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APPLICATION OF THE MEAN ERGODIC THEOREM TO CERTAIN SEMIGROUPS

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ABSTRACT. We study the asymptotic behaviour of solutions of the Cauchy problem $u' = (\sum_{j=1}^{n}(A_j + A_j^{-1}) - 2nI)u$, $u(0) = x$ as $t \to \infty$, for invertible isometries $A_1, \dots, A_n$.

1. Introduction

Let E be a complex Banach space, $L(E)$ the Banach algebra of all bounded linear operators on E, and let $A_1, \dots, A_n \in L(E)$ be invertible, pairwise commuting, and such that $\|A_k\| = \|A_k^{-1}\| = 1$ $(k = 1, \dots, n)$. Let $T_1, \dots, T_n \in L(E)$ be defined by $T_k = A_k + A_k^{-1} - 2I$, and let $T = T_1 + \cdots + T_n$. The aim of this paper is to clear the asymptotic behaviour of the Cauchy problem

$$u'(t) = Tu(t), \quad u(0) = u_0 \tag{1.1}$$

that is of $t \mapsto \exp(tT)u_0$ for $t \to \infty$. Such problems occur in a natural way by semidiscretization of the parabolic Cauchy problem $v_t = \Delta v$, $v(0, x) = v_0(x)$: For example, if $v_0 : \mathbb{R}^n \to \mathbb{R}$ is bounded, the longitudinal line method, see for example [**4**], with step size 1 leads to a linear Cauchy problem of type (1.1) in $l^\infty(\mathbb{Z}^n)$ with

$$A_k x = (x(j_1, j_2, \dots, j_{k-1}, j_k + 1, j_{k+1}, \dots, j_n))_{j \in \mathbb{Z}^n}.$$

The corresponding problem for the heat equation was studied in [**1**].

2. Notations and preliminaries

For $A \in L(E)$ let $N(A)$, $A(E)$, $\sigma(A)$ and $r(A)$ denote the kernel, the range, the spectrum and the spectral radius of A, respectively. Let $\mathbb{D}$ denote the complex unit circle $\{z \in \mathbb{C} : |z| < 1\}$.

PROPOSITION 2.1. *Let* $A \in L(E)$, $0 \notin \sigma(A)$ *and* $\|A\| = \|A^{-1}\| = 1$. *Then:*

(1) *A is an isometry;*

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(2) $\|A^n\| = \|A\|^n = 1$ $(n \in \mathbb{N})$, *hence A is normaloid;*
(3) $r(A) = 1$ *and* $\sigma(A) \subseteq \partial\mathbb{D}$*;*
(4) $N(A-I) \cap \overline{(A-I)(E)} = \{0\}$*;*
(5) $(A-I)(E) = (A^{-1}-I)(E)$*;*
(6) $N(A-I) = N((A-I)^2)$*;*
(7) $N(A-I) \oplus \overline{(A-I)(E)}$ *is closed;*
(8) *if* $(A-I)(E)$ *is closed then* $E = N(A-I) \oplus (A-I)(E)$.

Proof. (1) and (2) are obvious.
(3): From (2) we get $r(A) = 1$. Next, it is clear that

$$\sigma(A) \cup \sigma(A^{-1}) \subseteq \overline{\mathbb{D}}.$$

Since $\sigma(A) = \{z \in \mathbb{C} : z^{-1} \in \sigma(A^{-1})\}$, we conclude $\sigma(A) \subseteq \partial\mathbb{D}$.
(4): Let $x \in N(A-I) \cap \overline{(A-I)(E)}$, let $\varepsilon > 0$ and choose $z \in E$ such that $\|x-(A-I)z\| < \varepsilon$. According to [**2**, Satz 102.3], we have $\|x\| \leqslant \|x-(A-I)z\| < \varepsilon$, hence $x = 0$.
(5): Follows from $(A-I)x = (I-A^{-1})(Ax)$.
(6): Follows from [**2**, Satz 102.3].
(7): Choose a sequence (x_n) in $N(A-I) \oplus \overline{(A-I)(E)}$ with $x_n \to x_0$ and corresponding decompositions $x_n = y_n + z_n$. According to [**2**, Satz 102.3] we have

$$\|y_n - y_m\| \leqslant \|x_n - x_m\| \quad (n, m \in \mathbb{N}),$$

hence (y_n) is convergent to a vector $y_0 \in N(A-I)$. Thus

$$z_n = x_n - y_n \to x_0 - y_0 \in \overline{(A-I)(E)},$$

and therefore $x_0 \in N(A-I) \oplus \overline{(A-I)(E)}$.
(8): Follows from [**2**, Satz 72.4 and 102.4]. □

Proposition 2.2. *Let $A \in L(E)$ be as in Proposition 2.1, let $T = A + A^{-1} - 2I$, and let $c : [0,\infty) \to \mathbb{R}$ denote the function*

$$c(t) = \exp(-t)\left(1 + \sum_{n=0}^{\infty} \frac{t^n}{n!}\left|1 - \frac{t}{n+1}\right|\right).$$

We have

(1) $\|\exp(tT)\| \leqslant 1$ $(t \geqslant 0)$*;*
(2) $t \mapsto \sqrt{t}c(t)$ *is bounded on* $[0,\infty)$ *and*

$$\|\exp(tT)(A-I)x\| \leqslant c(t)\|x\| \quad (t \geqslant 0,\ x \in E);$$

(3) $\lim_{t\to\infty} \exp(tT)y = 0$ $(y \in (A-I)(E))$*;*
(4) *if* $y \in E$ *then*

$$\lim_{t\to\infty} \exp(tT)y = 0 \iff y \in \overline{(A-I)(E)};$$

(5) $N(A-I) = \{x \in E : \exp(tT)x = x \ (t \geqslant 0)\}$.

Proof. (1): For each $t \geqslant 0$

$$\begin{aligned}\|\exp(tT)\| &= \|\exp(-2t)\exp(tA)\exp(tA^{-1})\| \\ &\leqslant \exp(-2t)\exp(t\|A\|)\exp(t\|A^{-1}\|) = 1.\end{aligned}$$

(2): Since $T = (A^{-1} - I) + (A - I)$ we have

$$\exp(tT)(A-I) = \exp(t(A^{-1}-I))\exp(t(A-I))(A-I) \quad (t \in \mathbb{R}),$$

and

$$\begin{aligned}\exp(t(A-I))(A-I)x &= \exp(-t)\sum_{n=0}^{\infty}\frac{t^n}{n!}(A^{n+1}-A^n)x \\ &= \exp(-t)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(1-\frac{t}{n+1}\right)\right)A^{n+1}x - \exp(-t)x.\end{aligned}$$

Hence, since $\|A\| = 1$,

$$\begin{aligned}\|\exp(tT)(A-I)x\| &\leqslant \|\exp(t(A^{-1}-I))\|\ \|\exp(t(A-I))(A-I)x\| \\ &\leqslant c(t)\|x\| \quad (t \geqslant 0,\ x \in E).\end{aligned}$$

To see that $t \mapsto \sqrt{t}c(t)$ is bounded on $[0,\infty)$ let $N \in \mathbb{N}$ and $N \leqslant t \leqslant N+1$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left|1-\frac{t}{n+1}\right| &= \sum_{n=0}^{N-1}\frac{t^n}{n!}\left(\frac{t}{n+1}-1\right) + \sum_{n=N}^{\infty}\frac{t^n}{n!}\left(1-\frac{t}{n+1}\right) \\ &= 2\frac{t^N}{N!} - 1,\end{aligned}$$

and therefore

$$\begin{aligned}\sqrt{t}c(t) &= \sqrt{t}\exp(-t)\left(1+\sum_{n=0}^{\infty}\frac{t^n}{n!}\left|1-\frac{t}{n+1}\right|\right) \\ &\leqslant \sqrt{N+1}\exp(-N)\left(1+2\frac{(N+1)^N}{N!}-1\right) \\ &= \frac{2(N+1)^{N+1/2}\exp(-N)}{N!},\end{aligned}$$

which is bounded according to Stirling's formula.

(3): Follows from (2).

(4): The implication $\Leftarrow$ follows from (3). Now suppose that $\exp(tT)y \to 0$ as $t \to \infty$. Since

$$\begin{aligned}\exp(tT)y &= y + \sum_{n=1}^{\infty}\frac{t^n}{n!}T^n y \\ &= y + (A-I)\sum_{n=1}^{\infty}\frac{t^n}{n!}(A-I)^{n-1}(I-A^{-1})^n y\end{aligned}$$

we conclude $y \in \overline{(A-I)(E)}$.

(5): The inclusion

$$N(A-I) \subseteq \{x \in E : \exp(tT)x = x\ (t \geqslant 0)\}$$

is obvious. Now suppose that $x \in E$ and $\exp(tT)x = x$ $(t \geqslant 0)$. By differentiation $0 = T\exp(tT)x$ $(t \geqslant 0)$, thus $A^{-1}(A-I)^2x = Tx = 0$. Part (6) of Proposition 2.1 now shows that $x \in N(A-I)$. □

3. The asymptotic behaviour of $\exp(tT)$

THEOREM 3.1. *Let A and T be as in Proposition 2.1. For $x \in E$ the following assertions are equivalent:*

(1) $\lim_{t\to\infty} \exp(tT)x$ *exists in* E [*resp.* $\lim_{t\to\infty} \exp(tT)x = 0$]*;*
(2) $x \in N(A-I) \oplus \overline{(A-I)(E)}$ [*resp.* $x \in \overline{(A-I)(E)}$]*;*
(3) *the sequence*

$$\left(\frac{x + Ax + \cdots + A^m x}{m+1}\right)_{m\in\mathbb{N}}$$

is convergent [*resp. is convergent with limit* 0].

PROOF. That (2) implies (1) follows from Proposition 2.2.
Now, assume that (1) holds, and let $z = \lim_{t\to\infty} \exp(tT)x$. As in the proof of part (4) of Proposition 2.2

$$\exp(tT)x = x + (A-I)\sum_{n=1}^{\infty} \frac{t^n}{n!}(A-I)^{n-1}(I-A^{-1})^n x,$$

hence $x - z \in \overline{(A-I)(E)}$. [In particular, if $z = 0$ then $x \in \overline{(A-I)(E)}$.] From part (3) of Proposition 2.2 we obtain

$$(A-I)z = \lim_{t\to\infty} \exp(tT)(A-I)x = 0,$$

and therefore $x = z + (x-z) \in N(A-I) \oplus \overline{(A-I)(E)}$.
The equivalence of (2) and (3) is proved in [**3**, Ch.2, Theorem 1.3]. □

According to part (8) of Proposition 2.1 the following corollary shows, that $\lim_{t\to\infty} \exp(tT)x$ exists for each $x \in E$ if $T(E)$ is closed:

COROLLARY 3.1. *We have*

(1) $T(E) = (A-I)^2(E) \subseteq (A-I)(E) \subseteq \overline{T(E)}$*;*
(2) $T(E) = \overline{T(E)} \iff (A-I)^2(E) = (A-I)(E) \iff (A-I)(E) = \overline{(A-I)(E)}$*.*

PROOF. (1): Part (5) of Proposition 2.1 gives

$$T(E) = (A-I)^2(E) \subseteq (A-I)(E).$$

As in the proof of Theorem 3.1 we obtain $(A-I)(E) \subseteq \overline{T(E)}$. Now, (2) follows by [**2**, Satz 102.4]. □

4. The general case

Now, let $A_1, \dots, A_n, T_1, \dots, T_n$ and T be as in section 1. Moreover we introduce the following subspaces of E:

$$X_1 = \bigcap_{j=1}^{n} N(A_j - I), \quad X_2 = \overline{\sum_{j=1}^{n}(A_j - I)(E)}, \quad X = X_1 + X_2.$$

THEOREM 4.1. *Under the assumptions above*

(1) $X_2 = \{x \in E : \lim_{t\to\infty} \exp(tT)x = 0\}$;
(2) $X_1 = \{x \in E : \exp(tT)x = x \ (t \geqslant 0)\}$;
(3) $X = \{x \in E : \lim_{t\to\infty} \exp(tT)x$ *exists in* $E\}$;
(4) $X_1 \cap X_2 = \{0\}$, *and* X *is closed.*

PROOF. (1): Let $x \in X_2$. Then $x = \lim_{m\to\infty} x_m$ where

$$x_m \in \sum_{j=1}^{n}(A_j - I)(E).$$

By part (1) and part (3) of Proposition 2.2 we obtain

$$\lim_{t\to\infty} \exp(tT)x_m = 0 \quad (m \in \mathbb{N}).$$

Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\|x - x_N\| < \varepsilon/2$. Next, choose $t_0 \in [0, \infty)$ such that $\|\exp(tT)x_N\| < \varepsilon/2$ $(t \geqslant t_0)$. Then

$$\begin{aligned}\|\exp(tT)x\| &= \|\exp(tT)(x - x_N) + \exp(tT)x_N\| \\ &\leqslant \|x - x_N\| + \|\exp(tT)x_N\| < \varepsilon \quad (t \geqslant t_0).\end{aligned}$$

Thus $\lim_{t\to\infty} \exp(tT)x = 0$.

Now suppose that $x \in E$ and $\lim_{t\to\infty} \exp(tT)x = 0$. Set

$$h(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} T^n x.$$

Since $T_j = (A_j - I)(I - A_j^{-1})$ $(j = 1, \dots, n)$, we have

$$Tx = \sum_{j=1}^{n}(A_j - I)(I - A_j^{-1})x \in \sum_{j=1}^{n}(A_j - I)(E),$$

hence

$$h(t) \in \sum_{j=1}^{n}(A_j - I)(E).$$

Thus, $\exp(tT)x = x + h(t)$ and $\lim_{t\to\infty} \exp(tT)x = 0$ imply $x \in X_2$.
(2): The inclusion $\subseteq$ is obvious. For the reversed inclusion let $x \in E$ be such that $\exp(tT)x = x$ $(t \geqslant 0)$. Then by part (1) we obtain

$$(A_j - I)x = \exp(tT)(A_j - I)x \to 0 \ (t \to \infty) \quad (j = 1, \dots, n),$$

hence $x \in X_1$.
(3): Here, the inclusion $\subseteq$ follows from parts (1) and (2) directly. Now, assume that $x \in E$ is such that $\lim_{t\to\infty} \exp(tT)x = z$. As in the proof of part (1)

$$\exp(tT)x = x + h(t), \quad h(t) \in X_2.$$

Therefore $x - z \in X_2$. From part (1) we derive

$$(A_j - I)z = \lim_{t\to\infty} \exp(tT)(A_j - I)x = 0 \quad (j = 1, \dots, n).$$

Thus $z \in X_1$, and so $x = z + (x - z) \in X_1 \oplus X_2 = X$.
(4): Let $x \in X_1 \cap X_2$. Then, by parts (1) and (2), we have

$$\exp(tT)x = x \ (t \geqslant 0), \quad \exp(tT)x \to 0 \ (t \to \infty),$$

thus $x = 0$. Next, if (x_m) is a sequence in X with limit x_0, then there exist sequences (y_m) and (z_m) in X_1 and X_2, respectively, with $x_m = y_m + z_m$. From part (1) and part (2) we obtain

$$\exp(tT)(x_m - x_k) \to y_m - y_k \ (t \to \infty).$$

Since $\|\exp(tT)(x_m - x_k)\| \leqslant \|x_m - x_k\|$ $(t \geqslant 0)$, we have

$$\|y_m - y_k\| \leqslant \|x_m - x_k\|,$$

thus (y_m) is convergent. Let $y_0 = \lim_{m\to\infty} y_m$. Then $z_m = x_m - y_m \to x_0 - y_0$. Hence we have $y_0 \in X_1$, $x_0 - y_0 \in X_2$, and therefore $x_0 \in X_1 \oplus X_2 = X$. □

The following result provides sufficient conditions for the convergence of $\exp(tT)x$.

