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REMARKS ON COMMUTING EXPONENTIALS IN BANACH ALGEBRAS, II

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Abstract. Suppose that a and b are elements of a complex unital Banach algebra such that the spectrum of a is $2\pi i$-congruence-free and $e^a e^b = e^b e^a$. We show that then $ab-ba$ is the sum of nilpotent elements. If $r(b)$ denotes the spectral radius of b, then we show that the additional assumption $r(b) < 2\pi$ implies that

$$b(ab-ba)^2 = (ab-ba)^2 b.$$

The present paper is a continuation of our previous paper [4]. Let $\mathcal{A}$ denote a complex unital Banach algebra. By $\mathcal{L}(\mathcal{A})$ we denote the Banach algebra of all bounded linear operators on $\mathcal{A}$. For $x \in \mathcal{A}$ the spectrum and the spectral radius of x are denoted by $\sigma(x)$ and $r(x)$, respectively. The map $\delta_x : \mathcal{A} \to \mathcal{A}$, defined by

$$\delta_x(c) = cx - xc \quad (c \in \mathcal{A}),$$

is called the *inner derivation determined by* x. From $\|\delta_x(c)\| \leq 2\|c\|\|x\|$ it follows that $\delta_x \in \mathcal{L}(\mathcal{A})$.

If $T \in \mathcal{L}(\mathcal{A})$, then $N(T)$ denotes the kernel of T.

If Ω_1 and Ω_2 are subsets of $\mathbb{C}$, then $\Omega_1 - \Omega_2 := \{\lambda - \mu : \lambda_1 \in \Omega_1, \lambda_2 \in \Omega_2\}$.

Proposition 6.4.8 in [2] shows that

$$\sigma(\delta_x) \subseteq \sigma(x) - \sigma(x) = \{\lambda - \mu : \lambda, \mu \in \sigma(x)\} \tag{1}$$

and

$$e^{\delta_x}(c) = e^{-x} c e^x \quad \text{for all } c \in \mathcal{A}. \tag{2}$$

Define the entire function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(\lambda) = \begin{cases} \lambda^{-1}(e^\lambda - 1), & \text{if } \lambda \neq 0, \\ 1, & \text{if } \lambda = 0. \end{cases}$$

Since $\lambda f(\lambda) = f(\lambda)\lambda = e^\lambda - 1$, we get for $x \in \mathcal{A}$

$$f(\delta_x)\delta_x = e^{\delta_x} - I;$$

thus, by (2),

$$f(\delta_x)(\delta_x(c)) = e^{-x} c e^x - c \quad \text{for all } c \in \mathcal{A}. \tag{3}$$

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A set $\Omega \subseteq \mathbb{C}$ is called *$2\pi i$-congruence-free* if $\lambda, \mu \in \Omega$ and $\lambda \equiv \mu \pmod{2\pi i}$ imply that $\lambda = \mu$.

The following result is due to E. M. E. Wermuth ([6]). See also [4] for a very short proof using (3).

Theorem 1. *Let $a, b \in \mathcal{A}$. Suppose that $\sigma(a)$ and $\sigma(b)$ are $2\pi i$-congruence-free and that $e^a e^b = e^b e^a$. Then $ab = ba$.*

In [3, Corollary 1] we have shown that if A and B are bounded self-adjoint operators on a Hilbert space, then

$$e^A e^B = e^{A+B} \quad \text{if and only if } AB = BA. \tag{4}$$

Even in the case of two-dimensional matrices the equivalence (4) does *not* hold under the weaker assumption of $2\pi i$-congruence-free spectra ([5]). But we have:

Theorem 2. *Let $a, b \in \mathcal{A}$. Suppose that $\sigma(a+b)$ is $2\pi i$-congruence-free. If*

$$e^a e^b = e^{a+b} = e^b e^a,$$

then $ab = ba$.

Proof. Since $\sigma(a+b)$ is $2\pi i$-congruence-free, (1) shows that f does not vanish on $\sigma(\delta_{a+b})$; hence $f(\delta_{a+b})$ is invertible in $\mathcal{L}(\mathcal{A})$. From (3) we get

$$\delta_{a+b}\ (e^a) = f(\delta_{a+b})^{-1}\ (e^{-(a+b)} e^a e^{a+b} - e^a) = f(\delta_{a+b})^{-1}\ (e^{-b} e^{-a} e^a e^b e^a - e^a) = 0;$$

thus $e^a(a+b) = (a+b)e^a$. Therefore $be^a = e^a b$. This gives

$$\delta_{a+b}\ (b) = f(\delta_{a+b})^{-1}\ (e^{-(a+b)} b e^{a+b} - b) = f(\delta_{a+b})^{-1}\ (e^{-b} e^{-a} b e^a e^b - b) = 0\ .$$

