COMMUTATIVITY UP TO A FACTOR IN BANACH ALGEBRAS

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ABSTRACT. In this note we explore commutativity up to a factor $ab = \lambda ba$ for Hermitian or normal elements of a complex Banach algebra. Our results generalize results obtained for bounded linear operators on Hilbert spaces.

MSC classification (2000): 46 H 99

Key words and phrases: Hermitian, normal, commutativity.

1. INTRODUCTION.

The following result is due to Brooke, Busch and Pearson [3] (see also [6]).

Theorem 1.1. *Let A, B be bounded linear operators on a complex Hilbert space such that $AB = \lambda BA \neq 0\,(\lambda \in \mathbb{C})$. Then*

(a) *if A or B is self-adjoint, then $\lambda \in \mathbb{R}$;*

(b) *if both A and B are self-adjoint, then $\lambda \in \{-1, 1\}$;*

(c) *if A and B are self-adjoint and one of them is positive, then $\lambda = 1$.*

The aim of this note is a generalization of Theorem 1.1 to Hermitian or normal elements of a complex Banach algebra.

Throughout, $\mathcal{A}$ will be a complex Banach algebra with unit e. The *spectrum* of an element $a \in \mathcal{A}$ will be denoted by $\sigma(a)$ and the *spectral radius* of a by $r(a)$. Let $\mathcal{A}'$ denote the dual space of $\mathcal{A}$. The *numerical range* $V(a)$ is the set

$$V(a) = \{f(a) : f \in \mathcal{A}', \quad \|f\| = f(e) = 1\}.$$

Proposition 1.2. *Let $a, b \in \mathcal{A}$. Then*

(a) $r(ab) = r(ba)$;

(b) $\sigma(ab)\backslash\{0\} = \sigma(ba)\backslash\{0\}$;

(c) *if $ab = ba$, then $\sigma(ab) \subseteq \sigma(a)\sigma(b)$;*

(d) $\sigma(a) \subseteq V(a)$.

Proof. (a), (b) und (c): [5, §10, §11]. (d) is shown in [2, Theorem 2.6]. ■

We say that $a, b \in \mathcal{A}$ *commute up to a factor* if there is $\lambda \in \mathbb{C}\backslash\{0\}$ such that $ab = \lambda ba$. Our first result is an immediate consequence of Proposition 1.2(a).

Proposition 1.3. *If $a, b \in \mathcal{A}$, $r(ab) \neq 0$, $\lambda \in \mathbb{C}$ and $ab = \lambda ba$, then $|\lambda| = 1$.*

Example. This example shows that if $r(ab) = 0$, then any value of $\lambda \neq 0$ may occur.

Let $\mathcal{A} = \mathbb{C}^{3\times3}$, $\lambda \neq 0$, $a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \lambda & 0 \\ 1 & \lambda & \lambda^2 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Then ab is nilpotent and $ab = \lambda ba$.

An element $a \in \mathcal{A}$ is said to be *Hermitian* if $V(a) \subseteq \mathbb{R}$. We denote the set of all Hermitian elements of $\mathcal{A}$ by $H(\mathcal{A})$. From Proposition 1.2(d) we see that $\sigma(a) \subseteq \mathbb{R}$ if $a \in H(\mathcal{A})$. It is well-known (see [2, Example 5.3]), that if $\mathcal{A}$ is a B^*-algebra, then $a \in H(\mathcal{A})$ if and only if $a^* = a$.

Our next proposition collects the basic properties of Hermitian elements. For proofs see [2].

