

Boundary value problems for linear operators in ordered Banach spaces

Gerd Herzog, Christoph Schmoeger

Institut für Analysis, Universität Karlsruhe, D-76128 Karlsruhe, Germany

Gerd.Herzog@math.uni-karlsruhe.de,

Christoph.Schmoeger@math.uni-karlsruhe.de

Abstract

We study boundary value problems of the type $Ax = r$, $\varphi(x) = \varphi(b)$ ($\varphi \in M \subseteq E^*$) in ordered Banach spaces.

MSC classification (2000): 47A50, 47B60

Keywords: Ordered Banach spaces, quasimonotone increasing operators, Fredholm operators, boundary value problems.

1 Introduction.

Let E be a real ordered Banach space, $A : E \rightarrow E$ a continuous linear operator and M a subset of E^* , the topological dual of E . We study Dirichlet type boundary value problems of the form

$$\begin{cases} Ax = r \\ \varphi(x) = \varphi(b) \quad (\varphi \in M) \end{cases}$$

and we prove that under suitable assumptions on A there exists a natural choice of M such that this problem is uniquely solvable, and such that the solution x depends monotone on r and b . Problems of this type emerge for example by discretization of linear elliptic boundary value problems [4, Chapter 4.]. In this case the underlying space E is finite dimensional and

is ordered coordinatewise, in general. To illustrate our results we consider the following example. Let $E = C([0, 1], \mathbb{R})$ be endowed with the pointwise ordering, let B_n denote the Bernstein operator

$$(B_n f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k},$$

and let $m \in \mathbb{N}$. We will see that the problem

$$\begin{cases} (B_n - id)^m f = g \\ f(0) = \alpha, \quad f(1) = \beta \end{cases}$$

is uniquely solvable for each g in the range of $B_n - id$ and each $\alpha, \beta \in \mathbb{R}$, and that the solution depends monotone on α, β and g .

2 Main Result

Let E be a real Banach space ordered by a cone K . A cone K is a nonempty closed convex subset of E such that $\lambda K \subseteq K$ ($\lambda \geq 0$), and $K \cap (-K) = \{0\}$. As usual $x \leq y : \iff y - x \in K$. For $x \leq y$ let $[x, y]$ denote the order interval of all z with $x \leq z \leq y$. We assume in the sequel that K is normal (that is $0 \leq x \leq y \implies \|x\| \leq \gamma \|y\|$ for some constant $\gamma \geq 1$), and has nonempty interior $\text{int}K$. Let K^* denote the dual cone of K , that is the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ ($x \geq 0$). Let $L(E)$ denote the Banach algebra of all continuous linear operators $A : E \rightarrow E$, and for $A \in L(E)$ let A^* denote its adjoint.

An operator $A \in L(E)$ is called quasimonotone increasing [11] if

$$x \in K, \varphi \in K^*, \varphi(x) = 0 \implies \varphi(Ax) \geq 0.$$

It is well known [8] that $A \in L(E)$ is quasimonotone increasing if and only if $\exp(tA)(K) \subseteq K$ (that is $\exp(tA)$ is a positive operator) for each $t \geq 0$. In this case also $\exp(tA^*)(K^*) \subseteq K^*$ ($t \geq 0$).

Next, if any $p \in \text{int}K$ is fixed, then we may renorm E equivalently by the Minkowsky functional of the order interval $[-p, p]$. This norm $\|\cdot\|$ satisfies

$$-cp \leq x \leq cp \iff \|x\| \leq c.$$

Under these settings we have

$$\|\varphi\| = \varphi(p) \quad (\varphi \in K^*),$$

and we set

$$C^* := \{\varphi \in K^* : \varphi(p) = 1\} = \{\varphi \in E^* : \varphi(p) = \|\varphi\| = 1\}.$$

Moreover, we define a continuous sublinear functional $S : E \rightarrow \mathbb{R}$ by

$$S(x) = \min\{\lambda \in \mathbb{R} : x \leq \lambda p\}.$$

Note [6] that S is increasing with respect to the order given by K , that

$$S(x) = \max\{\varphi(x) : \varphi \in C^*\},$$

and that

$$\|x\| = \max\{S(-x), S(x)\} \quad (x \in E).$$

With $N(\cdot)$, $R(\cdot)$, and $\text{ext}(\cdot)$ we denote the kernel and the range of a linear operator, and the set of all extremal points of a subset of a Banach space, respectively.

