A note on Gâteaux differentials of hermitian elements in Banach algebras

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Abstract. Let $\mathcal{A}$ be a complex unital Banach algebra with unit $\mathbf{1}$. If $a \in \mathcal{A}$ is hermitian then we show that

$$\|a\|^2 = \lim_{t\to 0+} t^{-1}(\|\mathbf{1} + ta^2\| - 1)\,,$$

and we give a proof of an inequality due to J. Nieto.

1. Terminology and results.

In this note let $\mathcal{A}$ be a complex Banach algebra, with a unit element $\mathbf{1}$ of norm one. The *spectrum* of $a \in \mathcal{A}$ is denoted by by $\sigma(a)$, and $r(a)$ denotes its *spectral radius*. Let $\mathcal{A}'$ denote the dual space of $\mathcal{A}$. The set of *normalized states* on $\mathcal{A}$ is given by

$$D(\mathcal{A};\mathbf{1}) = \{\varphi \in \mathcal{A}' : \varphi(\mathbf{1}) = 1 = \|\varphi\|\}\,.$$

The *numerical range* of $a \in \mathcal{A}$ is defined by

$$V(\mathcal{A};a) = \{\varphi(a) : \varphi \in D(\mathcal{A};\mathbf{1})\}\,.$$

The set $V(\mathcal{A};a)$ is compact and convex. We say that $a \in \mathcal{A}$ is *hermitian* if $V(\mathcal{A};a) \subseteq \mathbb{R}$. $H(\mathcal{A})$ denotes the set of all hermitian elements in $\mathcal{A}$.

The Gâteaux differentials $n_+[\cdot]$, $n_-[\cdot] : \mathcal{A} \to \mathbb{R}$ are defined by

$$n_+[a] = \lim_{t\to 0+} \frac{\|\mathbf{1} + ta\| - 1}{t} \quad \text{and} \quad n_-[a] = \lim_{t\to 0-} \frac{\|\mathbf{1} + ta\| - 1}{t}\,.$$

The main results of this note read as follows:

1.1. Theorem. *Given $a, b \in H(\mathcal{A})$ and $\varphi \in D(\mathcal{A};1)$, then*

$$(\operatorname{Re}\varphi(ab))^2 \le \operatorname{Re}\varphi(a^2)\operatorname{Re}\varphi(b^2)\,.$$

1.2. Theorem. *If $a \in H(\mathcal{A})$, then*

$$n_+[a^2] = \|a\|^2\,.$$

Remark. If $a \in H(\mathcal{A})$, then in general it does not follow that $a^2 \in H(\mathcal{A})$ (see [**2**, Example 6.1].

2000 *Mathematics Subject Classification.* 47A12.
Key words and phrases. numerical range, hermitian element.

2. Proofs.

Note the following properties of n_+ and n_-: the function $a \mapsto n_+[a]$ is sublinear,

$$n_-[a] \le n_+[a], \; n_-[-a] = -n_+[a]$$

and

$$n_\pm[a + \lambda \mathbf{1}] = n_\pm[a] + \lambda$$

for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{R}$.

Furthermore we have

$$\sigma(a) \subseteq V(\mathcal{A}; a)$$

([**2**, Theorem 2.6]) and

$$\{\operatorname{Re} \lambda : \lambda \in V(\mathcal{A}; a)\} = [n_-[a], \; n_+[a]] \tag{2.1}$$

(see [**2**] or [**3**] or [**4**]).

2.1. Proposition. *Given $a \in \mathcal{A}$, the following statements are equivalent:*
(1) $a \in H(\mathcal{A})$;
(2) $n_+[ia] = n_-[ia] = 0$;
(3) $\|e^{ita}\| = 1$ *for all* $t \in \mathbb{R}$;
(4) $\sigma(a) \subseteq \mathbb{R}$ *and* $r(\alpha\mathbf{1} + \beta a) = \|\alpha\mathbf{1} + \beta a\| \quad (\alpha, \beta \in \mathbb{C})$.

Proof. [**2**, Lemma 5.2] and [**1**]. □

2.2. Proposition.
(1) *If $a, b \in H(\mathcal{A})$, then $i(ab - ba) \in H(\mathcal{A})$;*
(2) *$H(\mathcal{A})$ is a closed real subspace of $\mathcal{A}$ and*

$$H(\mathcal{A}) \cap iH(\mathcal{A}) = \{0\};$$

(3) *$V(\mathcal{A}; a) = \operatorname{co} \sigma(a)$ for all $a \in H(\mathcal{A})$ (where* co *denotes the convex hull).*

Proof. [**2**, 5]. □

Note that by (2.1),

$$V(\mathcal{A}; a) = [n_-[a], \; n_+[a]] \quad \text{if } a \in H(\mathcal{A}). \tag{2.2}$$

2.3. Proposition. *Let a, $b \in H(\mathcal{A})$, and $\varphi \in D(\mathcal{A}, 1)$. Then*

$$\operatorname{Re} \varphi(ab) = \operatorname{Re} \varphi(ba)$$

and

$$\operatorname{Re} \varphi(a^2) \ge 0,$$

thus $0 \le n_-[a^2] \le n_+[a^2]$.

Proof. Since $i(ab - ba) \in H(\mathcal{A})$,

$$i(\varphi(ab) - \varphi(ba)) = \varphi(i(ab - ba)) \in \mathbb{R},$$

hence $\operatorname{Re}(\varphi(ab) - \varphi(ba)) = 0$.
We have

$$\frac{a^2}{2} = \lim_{t \to 0} \frac{\mathbf{1} + ita - e^{ita}}{t^2}.$$

Since $\varphi(\mathbf{1}) = 1$ and $\varphi(ta) \in \mathbb{R}$,

$$\mathrm{Re}\,\varphi\left(\frac{a^2}{2}\right) = \lim_{t\to 0} \frac{1 - \mathrm{Re}\,\varphi(e^{ita})}{t^2}\,.$$

We have $\mathrm{Re}\,\varphi(e^{ita}) \le |\varphi(e^{ita})| \le \|\varphi\|\,\|e^{ita}\| = 1$, therefore $\mathrm{Re}\,\varphi\left(\frac{a^2}{2}\right) \ge 0$. □

Proof of Theorem 1.1. Let $t \in \mathbb{R}$. Then $a + tb \in H(\mathcal{A})$, thus, by Proposition 2.3, we have $\mathrm{Re}\,\varphi((a+tb)^2) \ge 0$ and $\mathrm{Re}\,\varphi(ab) = \mathrm{Re}\,\varphi(ba)$. Hence

$$0 \le \mathrm{Re}\,\varphi(a^2) + 2t\mathrm{Re}\,\varphi(ab) + t^2\mathrm{Re}\,\varphi(b^2) \qquad (t \in \mathbb{R})$$

with $\alpha = \mathrm{Re}\,\varphi(a^2)$, $\beta = \mathrm{Re}\,\varphi(ab)$ and $\gamma = \mathrm{Re}\,\varphi(b^2)$ we have $0 \le \alpha + 2\beta t + \gamma t^2$ for all $t \in \mathbb{R}$. If $\gamma = 0$, then $\alpha + \beta t \ge 0$ for all $t \in \mathbb{R}$, hence $\beta = 0$, and the result follows. Now assume that $\gamma \ne 0$. Thus $\gamma = \mathrm{Re}\ \varphi(b^2) > 0$. Since the quadratic equation $0 = \alpha + 2\beta t + \gamma t^2$ has at most one solution, we get $\beta^2 \le \alpha\gamma$. □

Proof of Theorem 1.2. From Theorem 1.1 and (2.1) we see that

$$(n_+[ab])^2 \le n_+[a^2]n_+[b^2] \qquad (a, b \in H(\mathcal{A}))\,.$$

Hence, with $b = \mathbf{1}$,

$$n_+[a]^2 \le n_+[a^2]\,. \tag{2.3}$$

The definition of $n_+[\cdot]$ gives $n_+[a^2] \le \|a^2\|$. Proposition 2.1 (4) shows that $\|a^2\| = \|a\|^2$ hence

$$n_+[a^2] \le \|a\|^2\,.$$

It remains to show that $\|a\|^2 \le n_+[a^2]$.

Case 1: $n_+[a] \ge -n_-[a]$. By Proposition 2.2 (3),

$$n_+[a] = r(a) = \|a\|\,,$$

thus, by (2.3)

$$\|a\|^2 = n_+[a]^2 \le n_+[a^2]\,.$$

Case 2: $n_+[a] < -n_-[a]$. Let $b = -a$, then $b \in H(\mathcal{A})$ and

$$n_+[b] = -n_-[-b] = -n_-[a] > n_+[a] = -n_-[-a] = -n_-[b]\,.$$

So Case 1 applies to b. Therefore

$$\|a\|^2 = \|b\|^2 = n_+[b^2] = n_+[a^2]\,.$$

□

In **[5]** the following subset of $\mathcal{A}$ is introduced:

$$\mathcal{A}_0 = \{x \in \mathcal{A} : n_+[x] = n_-[x]\}\,.$$

It is easy to see (use Proposition 2.1 (2)) that $x \in \mathcal{A}_0 \Leftrightarrow i(x - \alpha\mathbf{1}) \in H(\mathcal{A})$ for some $\alpha \in \mathbb{R}$.

In **[5]** J. Nieto has shown

$$\|x\| = r(x) \le (2n_+[x]^2 + n_+[-x^2])^{1/2} \qquad (x \in \mathcal{A}_0)\,. \tag{2.4}$$

Now let $a \in H(\mathcal{A})$, then $ia \in \mathcal{A}_0$, hence $n_+[ia] = 0$ and $-(ia)^2 = a^2$, thus by (2.4),

$$\|a\|^2 \le n_+[a^2]\,.$$

Hence (2.4) implies Theorem 1.2. On the other hand Theorem 1.2 implies (2.4):

Let $x \in \mathcal{A}_0$, hence there is $\alpha \in \mathbb{R}$ such that $a := i(x - \alpha\mathbf{1}) \in H(\mathcal{A})$. By Proposition 2.1, $\|x\| = r(x)$. We have $-x^2 = -\alpha^2\mathbf{1} + 2i\alpha a + a^2$, hence for $\varphi \in D(\mathcal{A};\mathbf{1})$

$$\mathrm{Re}\,\varphi(-x^2) = -\alpha^2 + 2\mathrm{Re}\,\varphi(i\alpha a) + \mathrm{Re}\,\varphi(a^2) = \mathrm{Re}\,\varphi(a^2) - \alpha^2 \,.$$

Consequently

$$n_+[-x^2] = n_+[a^2] - \alpha^2 = \|a\|^2 - \alpha^2 \,.$$

Since $n_+[x] = \alpha$, we have

$$2n_+[x]^2 + n_+[-x^2] = \|a\|^2 + \alpha^2 \,. \tag{2.5}$$

Now take $\lambda \in \sigma(x) = \sigma(\alpha\mathbf{1} - ia)$. Then $\lambda = \alpha - i\beta$ with $\beta \in \sigma(a) \subseteq \mathbb{R}$. It follows that

$$|\lambda|^2 = \alpha^2 + \beta^2 \leq \alpha^2 + \|a\|^2 \,,$$

therefore $r(x) \leq \alpha^2 + \|a\|^2$. Now use (2.5) to see that (2.4) holds.

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