Proposition 1.4. (a) $a \in H(\mathcal{A}) \Leftrightarrow \|\exp(ita)\| = 1$ *for all* $t \in \mathbb{R}$*;*

(b) *$H(\mathcal{A})$ is a closed real linear subspace of $\mathcal{A}$,*

(c) *$H(\mathcal{A}) \cap iH(\mathcal{A}) = \{0\}$;*

(d) *if $a, b \in H(\mathcal{A})$, then $i(ab - ba) \in H(\mathcal{A})$;*

(e) *if $a \in H(\mathcal{A})$, then $r(a) = \|a\|$ (Sinclair's Theorem);*

(f) *if $a \in H(\mathcal{A})$, it does not follow that $a^2 \in H(\mathcal{A})$.*

Let $J(\mathcal{A}) = \{h + ik : h, k \in H(\mathcal{A})\}$. It follows from Proposition 1.4(c) that each element of $J(\mathcal{A})$ has a unique representation of the form $h + ik$ with $h, k \in H(\mathcal{A})$. Therefore we may define a mapping * from $J(\mathcal{A})$ into itself by

$$(h + ik)^* = h - ik \quad (h, k \in H(\mathcal{A})).$$

It is easy to verify that * is a linear involution on $J(\mathcal{A})$.

We say that $a \in \mathcal{A}$ is *normal* if $a = h+ik$ with $h, k \in H(\mathcal{A})$ and $hk = kh$. Observe that $a \in \mathcal{A}$ is normal if and only if $a \in J(\mathcal{A})$ and $aa^* = a^*a$.
An element $a \in \mathcal{A}$ is *positive* if $V(a) \subseteq [0, \infty)$. By [2, Theorem 5.14], we have

$$a \in \mathcal{A} \ \text{ is positive } \ \Leftrightarrow a \in H(\mathcal{A}) \ \text{ and } \ \sigma(a) \subseteq [0, \infty).$$

2. Results.

We begin with a generalization of Fuglede's theorem (see [1, Theorem 1]). For the convenience of the reader we include a proof, which is an adaption of Rosenblum's proof of the Putnam-Fuglede theorem (see [5, Theorem 12.16]).

Proposition 2.1. *Assume that $a, b, c \in \mathcal{A}$, a, c are normal and that $ab = bc$. Then $a^*b = bc^*$.*

Proof. By induction we have $a^k b = bc^k$ for $k = 1, 2, 3, \ldots$. Therefore $\exp(a)b = b\exp(c)$, thus

$$b = \exp(-a)b\exp(c). \tag{1}$$

Put $u_1 = \exp(a^* - a)$ and $u_2 = \exp(c - c^*)$. There are $h, k \in H(\mathcal{A})$ with $a = h + ik$ and $hk = kh$. Then $a^* - a = -2ik$. Since $k \in H(\mathcal{A})$, Proposition 1.4(a) shows that $\|u_1\| = \|\exp(-2ik)\| = 1$. A similar argument gives $\|u_2\| = 1$. From $a^*a = aa^*$ and $cc^* = c^*c$ we derive

$$u_1 = \exp(a^*)\exp(-a) \quad \text{and} \quad u_2 = \exp(c)\exp(-c^*),$$

hence, by (1),

$$u_1 b u_2 = \exp(a^*)b\exp(-c^*),$$

thus

$$\|\exp(a^*)b\exp(-c^*)\| = \|u_1 b u_2\| \leq \|b\|. \tag{2}$$

Now define the entire function $f : \mathbb{C} \to \mathcal{A}$ by

$$f(z) = \exp(za^*)b\exp(-zc^*).$$

Since the hypotheses of the proposition hold with $\overline{z}a$ and $\overline{z}c$ in place of a and c, (2) implies that $\|f(z)\| \leq \|b\|$ for all $z \in \mathbb{C}$. By Liouville's theorem, $f(z) = f(0) = b$ for all $z \in \mathbb{C}$. Hence

$$\exp(za^*)b = b\exp(zc^*) \quad (z \in \mathbb{C}).$$

Thus

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}(a^*)^n b = \sum_{n=0}^{\infty} \frac{z^n}{n!} b(c^*)^n \quad (z \in \mathbb{C}).$$

Comparing coefficients gives $(a^*)^n b = b(c^*)^n$ for $n = 0, 1, 2, \ldots$, hence $a^*b = bc^*$. ∎

Corollary 2.2. *If $a, b, c \in \mathcal{A}$, $a, c \in H(\mathcal{A})$, $\lambda \in \mathbb{C}$ and $ab = \lambda bc \neq 0$, then $\lambda \in \mathbb{R}$.*