The aim of this paper is to prove the following results:

Let $A \in L(E)$ be quasimonotone increasing with

$$N(A) \cap \text{int}K \neq \emptyset$$

and fix $p \in N(A) \cap \text{int}K$. Let E be normed with respect to this p . Moreover we assume that $t \mapsto \exp(tA)$ is strongly Cesaro integrable, that is

$$(1) \quad \frac{1}{t} \int_0^t \exp(\tau A)x \, d\tau$$

is convergent in E as $t \rightarrow \infty$ for each $x \in E$.

Lemma 1 *Under the assumptions above we have: For each $x_0 \in E$ with $Ax_0 \geq 0$ there exists a unique $y_0 \in N(A)$ such that*

$$(2) \quad x_0 \leq y_0, \quad \varphi(x_0) = \varphi(y_0) \quad (\varphi \in \text{ext}(N(A^*) \cap C^*)),$$

and

$$(3) \quad S(x_0) = \max\{\varphi(x_0) : \varphi \in \text{ext}(N(A^*) \cap C^*)\}.$$

Theorem 1 *Let A be as in Lemma 1. Then for each $m \in \mathbb{N}$ the Dirichlet type boundary value problem*

$$(4) \quad \begin{cases} A^m x = r \\ \varphi(x) = \varphi(b) \end{cases} \quad (\varphi \in \text{ext}(N(A^*) \cap C^*))$$

is uniquely solvable in E for each $r \in R(A^m)$ and $b \in E$, and the solution depends increasingly on b , decreasingly on r if m is odd, and increasingly on r if m is even. If in addition $R(A^m)$ is closed, then there exists a constant c such that

$$(5) \quad \|x\| \leq c\|r\| + \|b\| \quad (r \in R(A^m), b \in E).$$

Remark: Part (3) of Lemma 1 and Theorem 1 for $m = 1$ can be considered in analogy to the classical maximum principle and to the solution behaviour of linear second order BVPs [10], or corresponding BVPs for difference equations [4, Chapter 4.4].

3 Preliminaries

We make use of the following Lemmata. We assume that $p \in \text{int}K$, and that E is normed by the Minkowsky functional of the order interval $[-p, p]$. With $m_+[x, y]$ we denote the right hand side directional derivative [9, Lemma II.5.6.]

$$m_+[x, y] = \lim_{h \rightarrow 0^+} \frac{\|x + hy\| - \|x\|}{h}.$$

Lemma 2 *Let $A \in L(E)$ be quasimonotone increasing. Then*

$$\|\exp(tA)x\| \leq \exp(tm_+[p, Ap])\|x\| \quad (x \in E, t \geq 0).$$

For the proof of Lemma 2 see [5].

Lemma 3 *Let $A \in L(E)$ be quasimonotone increasing with $Ap = 0$. Let $x \in E$ and $u(t) = \exp(tA)x$ ($t \geq 0$). Then*

1. $\|u(t)\| \leq \|x\|$ ($t \geq 0$),
2. $t \mapsto S(u(t))$ is decreasing on $[0, \infty)$,
3. $Ax \geq 0 \Rightarrow S(u(t)) = S(x)$ ($t \geq 0$).

Proof: 1.) follows by Lemma 2.

2.) We have $x \leq S(x)p$, thus

$$u(t) = \exp(tA)x \leq S(x)\exp(tA)p = S(x)p \quad (t \geq 0).$$

Therefore $S(u(t)) \leq S(x)$ ($t \geq 0$). Hence, if $0 \leq t_1 \leq t_2$ we obtain

$$S(u(t_2)) = S(\exp((t_2 - t_1)A) \exp(t_1 A)x) \leq S(\exp(t_1 A)x) = S(u(t_1)).$$

3.) If $Ax \geq 0$, then u is increasing on $[0, \infty)$ (since $u'(t) = \exp(tA)Ax \geq 0$ in this case). Since S is increasing on E we have that $t \mapsto S(u(t))$ is monotone increasing and (by 2.), $t \mapsto S(u(t))$ is monotone decreasing. \blacksquare

4 Proof of Lemma 1 and Theorem 1

For $x \in E$ we set

$$Qx := \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \exp(\tau A)x \, d\tau \quad (x \in E).$$

Then $Q \in L(E)$, and note that

$$(Q^* \varphi)(x) = \varphi(Qx) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(\exp(\tau A)x) \, d\tau$$

for each $x \in E$ and $\varphi \in E^*$.

