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REMARKS ON COMMUTING EXPONENTIALS IN BANACH ALGEBRAS

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ABSTRACT. Suppose that a and b are elements of a complex unital Banach algebra such that the spectra of a and b are $2\pi i$-congruence-free. E.M.E. Wermuth has shown that then

$$e^a e^b = e^b e^a \quad \text{implies that} \quad ab = ba.$$

In this note we use two elementary facts concerning inner derivations on Banach algebras to give a very short proof of Wermuth's result.

Let $\mathcal{A}$ denote a complex unital Banach algebra. For $x \in \mathcal{A}$ the spectrum of x is denoted by $\sigma(x)$. The map $\delta_x : \mathcal{A} \to \mathcal{A}$, defined by

$$\delta_x(c) = cx - xc \qquad (c \in \mathcal{A}),$$

is called the *inner derivation determined by* x. From $\|\delta_x(c)\| \leq 2\|c\|\,\|x\|$ it follows that δ_x is a bounded linear operator on $\mathcal{A}$. Proposition 6.4.8 in [1] shows that

$$\sigma(\delta_x) \subseteq \{\lambda - \mu : \lambda, \mu \in \sigma(x)\} \tag{1}$$

and

$$e^{\delta_x}(c) = e^{-x} c e^{x} \ \text{ for all } \ c \in \mathcal{A}. \tag{2}$$

Define the entire function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) = \begin{cases} \lambda^{-1}(e^\lambda - 1), & \text{if } \lambda \neq 0, \\ 1, & \text{if } \lambda = 0. \end{cases}$$

Since $\lambda f(\lambda) = f(\lambda)\lambda = e^\lambda - 1$, we obtain for $x \in \mathcal{A}$

$$f(\delta_x)\delta_x = e^{\delta_x} - I;$$

hence, by (2),

$$(f(\delta_x)\delta_x)(c) = e^{-x} c e^{x} - c \ \text{ for all } \ c \in \mathcal{A}. \tag{3}$$

A set $\Omega \subseteq \mathbb{C}$ is called $2\pi i$-*congruence-free* if $\lambda_1, \lambda_2 \in \Omega$ and $\lambda_1 \equiv \lambda_2 \pmod{2\pi i}$ implies that $\lambda_1 = \lambda_2$.

Theorem. *Let $a, b \in \mathcal{A}$. Suppose that $\sigma(a)$ and $\sigma(b)$ are $2\pi i$-congruence-free and that $e^a e^b = e^b e^a$. Then $ab = ba$.*

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Proof. Let $x \in \{a, b\}$. Since $\sigma(x)$ is $2\pi i$-congruence-free, (1) shows that f does not vanish on $\sigma(\delta_x)$; hence $f(\delta_x)$ is bijective. From (3) it follows that

$$\delta_x(c) = f(\delta_x)^{-1}(e^{-x}ce^x - c) \quad \text{for all} \quad c \in \mathcal{A}. \tag{4}$$

Therefore we get

$$\delta_a(e^b) = f(\delta_a)^{-1}(e^{-a}e^be^a - e^b) = f(\delta_a)^{-1}(e^{-a}e^ae^b - e^b) = 0.$$

Thus $ae^b = e^ba$. Use (4) again to obtain

$$\delta_b(a) = f(\delta_b)^{-1}(e^{-b}ae^b - a) = f(\delta_b)^{-1}(e^{-b}e^ba - a) = 0;$$

hence $ab = ba$. □

Corollary 1. *Suppose that $\mathcal{H}$ is a complex Hilbert space and $\mathcal{A}$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$. For self-adjoint operators $A, B \in \mathcal{A}$ the following assertions are equivalent:*

(i) $e^Ae^B = e^Be^A$.
(ii) $e^Ae^B = e^{A+B}$.
(iii) $AB = BA$.

Proof. It is clear that (iii) implies (i) and (ii). Since $\sigma(A), \sigma(B) \subseteq \mathbb{R}$, it follows that $\sigma(A)$ and $\sigma(B)$ are $2\pi i$-congruence-free, and so (i) implies (iii). If (ii) holds, we get $e^Be^A = (e^B)^*(e^A)^* = (e^Ae^B)^* = (e^{A+B})^* = e^{A+B} = e^Ae^B$, and thus (i) holds. □

Corollary 2. *Suppose that $\mathcal{A}$ is as in Corollary 1 and that the spectrum of $A \in \mathcal{A}$ is $2\pi i$-congruence-free. Then*

$$e^A \text{ is normal if and only if } A \text{ is normal.}$$

Proof. If A is normal, then $e^Ae^{A^*} = e^{A+A^*} = e^{A^*}e^A$; thus e^A is normal.

Now suppose that e^A is normal. Since $\sigma(A^*) = \{\bar{\lambda} : \lambda \in \sigma(A)\}$, it follows that $\sigma(A)$ and $\sigma(A^*)$ are $2\pi i$-congruence-free. The theorem gives $AA^* = A^*A$. □

Corollary 3. *Suppose that $\mathcal{A}$ is as in Corollary 1. If $A \in \mathcal{A}$ and $\|A\| < \pi$, then*

$$e^A \text{ is normal if and only if } A \text{ is normal.}$$

Proof. $\|A\| < \pi$ implies that $\sigma(A)$ is $2\pi i$-congruence-free. Now use Corollary 2. □

References

[1] T.W. Palmer: *Banach algebras and the general theory of *-algebra*, Vol. I. Cambridge, 1994. MR **95c:**46002

[2] E.M.E. Wermuth: *A remark on commuting operator exponentials*, Proc. Amer. Math. Soc. **125** (1997), 1685–1688. MR **97g:**39011

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