Proof. By Proposition 2.1, $a^*b = \overline{\lambda}bc^*$. Since $a = a^*$ and $c = c^*$, $\lambda bc = ab = a^*b = \overline{\lambda}bc^* = \overline{\lambda}bc$, thus $(\lambda - \overline{\lambda})bc = 0$, hence $\lambda = \overline{\lambda}$. ∎

Proposition 2.3. *Let $a \in H(\mathcal{A})$, $x \in \mathcal{A}$ and $a^2x = 0$. Then $ax = 0$.*

Proof. We can assume that $\|x\| = 1$. Then, by Proposition 1.4(a), for all $t \in \mathbb{R}$

$$\begin{aligned} 1 = \|x\| &= \|\exp(-ita)\exp(ita)x\| \le \|\exp(-ita)\|\,\|\exp(ita)x\| \\ &= \|\exp(ita)x\| \le \|\exp(ita)\|\,\|x\| = \|x\| = 1, \end{aligned}$$

thus, since $a^n x = 0$ for $n \ge 2$,

$$1 = \|\exp(ita)x\| = \|x + itax\|.$$

Therefore $|t|\,\|ax\| - 1 \le \|x + itax\| = 1$ for all $t \in \mathbb{R}$, hence $ax = 0$. ■

Proposition 2.4. *Suppose that $a, b \in \mathcal{A}$, b is normal, $b = h+ik$, $h, k \in H(\mathcal{A})$, $hk = kh$, $ab = \lambda ba$ and $\lambda \in \mathbb{R}$. Then*

$$ah = \lambda ha \quad \text{and} \quad ak = \lambda ka.$$

Proof. Because of Proposition 2.1 we have $ab^* = \lambda b^* a$. Therefore $a(b + b^*) = \lambda(b + b^*)a$ and $a(b - b^*) = \lambda(b - b^*)a$. Since $h = \frac{1}{2}(b + b^*)$ and $k = \frac{1}{2i}(b - b^*)$, the result follows. ■

Now we are in a position to state the main result of this paper, a generalization of Theorem 1.1.

Theorem 2.5. *Let $a, b \in \mathcal{A}$ such that $ab = \lambda ba \neq 0$, $\lambda \in \mathbb{C}$.*

(a) *If $a \in H(\mathcal{A})$, then $\lambda \in \mathbb{R}$.*

(b) *If $a \in H(\mathcal{A})$ and b is normal, then $\lambda \in \{-1, 1\}$.*

(c) *If $a \in H(\mathcal{A})$ is positive and b is normal then $\lambda = 1$.*

Proof. (a) follows from Corollary 2.2.
For the proof of (b) and (c) let $b = h + ik$, $h, k \in H(\mathcal{A})$ and $hk = kh$. Then, by (a) and Proposition 2.4, $\lambda \in \mathbb{R}$, $ah = \lambda ha$ and $ak = \lambda ka$. Since $ab \neq 0$, we have $ah \neq 0$ or $ak \neq 0$. *Thus we can assume that b is Hermitian.*
(b) If $\lambda = 1$, we are done. So assume that $\lambda \neq 1$. Then

$$i(ab - ba) = i(\lambda ba - ba) = i(\lambda - 1)ba \neq 0.$$

Proposition 1.4(d) shows that $i(\lambda - 1)ba \in H(\mathcal{A})$, thus, by Proposition 1.4(e)

$$|\lambda - 1| r(ba) = r(i(\lambda - 1)ba) = \|i(\lambda - 1)ba\| = |\lambda - 1|\,\|ba\| \neq 0,$$

hence $r(ba) \neq 0$. Because of Proposition 1.3 we derive $|\lambda| = 1$. Since $\lambda \in \mathbb{R}\backslash\{1\}$, $\lambda = -1$.
(c) From (b) we get $\lambda = 1$ or $\lambda = -1$. Assume that $\lambda = -1$, thus $ab = -ba$. Then $bab = -b^2a$ and $ab^2 = -bab$, hence