By means of Lemma 2 we find $\|\exp(tA)\| = 1$ ($t \geq 0$). Therefore

$$\begin{aligned} QA x &= A Q x = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A \exp(\tau A)x \, d\tau \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} (\exp(tA)x - x) = 0 \quad (x \in E). \end{aligned}$$

Thus $Q(E) \subseteq N(A)$, and since $Q(x) = x$ ($x \in N(A)$) we have that Q is a projection onto $N(A)$. From the definition of Q we immediately get

$$Q(K) \subseteq K, \quad \|Q\| = 1.$$

Thus

$$Q^*(K^*) \subseteq K^*, \quad \|Q^*\| = 1.$$

Moreover Q^* is a projection onto $N(A^*)$. Indeed, if $\varphi = Q^* \psi$, then

$$(A^* \varphi)(x) = (A^* Q^* \psi)(x) = \psi(QAx) = 0 \quad (x \in E),$$

thus $Q^*(E^*) \subseteq N(A^*)$, and if $\varphi \in N(A^*)$, then

$$(Q^* \varphi)(x) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(\exp(\tau A)x) \, d\tau$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(x) d\tau = \varphi(x) \quad (x \in E),$$

therefore $Q^*\varphi = \varphi$.

We set $u(t) = \exp(tA)x_0$ ($t \geq 0$), $g(0) = x_0$ and

$$g(t) = \frac{1}{t} \int_0^t u(\tau) d\tau \quad (t > 0).$$

Then g is continuous and increasing on $[0, \infty)$ (since u is increasing), and therefore $y_0 := Qx_0 \geq x_0$. Moreover, by Lemma 3, we know that $S(u(t)) = S(x_0)$ ($t \geq 0$). Since S is sublinear we have

$$S(g(t)) \leq \frac{1}{t} \int_0^t S(u(\tau)) d\tau = S(x_0) \quad (t > 0).$$

Thus $S(y_0) \leq S(x_0)$. Since S is increasing, in addition $S(x_0) \leq S(y_0)$, and so $S(x_0) = S(y_0)$. Since

$$\varphi(y_0) = \varphi(Qx_0) = (Q^*\varphi)(x_0) = \varphi(x_0) \quad (\varphi \in N(A^*))$$

we have $y_0 \in N(A)$ satisfying (2). Uniqueness of y_0 will follow from unique solvability of (4) (with $m = 1$, $r = 0$ and $b = x_0$).

Next, we choose $\psi_0 \in C^*$ such that $\psi_0(x_0) = S(x_0)$, and we set $\varphi_0 := Q^*\psi_0$. Then $\varphi_0 \in N(A^*) \cap K^*$, and

$$\varphi_0(p) = (Q^*\psi_0)(p) = \psi_0(Qp) = \psi_0(p) = 1,$$

thus $\varphi_0 \in N(A^*) \cap C^*$.

We set $v(t) = \exp(tA^*)\psi_0$ ($t \geq 0$), $h(0) = \psi_0$ and

$$h(t) = \frac{1}{t} \int_0^t v(\tau) d\tau \quad (t > 0).$$

Now h is continuous, and with u also $\psi_0 \circ u$ and $(h(\cdot))(x_0)$ are increasing on $[0, \infty)$. For $t > 0$ we have

$$\begin{aligned} (h(t))(x_0) &= \frac{1}{t} \int_0^t (\exp(\tau A^*)\psi_0)(x_0) d\tau \\ &= \frac{1}{t} \int_0^t \psi_0(u(\tau)) d\tau \rightarrow \psi_0(Qx_0) = \varphi_0(x_0) \quad (t \rightarrow \infty). \end{aligned}$$

Thus $\psi_0(x_0) = (h(0))(x_0) \leq \varphi_0(x_0)$. Now

$$S(x_0) = \psi_0(x_0) \leq \varphi_0(x_0) \leq S(x_0),$$

and therefore $\varphi_0(x_0) = S(x_0)$. At this point we know that

$$S(x_0) = \max\{\varphi(x_0) : \varphi \in N(A^*) \cap C^*\}.$$

Since $N(A^*) \cap C^*$ is a convex and weak-* compact subsets of E^* , and since $\varphi \mapsto \varphi(x_0)$ is an affine function on $N(A^*) \cap C^*$, its maximum is attained at an extremal point (see [2, Prop.7.9.]). Thus we have (3).