Hence $b(a+b) = (a+b)b$. This shows that $ba = ab$. □

Corollary 1. *If A is a bounded linear operator on a complex Hilbert space, then*

$$e^A e^{A^*} = e^{A+A^*} = e^{A^*} e^A \quad \textit{if and only if } A \textit{ is normal.}$$

Proof. Since $A + A^*$ is self-adjoint, $\sigma(A + A^*) \subseteq \mathbb{R}$. Hence $\sigma(A + A^*)$ is $2\pi i$-congruence-free. □

For our main results in this paper we need the following propositions.

Proposition 1. *Let $c, d, x \in \mathcal{A}$ and $\lambda_0, \mu_0 \in \mathbb{C}\backslash\{0\}$. If $cx - xc = \lambda_0 c$ and $dx - xd = \mu_0 d$, then:*

(a) $(dc)x - x(dc) = (\mu_0 + \lambda_0)dc$;
(b) $c^n x - xc^n = n\lambda_0 c^n$ *for all* $n \in \mathbb{N}$;
(c) $c^n = 0$ *for all* $n \in \mathbb{N}$ *with* $n > 2\ \|x\|\ |\lambda_0|^{-1}$.

Proof. (a) From $xc = cx - \lambda_0 c$ and $xd = dx - \mu_0 d$ we derive

$$\begin{aligned}(dc)x - x(dc) &= (dc)x - (xd)c \\ &= (dc)x - (dx - \mu_0 d)c \\ &= (dc)x - d(xc) + \mu_0 dc \\ &= (dc)x - d(cx - \lambda_0 c) + \mu_0 dc \\ &= (dc)x - (dc)x + (\lambda_0 + \mu_0)dc = (\lambda_0 + \mu_0)dc.\end{aligned}$$

(b) follows from (a) by induction.

(c) If $n|\lambda_0| > 2\|x\|$, then $n|\lambda_0| > \|\delta_x\|$; thus $n\lambda_0 \notin \sigma(\delta_x)$. Hence, by (b), $c^n = 0$. □

Remark. Proposition 1 is valid for not necessarily inner derivations $D : \mathcal{A} \to \mathcal{A}$.

Recall that the entire function f is defined by $f(\lambda) = \lambda^{-1}(e^{\lambda} - 1)$ if $\lambda \neq 0$ and $f(0) = 1$. We use the following notation for $x \in \mathcal{A}$:

$$M_x = \{\lambda \in \sigma(\delta_x) : \ f(\lambda) = 0\}.$$

Observe that if $M_x = \emptyset$, then $f(\delta_x)$ is invertible in $\mathcal{L}(\mathcal{A})$. Observe also that if $\sigma(x)$ is $2\pi i$-congruence-free, then, by (1), $M_x = \emptyset$.

Since $\sigma(\delta_x)$ is compact it follows that M_x is a finite set. From the definition of f it is clear that each $\lambda \in M_x$ is a simple zero of f and that there is $k \in \mathbb{Z} \setminus \{0\}$ and that $\lambda = 2k\pi i$.

Proposition 2. *Let $x \in \mathcal{A}$ and let n denote the smallest integer with $n > \frac{r(x)}{\pi}$. We have:*

(a) *M_x has at most $2n - 2$ elements;*

(b) *if $M_x \neq \emptyset$ and $M_x = \{\lambda_1, \dots, \lambda_k\}$ with $k \leq 2n - 2$ and $\lambda_j \neq \lambda_l$ for $j \neq l$, then*

$$N(f(\delta_x)) = N(\delta_x - \lambda_1 I) \ \oplus \cdots \oplus \ N(\delta_x - \lambda_k I)$$

and for each $z \in N(f(\delta_x))$ there are $u_1, \dots, u_k \in \mathcal{A}$ such that

$$z = u_1 + \cdots + u_k, \quad u_j \in N(\delta_x - \lambda_j I) \ \text{ and } \ u_j^n = 0$$

for $j = 1, \dots, k$.