$$b^2a = ab^2 = -bab. \tag{3}$$

Since $\sigma(b^2) \subseteq [0, \infty)$ (Proposition 1.2(d)), $\sigma(a) \subseteq [0, \infty)$ and $ab^2 = b^2 a$, we derive from Proposition 1.2(c) that

$$\sigma(ab^2) \subseteq \sigma(a)\sigma(b^2) \subseteq [0, \infty). \tag{4}$$

From $i(ab - ba) = 2iab$ we see that $iab \in H(\mathcal{A})$ (Proposition 1.4(d)).

Furthermore we have by (3) that

$$i((iab)b - b(iab)) = -ab^2 + bab = -2ab^2.$$

Use again Proposition 1.4(d) to get $ab^2 \in H(\mathcal{A})$. It follows from (4) that $ab^2 = -bab$ is positive. Proposition 1.2(b) and (3) show that

$$\sigma(ab^2)\backslash\{0\} = \sigma(-b(ab))\backslash\{0\} = \sigma(-ab^2)\backslash\{0\},$$

thus

$$\sigma(ab^2) = \sigma(-ab^2),$$

therefore, by (4), $\sigma(ab^2) = \{0\}$. Since $ab^2 \in H(\mathcal{A})$, we derive from Proposition 1.4(e) that $\|ab^2\| = r(ab^2) = 0$. Hence $b^2 a = ab^2 = 0$. By Proposition 2.3 we obtain the contradiction $ab = -ba = 0$. Thus $\lambda = 1$. ∎

Corollary 2.6. *If $a \in H(\mathcal{A})$ is positive and $b \in \mathcal{A}$ is normal, then the following statements are equivalent:*

(a) $ab = ba$;

(b) $a^2 b = ba^2$.

Proof. The implication (a) ⇒ (b) is clear.

(b) ⇒ (a): Proposition 2.1 gives $a^2 b^* = b^* a^2$. Let $b = h + ik$ with $h, k \in H(\mathcal{A})$ and $h, k = kh$. Then

$$a^2 h = a^2 \tfrac{1}{2}(b + b^*) = \tfrac{1}{2}(b + b^*) a^2 = ha^2 \tag{5}$$

and

$$a^2 k = a^2 \tfrac{1}{2i}(b - b^*) = \tfrac{1}{2i}(b - b^*) a^2 = ka^2. \tag{6}$$

Thus, by (5),

$$a(ah - ha) + (ah - ha)a = a^2 h - aha + aha - ha^2 = 0.$$

Put $c = i(ah - ha)$. Then $c \in H(\mathcal{A})$ (Proposition 1.4(d)) and $ac = -ca$. Theorem 2.5(c) shows now that $ac = ca = 0$. Thus $a(ah - ha) = 0 = (ah - ha)a$. Now use the Kleinecke-Shirokov theorem ([4, Theorem 1.3.1]) to conclude that $r(ah - ha) = 0$. Therefore $r(c) = 0$, thus, since $c \in H(\mathcal{A})$, $c = 0$. This gives $ah = ha$. A similar argument, use (6), shows that $ak = ka$. Hence $ab = a(h + ik) = (h + ik)a = ba$. ∎

Corollary 2.7. *If $a, b \in H(\mathcal{A})$ and a is positive, then the following statements are equivalent:*

(a) $ab = ba$;

(b) $a^2 b = ba^2$;

(c) $a^2(ab - ba) = (ab - ba)a^2$.

Proof. Because of Corollary 2.6 we only have to show that (c) implies (a). So suppose that (c) holds. Then, if $c = i(ab - ba)$, we have $c \in H(\mathcal{A})$ and $a^2 c = ca^2$. By Corollary 2.6, $ac = ca$, thus $a(ab - ba) = (ab - ba)a$. Apply the Kleinecke-Shirokov theorem to get $r(c) = r(ab - ba) = 0$. Proposition 1.4(e) shows now that $ab = ba$. ■

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