To prove that (4) has at most one solution first note that $N(A^2) = N(A)$. Indeed if $y \in N(A^2)$ then

$$\|\exp(tA)y\| = \|y + tAy\| \leq \|y\| \quad (t \geq 0).$$

Thus $Ay = 0$. Consequently $N(A^n) = N(A)$ ($n \in \mathbb{N}$). Now, consider a solution $x \in E$ of the homogeneous problem

$$\begin{cases} A^m x = 0 \\ \varphi(x) = 0 \quad (\varphi \in \text{ext}(N(A^*) \cap C^*)). \end{cases}$$

Then $x \in N(A^m) = N(A)$, and according to (3) we have $S(x) = S(-x) = 0$. Thus $x = 0$.

To prove the existence of a solution of (4) we consider

$$B : (I - Q)(E) \rightarrow R(A^m)$$

defined by $Bx = A^m x$ ($x \in (I - Q)(E)$). Then B is bijective, and $A^m B^{-1}r = r$. Moreover

$$\varphi(Qx) = (Q^* \varphi)(x) = \varphi(x) \quad (x \in E, \varphi \in N(A^*)).$$

Now,

$$x = B^{-1}r + Qb$$

satisfies $A^m x = r$, and for each $\varphi \in N(A^*)$ we have

$$\varphi(x) = \varphi(B^{-1}r) + \varphi(Qb) = 0 + \varphi(b) = \varphi(b).$$

In particular x is the solution of (4), and since Q is a positive operator, we see that the solution of (4) depends increasingly on b .

Next, let $r_1, r_2 \in R(A^m)$ with $r_1 \leq r_2$, and let x_1, x_2 be the solutions of

$$\begin{cases} A^m x_i = r_i \\ \varphi(x_i) = \varphi(b) \quad (\varphi \in \text{ext}(N(A^*) \cap C^*)) \end{cases}$$

for $i = 1, 2$. Then $z = x_2 - x_1$ is the solution of

$$\begin{cases} A^m z = r_2 - r_1 \geq 0 \\ \varphi(z) = 0 \quad (\varphi \in \text{ext}(N(A^*) \cap C^*)) \end{cases}$$

and

$$\varphi(A^{m-1}z) = ((A^*)^{m-1}\varphi)(z) = 0 \quad (\varphi \in \text{ext}(N(A^*) \cap C^*)).$$

By means of (3) we have $S(A^{m-1}z) = 0$, so $A^{m-1}z \leq 0$. Thus, $-z$ satisfies

$$\begin{cases} A^{m-1}(-z) \geq 0 \\ \varphi(-z) = 0 \end{cases} \quad (\varphi \in \text{ext}(N(A^*) \cap C^*))$$

and repeating this step we obtain

$$A^{m-2}(-z) \leq 0, \quad A^{m-3}z \leq 0, \quad \dots, \quad (-1)^{m-1}z \leq 0,$$

which means

$$(-1)^m x_1 \leq (-1)^m x_2.$$

To prove (5) we assume in addition that $R(A^m)$ is closed. Since $(I - Q)(E)$ is a closed subspace of E , and since $B : (I - Q)(E) \rightarrow R(A^m)$ is bijective, we have a continuous inverse $B^{-1} : R(A^m) \rightarrow (I - Q)(E)$ in this case. Set $c := \|B^{-1}\|$. Now the solution $x = B^{-1}r + Qb$ of (4) satisfies

$$\|x\| \leq \|B^{-1}r\| + \|Qb\| \leq c\|r\| + \|b\|.$$

■

5 Fredholm Operators

In special cases the convergence of the integral (1) for each $x \in E$ is automatically fulfilled under the remaining assumptions of Lemma 1. This is the case for example if E is reflexive [3, VIII.7.3]. For general E we can even prove a bit more if A is a Fredholm operator.

An operator $A \in L(E)$ is called Fredholm operator if

$$\alpha(A) := \dim N(A) < \infty, \quad \beta(A) := \text{codim} R(A) < \infty.$$

In this case $R(A)$ is closed [7, Prop.36.3], A^n is a Fredholm operator ($n \in \mathbb{N}$) [7, Prop.25.3], and A^* is a Fredholm operator [7, Prop.27.3]. Moreover $\text{ind}(A) := \alpha(A) - \beta(A)$ is called the index of A .

