theorem 3.22]. ■

Corollary 3. Let $A, B \in \mathscr{L}(X)$.
(1) *If* $A^n \in \Phi_g(X)$ *for some integer* $n \geq 1$, *then* $A^{n+k} \in \Phi_g(X)$ *for* $k = 1, 2, \ldots$.
(2) *If* $(AB)^n \in \Phi_g(X)$ *for some integer* $n \geq 1$ *then* $(BA)^{n+k} \in \Phi_g(X)$ *for* $k = 1, 2, \ldots$.

PROOF. Theorem 5. ■

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