Proof. (a) Suppose that $M_x \neq \emptyset$. If $\lambda \in M_x$, then $\lambda = 2k\pi i$ for some $k \in \mathbb{Z} \setminus \{0\}$. By (1) there are $\alpha, \beta \in \sigma(x)$ with $\lambda = \alpha - \beta$. Thus

$$2\ |k|\ \pi = |\lambda| = |\alpha - \beta| \ \leq \ |\alpha| + |\beta| \ \leq \ 2r(x) < 2n\pi;$$

hence $|k| \ \leq \ n - 1$. Therefore

$$M_x \ \subseteq \ \{ \ \pm 2\pi i, \ \pm 4\pi i, \dots, \ \pm 2(n-1)\pi i \ \}.$$

(b) Recall that each λ_j is a simple zero of f; hence there is an entire function g such that

$$f(\lambda) = g(\lambda)(\lambda - \lambda_1) \cdot \dots \cdot (\lambda - \lambda_k).$$

It follows that $g(\delta_x)$ is invertible in $\mathcal{L}(\mathcal{A})$ and

$$f(\delta_x) = g(\delta_x)(\delta_x - \lambda_1 I) \cdot \dots \cdot \ (\delta_x - \lambda_k I).$$

Satz 80.3 in [1] gives now

$$N(f(\delta_x)) = N(\delta_x - \lambda_1 I) \ \oplus \cdots \oplus \ N(\delta_x - \lambda_k I).$$

Take $z \in N(f(\delta_x))$. Then there are $u_j \in N(\delta_x - \lambda_j I)$ $(j = 1, \dots, k)$ with $z = u_1 + \cdots + u_k$. Let $j \in \{1, \dots, k\}$. Proposition 1 (b) shows that

$$u_j^n x - x u_j^n = n \lambda_j u_j^n.$$

Suppose that $u_j^n \neq 0$. Then $n\lambda_j \in \sigma(\delta_x)$. As in the proof of (a) we derive

$$n\ |\lambda_j| \ \leq \ n\ 2r(x) \ < \ 2n\pi;$$

hence $|\lambda_j| < 2\pi$, a contradiction since $\lambda_j \neq 0$. □

Let $x \in \mathcal{A}$ and $r(x) < \pi$. Then $\sigma(x)$ is $2\pi i$-congruence-free and hence $N(f(\delta_x)) = \{0\}$. The following theorem deals with the case where $\pi \leq r(x) < 2\pi$.

Theorem 3. *Let $x \in \mathcal{A}$ and suppose that $\pi \le r(x) < 2\pi$. Then*

$$xz^2 = z^2 x \quad \text{for each } z \in N(f(\delta_x)). \tag{5}$$

There are the following three possibilities:

(a) $M_x = \emptyset$. In this case we have $N(f(\delta_x)) = \{0\}$.

(b) $M_x = \{\lambda\}$ with $\lambda = 2\pi i$ or $\lambda = -2\pi i$. In this case we have $z^2 = 0$ for each $z \in N(f(\delta_x))$.

(c) $M_x = \{\lambda_1, \lambda_2\}$ with $\lambda_1 = 2\pi i = -\lambda_2$. In this case (5) holds.

Proof. From Proposition 2 we know that M_x has at most two elements. Suppose that $M_x \neq \emptyset$. Take $\lambda \in M_x$; hence $\lambda = 2k\pi i$ $(k \in \mathbb{Z} \setminus \{0\})$. As in the proof of Proposition 2 (a) we see that $|k| \le 1$; thus $\lambda = \pm 2\pi i$.

Case 1: $M = \{\lambda_1\}$. Proposition 2 (b) gives

$$N(f(\delta_x)) = N(\delta_x - \lambda_1 I) \quad \text{and} \quad z^2 = 0 \quad \text{for each} \quad z \in N(f(\delta_x)).$$

Case 2: $M = \{\lambda_1, \lambda_2\}$. Then $\lambda_1 = 2\pi i = -\lambda_2$. Let $z \in N(f(\delta_x))$. By Proposition 2 (b) there are $u_j \in N(\delta_x - \lambda_j I)$ with $u_j^2 = 0$ $(j = 1, 2)$ and $z = u_1 + u_2$.

Proposition 1 (a) gives

$$u_1 u_2 x - x u_1 u_2 = (\lambda_1 + \lambda_2) u_1 u_2 = 0.u_1 u_2 = 0;$$

thus $x u_1 u_2 = u_1 u_2 x$. A similar argument shows that $x u_2 u_1 = u_2 u_1 x$. From $z^2 = u_1^2 + u_1 u_2 + u_2 u_1 + u_2^2 = u_1 u_2 + u_2 u_1$ it follows then that

$$xz^2 = x u_1 u_2 + x u_2 u_1 = u_1 u_2 x + u_2 u_1 x = z^2 x. \quad \square$$

Theorem 4. *Let $a, b \in \mathcal{A}$ and $e^a e^b = e^b e^a$. Suppose that $\sigma(a)$ is $2\pi i$-congruence-free. Then $ab - ba$ is the sum of nilpotent elements. If in addition $r(b) < 2\pi$, then*

$$b(ab - ba)^2 = (ab - ba)^2 b.$$

Proof. Since $f(\delta_a)$ is invertible in $\mathcal{L}(\mathcal{A})$ we get from

$$f(\delta_a)(\delta_a(e^b)) = e^{-a} e^b e^a - e^b = 0$$

that $ae^b = e^b a$. Therefore

$$f(\delta_b)(\delta_b(a)) = e^{-b} a e^b - a = 0;$$

hence $ab - ba = \delta_b(a) \in N(f(\delta_b))$. Use Proposition 2, Theorem 1 and Theorem 3 to complete the proof. $\square$

With the aid of Theorem 3 we can prove a further result concerning exponentials in Banach algebras.

Theorem 5. *Suppose that $a, b \in \mathcal{A}$ and $e^a = e^b$.*

(a) If $\sigma(a)$ is $2\pi i$-congruence-free, then $ab = ba$.

(b) If $\pi \le r(a) < 2\pi$, then $a(ab - ba)^2 = (ab - ba)^2 a$.

Proof. From (3) we get

$$f(\delta_a)(ba - ab) = f(\delta_a)(\delta_a(b)) = e^{-a} b e^a - b = e^{-b} b e^b - b = 0.$$

Hence $ab - ba \in N(f(\delta_a))$.

(a) If $\sigma(a)$ is $2\pi i$-congruence-free we have $N(f(\delta_a)) = \{0\}$; thus $ab = ba$.

(b) follows from Theorem 3. $\square$

For our next result we denote by $\mathbf{1}$ the unit of $\mathcal{A}$.

Theorem 6. *Let $a, b \in \mathcal{A}$, let $e^a = e^b$ and let $\sigma(a)$ and $\sigma(a) - \sigma(b)$ be $2\pi i$-congruence-free. Then there is some $k \in \mathbb{Z}$ with*

$$a - b = (2k\pi i)\mathbf{1}.$$

Proof. From Theorem 5 (a) we get $ab = ba$. Thus $e^{a-b} = \mathbf{1}$. Put $c = a - b$ and define $g : \mathbb{C} \to \mathbb{C}$ by $g(\lambda) = e^\lambda - 1$. Then $g(c) = 0$. Take $\lambda_1, \lambda_2 \in \sigma(c)$. Since a and b commute, we get from [3, Theorem 11.23] that $\sigma(c) \subseteq \sigma(a) - \sigma(b)$; thus $\lambda_1, \lambda_2 \in \sigma(a) - \sigma(b)$. The spectral mapping theorem gives $e^{\lambda_1} = 1 = e^{\lambda_2}$; thus $\lambda_1 = 2k\pi i$ and $\lambda_2 = 2j\pi i$ for some $k, j \in \mathbb{Z}$. This shows that $\lambda_1 \equiv \lambda_2 \pmod{2\pi i}$. Since $\sigma(a) - \sigma(b)$ is $2\pi i$-congruence-free, we get $\lambda_1 = \lambda_2$. Hence there is $k \in \mathbb{Z}$ such that $\sigma(c) = \{2k\pi i\}$. Since $2k\pi i$ is a simple zero of g, there is an entire function h with

$$g(\lambda) = h(\lambda)\ (\lambda - 2k\pi i)$$

and $h(2k\pi i) \neq 0$. This gives

$$0 = g(c) = h(c)\ (c - (2k\pi i)\mathbf{1}).$$

Since $h(c)$ is invertible in $\mathcal{A}$, $c = (2k\pi i)\mathbf{1}$. □

As an immediate consequence of Theorem 6 we have the following well-known result:

Corollary 2. *If A and B are bounded, self-adjoint operators on a complex Hilbert space and if $e^A = e^B$, then $A = B$.*

Proof. From $\sigma(A)$, $\sigma(A) - \sigma(B) \subseteq \mathbb{R}$ we see that $\sigma(A)$ and $\sigma(A) - \sigma(B)$ are $2\pi i$-congruence-free. Theorem 6 now shows that $A - B = (2k\pi i)I$ for some $k \in \mathbb{Z}$. From $A - B = (A - B)^* = -(2k\pi i)I = -(A - B)$ we get $A = B$. □

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