For $A \in L(E)$ we define

$$a(A) := \min\{n \geq 0 : N(A^n) = N(A^{n+1})\},$$

$$d(A) := \min\{n \geq 0 : R(A^n) = R(A^{n+1})\},$$

with $\min \emptyset := \infty$. Now, A is called chain-finite if $a(A) < \infty$ and $d(A) < \infty$. In this case $a(A) = d(A)$ [7, Prop.38.3]. In the sequel we will use the following facts from the Riesz-Schauder Theory of compact operators [7, Prop.40.1]:

If $T \in L(E)$ is compact and $A = T - id$, then A and A^* are Fredholm operators with,

$$\alpha(A) = \beta(A) = \alpha(A^*) = \beta(A^*),$$

and

$$a(A) = d(A) = a(A^*) = d(A^*) < \infty.$$

If $T \in L(E)$, if T^k is compact for some $k \in \mathbb{N}$ (we call T power compact in this case), $A = T - id$, $A_1 = T^k - id$, and

$$A_2 = id + T + \cdots + T^{k-1},$$

then A_1 is a Fredholm operator of index 0, $a(A_1) = d(A_1) < \infty$, and

$$A_1 = AA_2 = A_2A.$$

Therefore

$$N(A) \subseteq N(A_1), \quad R(A_1) \subseteq R(A),$$

and so

$$\alpha(A) \leq \alpha(A_1) < \infty, \quad \beta(A) \leq \beta(A_1) < \infty.$$

Thus A is a Fredholm operator, and by [7, Exerc.2,Ch.38] we have

$$\text{ind}(A) = 0, \quad a(A) = d(A) < \infty.$$

The following results are stated for operators of the form $A = T - id$, but also hold for $A = T - \lambda id$ with $\lambda > 0$. We assume without loss of generality that $\lambda = 1$, since division by λ does not affect the following considerations.

Theorem 2 *Let $T \in L(E)$ be power compact, let $A = T - id$ be quasimonotone increasing, and let $N(A) \cap \text{int}K \neq \emptyset$. Then (1) is convergent as $t \rightarrow \infty$ for each $x \in E$. Moreover A is a Fredholm operator of index 0 and*

$$a(A) = d(A) = 1.$$

By Theorems 1 and 2 we get the following result on problem (4).

Theorem 3 *Let $T \in L(E)$ be power compact, let $A = T - id$ be quasimonotone increasing, let $N(A) \cap \text{int}K \neq \emptyset$, and let $m \in \mathbb{N}$. Then problem (4) is uniquely solvable in E for each $r \in R(A)$ and $b \in E$, and the solution depends increasingly on b , decreasingly on r if m is odd, and increasingly on r if m is even, and there exists a constant c such that*

$$\|x\| \leq c\|r\| + \|b\| \quad (r \in R(A), b \in E).$$

Proof: By Theorem 2 A is a Fredholm operator of index 0, hence $R(A^m)$ is closed. Since $d(A) = 1$ we have $R(A) = R(A^m)$. The assertion now follows from Theorem 1. \blacksquare

Remark: Theorem 3 applies to our introductory example. There $E = C([0, 1], \mathbb{R})$ is ordered by the cone

$$K = \{f \in E : f(t) \geq 0 \ (t \in [0, 1])\}.$$

The operator B_n is compact, and since B_n is increasing $A = B_n - id$ is quasimonotone increasing. The functions $p(t) = 1 \ (t \in [0, 1])$ is in $\text{int}K$, and $Ap = 0$. The norm induced by p is the maximum norm. Since $B_n^k \rightarrow B_1$ pointwise on E as $k \rightarrow \infty$ [1] we see that $N(A) = \{1, t\}$. Hence $\alpha(A^*) = \alpha(A) = 2$. Let φ_0, φ_1 denote the functionals $\varphi_0(f) = f(0)$, $\varphi_1(f) = f(1)$ ($f \in E$). Then $\varphi_0, \varphi_1 \in N(A^*)$, and since $\alpha(A^*) = 2$ we have $N(A^*) = \text{span}\{\varphi_0, \varphi_1\}$. Thus

$$N(A^*) \cap C^* = \{\mu\varphi_0 + (1 - \mu)\varphi_1 : \mu \in [0, 1]\},$$

and therefore

$$\text{ext}(N(A^*) \cap C^*) = \{\varphi_0, \varphi_1\}.$$

6 Proof of Theorem 2

Fix $p \in N(A) \cap \text{int}K$ and again let E be normed with respect to this p . Let $x \in E$. We have

$$\exp(tA)x = \exp(-t) \left(\sum_{j=0}^{k-1} \frac{t^j T^j x}{j!} + T^k \sum_{j=k}^{\infty} \frac{t^j T^{j-k} x}{j!} \right).$$

