PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 2, Pages 511–518
S 0002-9939(04)07619-1
Article electronically published on August 4, 2004

# A NOTE ON MEROMORPHIC OPERATORS

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(Communicated by Joseph A. Ball)

ABSTRACT. Let $X$ be a complex Banach space and $T$ a bounded linear operator on $X$. $T$ is called meromorphic if the spectrum $\sigma(T)$ of $T$ is a countable set, with 0 the only possible point of accumulation, such that all the nonzero points of $\sigma(T)$ are poles of $(\lambda I - T)^{-1}$. By means of the analytical core $K(T)$ we give a spectral theory of meromorphic operators. Our results are a generalization of some results obtained by Gong and Wang (2003).

## 1. Introduction and terminology

Throughout this paper, $X$ will denote an *infinite-dimensional* complex Banach space. By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators on $X$. Let $T \in \mathcal{L}(X)$. The kernel and the range of $T$ will be denoted by $N(T)$ and $T(X)$, respectively. The spectrum, the set of eigenvalues, and the resolvent set of $T$ are denoted by $\sigma(T), \sigma_p(T)$ and $\rho(T)$, respectively. For the resolvent $(\lambda I - T)^{-1}$ we write $R_\lambda(T)$ $(\lambda \in \rho(T))$.

The *nullity* $\alpha(T)$ of $T$ is the dimension of $N(T)$. The *defect* $\beta(T)$ of $T$ is the codimension of $T(X)$. The *ascent* $p(T)$ and the *descent* $q(T)$ are the extended integers given by

$$p(T) = \inf\{n \geq 0 : \ N(T^n) = N(T^{n+1})\},$$
$$q(T) = \inf\{n \geq 0 : \ T^n(X) = T^{n+1}(X)\}.$$

The infimum over the empty set is taken to be $\infty$. It follows from [4, Satz 72.3] that if $p(T)$ and $q(T)$ are both finite, then they are equal. If $\lambda_0$ is an isolated point in $\sigma(T)$, the spectral projection corresponding to $\lambda_0$ will denoted by $P_{\lambda_0}$. We have $X = P_{\lambda_0}(X) \oplus N(P_{\lambda_0})$. From [4, Satz 101.2] we have the following characterization of the poles of $R_\lambda(T)$:

**Theorem 1.** *The complex number $\lambda_0$ is a pole of $R_\lambda(T)$ if and only if $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$. In this case we have*

$$P_{\lambda_0}(X) = N((\lambda_o I - T)^p) \text{ and } N(P_{\lambda_0}) = (\lambda_0 I - T)^p(X),$$

*where $p = p(\lambda_0 I - T)$ is the order of the pole $\lambda_0$, and $\lambda_0 \in \sigma_p(T)$.*

Received by the editors August 15, 2003 and, in revised form, October 20, 2003.
2000 *Mathematics Subject Classification.* Primary 47A10, 47A11.
*Key words and phrases.* Meromorphic operators, analytical core.

We now list various classes of operators, which will be discussed in this note:

$$\begin{aligned}\mathcal{F}(X) &= \{T \in \mathcal{L}(X) : \dim T(X) < \infty\};\\ \mathcal{K}(X) &= \{T \in \mathcal{L}(X) : T \text{ is compact}\};\\ \Phi(X) &= \{T \in \mathcal{L}(X) : \alpha(T) < \infty,\ \beta(T) < \infty\}.\end{aligned}$$

Operators in $\Phi(X)$ are called *Fredholm operators.*

Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. $\lambda$ is called a *Riesz point* of $T$ if

$$\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty \text{ and } p(\lambda I - T) = q(\lambda I - T) < \infty.$$

$T \in \mathcal{L}(X)$ is called a *Riesz operator* if each $\lambda \neq 0$ is a Riesz point of $T$. We denote by $\mathcal{R}(X)$ the class of all Riesz operators in $\mathcal{L}(X)$.

We have the following characterization of Riesz operators (see [4, §105]):

**Theorem 2.** *Let $T \in \mathcal{L}(X)$. Then:*

$$T \in \mathcal{R}(X) \Leftrightarrow \text{ each } \lambda_0 \in \sigma(T)\backslash\{0\} \text{ is an isolated point of } \sigma(T) \text{ and } P_{\lambda_0} \in \mathcal{F}(X).$$

The class $\mathcal{M}(X)$ of *meromorphic operators* is defined as follows:

$$\mathcal{M}(X) = \{T \in \mathcal{L}(X) : \text{ each } \lambda_0 \in \sigma(T)\backslash\{0\} \text{ is a pole of } R_\lambda(T)\}.$$

We have the following inclusions:

$$\mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{R}(X) \subseteq \mathcal{M}(X).$$

Two subclasses of $\mathcal{M}(X)$ are also considered in this note:

$$\mathcal{Q}(X) = \{T \in \mathcal{L}(X) : \sigma(T) = \{0\}\}$$

and

$$\mathcal{M}_0(X) = \{T \in \mathcal{M}(X) : \sigma(T) \text{ is finite}\}.$$

An operator in $\mathcal{Q}(X)$ is called *quasinilpotent.*

In [5] Mbekhta introduced two important subspaces for $T \in \mathcal{L}(X)$: the *analytical core* $K(T)$ of $T$ is defined by

$$\begin{aligned}K(T) = \{x \in X : \ &\text{there exist } c > 0 \text{ and a sequence } (x_n)_{n\geq 1} \text{ in } X \text{ such that}\\ &Tx_1 = x,\ Tx_{n+1} = x_n \text{ and } ||x_n|| \leq c^n ||x|| \text{ for all } n \in \mathbb{N}\}.\end{aligned}$$

Observe that if $Y$ is a closed subspace of $X$ such that $T(Y) = