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ON ISOLATED POINTS OF THE SPECTRUM OF A BOUNDED LINEAR OPERATOR

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ABSTRACT. For a bounded linear operator A on a Banach space we characterize the isolated points in the spectrum of A , the Riesz points of A , and the poles of the resolvent of A .

1. TERMINOLOGY AND INTRODUCTION

Throughout this paper E will be an infinite-dimensional complex Banach space and A will be a bounded linear operator on E . We denote by $N(A)$ the kernel and by $A(E)$ the range of A . The spectrum of A will be denoted by $\sigma(A)$. The resolvent set $\rho(A)$ of A is the complement of $\sigma(A)$ in the complex plane \mathbb{C} . For any λ in $\rho(A)$ the resolvent operator $(\lambda I - A)^{-1}$ is denoted by $R_\lambda(A)$.

Let λ_0 be an isolated point in $\sigma(A)$. The spectral projection corresponding to λ_0 will be denoted by P_{λ_0} . We have $E = P_{\lambda_0}(E) \oplus N(P_{\lambda_0})$.

In [3] Mbekhta introduced two important subspaces of E :

$$K(A) = \{x \in E : \text{there exist } c > 0 \text{ and a sequence } (x_n)_{n \geq 1} \subseteq E \\ \text{such that } Ax_1 = x, Ax_{n+1} = x_n \text{ for all } n \in \mathbb{N}, \\ \text{and } \|x_n\| \leq c^n \|x\| \text{ for all } n \in \mathbb{N}\},$$

$$H_0(A) = \left\{x \in E : \lim_{n \rightarrow \infty} \|A^n x\|^{1/n} = 0\right\}$$

and proved the following

Theorem 1. *A point $\lambda_0 \in \sigma(A)$ is isolated in $\sigma(A)$ if and only if there is a bounded projection P on E such that*

$$P(E) = H_0(\lambda_0 I - A) \quad \text{and} \quad N(P) = K(\lambda_0 I - A).$$

In the present paper we shall prove that $\lambda_0 \in \sigma(A)$ is an isolated point of $\sigma(A)$ if and only if $K(\lambda_0 I - A)$ is closed and $E = K(\lambda_0 I - A) \oplus H_0(\lambda_0 I - A)$ (where \oplus denotes the algebraically direct sum). This characterization leads to

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a characterization of the poles of the resolvent of A and to a characterization of the Riesz points of A . This will be done in §3 of this paper.

2. PRELIMINARY RESULTS

The operator A is said to have the *single-valued extension property* (SVEP) in $\lambda_0 \in \mathbb{C}$ if for any holomorphic function $f: U \rightarrow E$, where U is a neighbourhood of λ_0 , with $(\lambda I - A)f(\lambda) \equiv 0$, the result is $f(\lambda) \equiv 0$. We say that A has the SVEP if A has the SVEP in each $\lambda \in \mathbb{C}$.

The following theorem collects some results due to Mbekhta (see [4]).

Theorem 2. (a) $A(K(A)) = K(A)$ and $A(H_0(A)) \subseteq H_0(A)$;
 (b) A has the SVEP in λ_0 if $H_0(\lambda_0 I - A)$ is closed;
 (c) A has the SVEP in λ_0 if and only if $K(\lambda_0 I - A) \cap H_0(\lambda_0 I - A) = \{0\}$.

The proof of the next result is immediate.

Proposition 1. Let $x \in H_0(A)$ and define the function g on $\mathbb{C} \setminus \{0\}$ by

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{A^n x}{\lambda^{n+1}}.$$

Then g is holomorphic and $(\lambda I - A)g(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Proposition 2. Let F be a closed subspace of E such that $A(F) = F$. Then $F \subseteq K(A)$.

Proof. Since F is a Banach space and $A(F) = F$, the open mapping theorem shows the existence of a constant $c > 0$ so that

$$(2.1) \quad \text{for each } u \in F \text{ there exists } v \in F \text{ such that} \\ Av = u \quad \text{and} \quad \|v\| \leq c\|u\|.$$

Let $x \in F$. Use (2.1) to construct a sequence $(x_n)_{n \geq 1} \subseteq F$ such that $Ax_1 = x$, $Ax_{n+1} = x_n$, and $\|x_n\| \leq c^n \|x\|$. It follows that $x \in K(A)$. \square

Let us review the classical definitions of ascent and descent. The *ascent* $p(A)$ and the *descent* $q(A)$ are the extended integers given by

$$p(A) = \inf\{n \geq 0 : N(A^n) = N(A^{n+1})\}, \\ q(A) = \inf\{n \geq 0 : A^n(E) = A^{n+1}(E)\}.$$

The infimum over the empty set is taken to be ∞ . It follows from [2, Satz 72.3] that if $p(A)$ and $q(A)$ are both finite then they are equal.

We have the following characterization of the poles of the resolvent of A (see [2, Satz 101.2]):

Theorem 3. The complex number λ_0 is a pole of $R_\lambda(A)$ if and only if $0 < p(\lambda_0 I - A) = q(\lambda_0 I - A) < \infty$. In this case we have

$$P_{\lambda_0}(E) = N((\lambda_0 I - A)^p) \quad \text{and} \quad N(P_{\lambda_0}) = (\lambda_0 I - A)^p(E),$$

where $p = p(\lambda_0 I - A)$ is the order of the pole λ_0 .

The next proposition is a generalization of [1, Theorem 2].

Proposition 3. *Suppose that A has the SVEP in $\lambda_0 = 0$ and $q(A) < \infty$. Then $p(A) = q(A)$.*

Proof. Let $q = q(A)$, $B = A^q$, and $\widehat{E} = E/N(B)$. Since $N(B)$ is closed, \widehat{E} is a Banach space. Let $\widehat{B}: \widehat{E} \rightarrow E$ be the corresponding canonical injection. It is easy to see that the operator $\widehat{B}^{-1}: A^q(E) \rightarrow \widehat{E}$ is closed, thus $A^q(E)$ is the domain of a closed linear operator. Since $A(A^q(E)) = A^q(E)$ and A has the SVEP in 0, [1, Corollary 4] shows that $N(A) \cap A^q(E) = \{0\}$. Use [2, Satz 72.1] to derive $p(A) < \infty$. \square

Corollary 1. *The following assertions are equivalent:*

- (a) 0 is a pole of $R_\lambda(A)$;
- (b) A has the SVEP in 0 and $q(A) < \infty$.

Proof. (a) implies (b). Since 0 is isolated in $\sigma(A)$, A has the SVEP in 0. Theorem 3 shows that $q(A) < \infty$.

(b) implies (a). Proposition 3 and Theorem 3. \square

3. ISOLATED POINTS OF THE SPECTRUM

The starting point of our investigation is

Proposition 4. *Suppose that 0 is an isolated point in $\sigma(A)$. Then*

- (a) $P_0(E) = H_0(A)$;
- (b) $N(P_0) = K(A)$.

Proof. (a) follows from [2, Satz 100.2].

(b) Since 0 is isolated in $\sigma(A)$, $\sigma(A|_{P_0(E)}) = \{0\}$ and $0 \in \varrho(A|_{N(P_0)})$ [2, Satz 100.1]. Then $N(P_0)$ is closed and $A(N(P_0)) = N(P_0)$. Hence, by Proposition 2, $N(P_0) \subseteq K(A)$. By Theorem 2(c), $K(A) \cap H_0(A) = \{0\}$. Therefore,

$$\begin{aligned} K(A) &= K(A) \cap E = K(A) \cap [N(P_0) \oplus P_0(E)] \\ &= N(P_0) + K(A) \cap H_0(A) = N(P_0). \quad \square \end{aligned}$$

Theorem 4. *The following assertions are equivalent:*

- (a) 0 is an isolated point in $\sigma(A)$;
- (b) $K(A)$ is closed and $E = K(A) \oplus H_0(A)$ (\oplus denotes the algebraically direct sum).

Proof. (a) implies (b). Use Proposition 4 or Theorem 1.

(b) implies (a). Since $K(A)$ is closed, $A(K(A)) = K(A)$ (Theorem 2(a)), and $N(A) \subseteq H_0(A)$, the operator $A: K(A) \rightarrow K(A)$ is invertible. Hence there exists $\varepsilon > 0$ such that $\lambda I - A|_{K(A)}$ is invertible if $|\lambda| < \varepsilon$. In particular,

$$(3.1) \quad (\lambda I - A)(K(A)) = K(A) \quad \text{if } |\lambda| < \varepsilon.$$

Since for all $\lambda \neq 0$, $N(\lambda I - A) \subseteq K(A)$, we have

$$(3.2) \quad N(\lambda I - A) = \{0\} \quad \text{if } 0 < |\lambda| < \varepsilon.$$

By Proposition 1, for all $\lambda \neq 0$,

$$(3.3) \quad H_0(A) \subseteq (\lambda I - A)(E).$$

Now, (3.1) and (3.3) imply

$$E = K(A) \oplus H_0(A) \subseteq (\lambda I - A)(E) \quad \text{if } 0 < |\lambda| < \varepsilon.$$

Consequently, $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \varepsilon\} \subseteq \varrho(A)$ and the proof is complete. \square

Now we are in a position to present the announced characterization of the poles of the resolvent of A .

Theorem 5. *The following assertions are equivalent:*

- (a) 0 is a pole of the resolvent of A ;
- (b) A has the SVEP in 0 and $q(A) < \infty$;
- (c) There exists $p \in \mathbb{N}$ such that

$$N(A^p) = H_0(A) \quad \text{and} \quad A^p(E) = K(A);$$

- (d) A has the SVEP in 0 and there exists $p \in \mathbb{N}$ such that $K(A) = A^p(E)$;
- (e) $q(A) < \infty$ and $H_0(A)$ is closed.

Proof. By Corollary 1, (a) and (b) are equivalent.

(a) implies (c). Use Theorem 3 and Proposition 4.

(c) implies (a). By Theorem 3, we have to show that $p(A)$ and $q(A)$ are both finite. Since

$$N(A^{p+1}) \subseteq H_0(A) = N(A^p) \subseteq N(A^{p+1}),$$

we have $p(A) \leq p$. Use Theorem 2(a) to derive $A^{p+1}(E) = A(A^p(E)) = A(K(A)) = K(A) = A^p(E)$. Thus $q(A) \leq p$.

(a) implies (d). Use (b) and (c).

(d) implies (b). As in the proof of “(c) implies (a),” we have $A^p(E) = A^{p+1}(E)$, hence $q(A) < \infty$.

(a) implies (e). Clear.

(e) implies (b). By Theorem 2(b), A has the SVEP in 0 . \square

The remainder of this paper deals with Riesz points and Riesz operators. A complex number λ_0 is called a *Riesz point* of A , if

$$p(\lambda_0 I - A) = q(\lambda_0 I - A) < \infty \quad \text{and} \quad \dim N(\lambda_0 I - A) = \text{codim}(\lambda_0 I - A)(E) < \infty.$$

Note that a Riesz point of A is either a pole of the resolvent (and hence isolated in $\sigma(A)$) or a point in the resolvent set $\varrho(A)$.

Proposition 5. *The complex number $\lambda_0 \in \sigma(A)$ is a Riesz point of A if and only if λ_0 is isolated in $\sigma(A)$ and the corresponding spectral projection is finite dimensional.*

Proof. [2, Satz 105.3]. \square

The next theorem uses the subspaces $K(A)$ and $H_0(A)$ and the SVEP to characterize the Riesz points of A .

Theorem 6. *The following assertions are equivalent:*

- (a) 0 is a Riesz point of A ;
- (b) $K(A)$ is closed, $\dim H_0(A) < \infty$, and $E = K(A) \oplus H_0(A)$, where \oplus denotes the algebraically direct sum;
- (c) $q(A) < \infty$ and $\dim H_0(A) < \infty$;
- (d) $\dim H_0(A) < \infty$ and $K(A) = A^p(E)$ for some $p \in \mathbb{N}$;
- (e) A has the SVEP in 0 , $q(A) < \infty$, and $\dim N(A) < \infty$.

Proof. (a) \Leftrightarrow (b). Proposition 4 and Theorem 4.

(c) \Rightarrow (a). Since $N(A^n) \subseteq N(A^{n+1}) \subseteq H_0(A)$ and $\dim H_0(A) < \infty$, there exists $p \in \mathbb{N}$ such that $\dim N(A^p) = \dim N(A^{p+1}) < \infty$. This gives $N(A^p) = N(A^{p+1})$, thus $p(A) < \infty$. By Theorem 3, 0 is a pole of $R_\lambda(A)$, hence 0 is isolated in $\sigma(A)$. Proposition 4 shows that $\dim P_0(E) < \infty$. Now use Proposition 5.

(a) \Rightarrow (d). Propositions 4 and 5 show that $\dim P_0(E) = \dim H_0(A) < \infty$ and $N(P_0) = K(A)$. Since 0 is a pole of $R_\lambda(A)$, we conclude from Theorem 3 that $K(A) = A^p(E)$ for some $p \in \mathbb{N}$.

(d) \Rightarrow (c). $A^{p+1}(E) = A(K(A)) = K(A) = A^p(E)$, thus $q(A) < \infty$.

(a) \Rightarrow (e). Clear.

(e) \Rightarrow (a). By Proposition 3, $p(A) = q(A) < \infty$. [2, Satz 72.6] shows that $\dim N(A) = \text{codim } A(E) < \infty$, thus 0 is a Riesz point of A . \square

The operator A is called a *Riesz operator* if every $\lambda \in \sigma(A) \setminus \{0\}$ is a Riesz point of A .

An immediate consequence of Theorem 6 is

Theorem 7. *The following assertions are equivalent:*

- (a) A is a Riesz operator;
- (b) $\dim H_0(\lambda I - A) < \infty$, $E = K(\lambda I - A) \oplus H_0(\lambda I - A)$, and $K(\lambda I - A)$ is closed for all $\lambda \in \sigma(A) \setminus \{0\}$;
- (c) $q(\lambda I - A) < \infty$ and $\dim H_0(\lambda I - A) < \infty$ for all $\lambda \in \sigma(A) \setminus \{0\}$;
- (d) $\dim H_0(\lambda I - A) < \infty$ for all $\lambda \in \sigma(A) \setminus \{0\}$ and for each $\lambda \in \sigma(A) \setminus \{0\}$ there exists $p(\lambda) \in \mathbb{N}$ such that $K(\lambda I - A) = (\lambda I - A)^{p(\lambda)}(E)$.

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REFERENCES

1. J. K. Finch, *The single-valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.
2. H. Heuser, *Funktionalanalysis*, 2nd ed., Teubner, Stuttgart, 1986.
3. M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29** (1987), 159–175.
4. —, *Sur la théorie spectrale locale et limite des nilpotents*, Proc. Amer. Math. Soc. **110** (1990), 621–631.

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