Since

$$\frac{d^k}{dt^k} \left(\sum_{j=k}^{\infty} \frac{t^j T^{j-k} x}{j!} \right) = \exp(tT)$$

we have

$$\begin{aligned} h(t) &:= \exp(-t) \left(\sum_{j=k}^{\infty} \frac{t^j T^{j-k} x}{j!} \right) \\ &= \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \exp(t_k T - tid)x \ dt_k \dots dt_2 dt_1. \end{aligned}$$

Since $t_k T - tid$ is quasimonotone increasing for $t_k, t \geq 0$, and since

$$m_+[p, (t_k T - tid)p] = m_+[p, (t_k - t)p] = t_k - t$$

we obtain by Lemma 2

$$\|\exp(t_k T - t id)x\| \leq \exp(t_k - t)\|x\| \quad (t_k, t \geq 0).$$

Thus

$$\|h(t)\| \leq \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \exp(t_k - t)\|x\| dt_k \dots dt_2 dt_1,$$

and evaluation of this integral proves that h is bounded on $[0, \infty)$. Now

$$\begin{aligned} & \frac{1}{t} \int_0^t \exp(\tau A)x d\tau \\ &= \frac{1}{t} \int_0^t \exp(-\tau) \left(\sum_{j=0}^{k-1} \frac{\tau^j T^j x}{j!} \right) d\tau + T^k \left(\frac{1}{t} \int_0^t h(\tau) d\tau \right) \quad (t > 0) \end{aligned}$$

proves that

$$\left\{ \frac{1}{t} \int_0^t \exp(\tau A)x d\tau : t > 0 \right\}$$

is relative compact, and according to [3, VIII.7.1]

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \exp(\tau A)x d\tau$$

exists.

Next, we have already seen that A is a Fredholm operator with

$$\text{ind}(A) = 0, \quad a(A) = d(A) < \infty,$$

since T is power compact. In the proof of Theorem 1 we have seen that $N(A^2) = N(A)$. Thus $a(A) \leq 1$, and since A is not injective $a(A) = 1$. Thus $d(A) = 1$ too. \blacksquare

References

- [1] Abel, U., Ivan, M.: *Over-iterates of Bernstein's Operators: A Short and Elementary Proof*. Amer. Math. Monthly **116** (2009), 535-538.
- [2] Conway, J.B.: *A course in functional analysis*. Graduate Texts in Mathematics, 96. New York etc.: Springer-Verlag. XIV (1985).
- [3] Dunford, N., Schwartz, J.T.: *Linear Operators. I. General theory*. (Pure and Applied Mathematics. Vol. 6) New York and London: Interscience Publishers. XIV (1958).

- [4] Hackbusch, W.: *Elliptic differential equations: theory and numerical treatment*. Springer Series in Computational Mathematics. 18. Berlin: Springer-Verlag. xiv (1992).
- [5] Herzog, G.: *One-sided estimates for linear quasimonotone increasing operators*. Numer. Funct. Anal. Optimization **19** (1998), 549-555.
- [6] Herzog, G., Lemmert, R.: *One-sided estimates for quasimonotone increasing functions*. Bull. Aust. Math. Soc. **67** (2003), 383-392.
- [7] Heuser, H.G.: *Functional analysis*. A Wiley-Interscience Publication. Chichester etc.: John Wiley & Sons. XV (1982).
- [8] Lemmert, R., Volkmann, P.: *On the positivity of semigroups of operators*. Commentat. Math. Univ. Carol. **39** (1998), 483-489.
- [9] Martin, R.H.: *Nonlinear Operators and Differential Equations in Banach spaces*. Robert E. Krieger Publ. Company, Malabar, 1987.
- [10] Protter, M.H., Weinberger, H.F.: *Maximum-principles in differential equations*. Prentice-Hall Partial Differential Equations Series. Englewood Cliffs, N.J.: Prentice-Hall, Inc. X (1967).
- [11] Volkmann, P.: *Gewöhnliche Differentialungleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen*. Math. Z. **127** (1972), 157-164.