THEOREM 4.2. *Let $(k_1, \dots, k_n) \in \mathbb{N}_0^n$, and set $B = A_1^{k_1} \dots A_n^{k_n}$. We have*

1. $\bigcap_{j=1}^n \left(N(A_j - I) \oplus \overline{(A_j - I)(E)} \right) \subseteq X$;
2. $\overline{(B - I)(E)} \subseteq X_2$;
3. *if $x \in E$ and if the sequences*
$$\left(\frac{x + A_j x + \cdots + A_j^m x}{m+1} \right)_{m\in\mathbb{N}}$$
are convergent $(j = 1, \dots, n)$, then $\lim_{t\to\infty} \exp(tT)x$ exists in E;
4. *if $x \in E$ and if the sequence*
$$\left(\frac{x + Bx + \cdots + B^m x}{m+1} \right)_{m\in\mathbb{N}}$$
is convergent to 0 in E, then $\lim_{t\to\infty} \exp(tT)x = 0$.

PROOF. According to Theorem 3.1 we see that (3) follows from (1), and (4) follows from (2).
For the proof of (1) we use induction. If $n = 1$ the result follows by Theorem 3.1.

Suppose that $n \in \mathbb{N}$ and that (1) holds. In the case of $n+1$ operators $T_1, \dots, T_{n+1}$ we write $T_0 = T_1 + \cdots + T_n$, so $T = T_0 + T_{n+1}$. Let

$$x \in \bigcap_{j=1}^{n+1} \Big(N(A_j - I) \oplus \overline{(A_j - I)(E)} \Big).$$

Then

$$x \in \bigcap_{j=1}^{n} \Big(N(A_j - I) \oplus \overline{(A_j - I)(E)} \Big), \quad x \in N(A_{n+1} - I) \oplus \overline{(A_{n+1} - I)(E)},$$

and therefore the limits $\lim_{t\to\infty} \exp(tT_0)x$ and $\lim_{t\to\infty} \exp(tT_{n+1})x$ exist in E. From

$$\|\exp(tT)x - \exp(sT)x\| = \|\exp(tT_0)\exp(tT_{n+1})x - \exp(sT_0)\exp(sT_{n+1})x\|$$

$$= \|\exp(tT_0)(\exp(tT_{n+1}) - \exp(sT_{n+1}))x + \exp(sT_{n+1})(\exp(tT_0) - \exp(sT_0))x\|$$

$$\leqslant \|\exp(tT_{n+1})x - \exp(sT_{n+1})x\| + \|\exp(tT_0)x - \exp(sT_0)x\|$$

we see that $\lim_{t\to\infty} \exp(tT)x$ exists.

Next, we prove (2) for $(k_1, \dots, k_n) \in \mathbb{N}^n$, without loss of generality. Let $p(z) = z_1^{k_1} \dots z_n^{k_n} - 1$ ($z = (z_1, \dots, z_n)$), and note that there are polynomials $q_1, \dots, q_n \in \mathbb{C}[z_1, \dots, z_n]$ such that

$$p(z) = (z_1 - 1)q_1(z) + \cdots + (z_n - 1)q_n(z).$$

Hence

$$(B - I)x \in \sum_{j=1}^{n} (A_j - I)(E) \quad (x \in E),$$

and therefore $\overline{(B - I)(E)} \subseteq X_2$. □

5. Example

Let us return to the semidiscretization of $v_t = \Delta v$ in $\mathbb{R}^2$, that is we consider $E = l^\infty(\mathbb{Z}^2)$ and

$$A_1 x = (x(i+1, j))_{(i,j)\in\mathbb{Z}^2}, \quad A_2 x = (x(i, j+1))_{(i,j)\in\mathbb{Z}^2}.$$

Let $k_1, k_2 \in \mathbb{N}$, and assume that $x \in l^\infty(\mathbb{Z}^2)$ is such that the sequence

$$\left(\left(\frac{x(i,j) + x(i+k_1, j+k_2) + \cdots + x(i+mk_1, j+mk_2)}{m+1} \right)_{(i,j)\in\mathbb{Z}^2} \right)_{m\in\mathbb{N}}$$

tends to 0 as $m \to \infty$ in $l^\infty(\mathbb{Z}^2)$. Then

$$\exp(tT)x \to 0 \quad (t \to \infty)$$

(apply part (4) of Theorem 4.2 with $B = A_1^{k_1} A_2^{k_2}$).

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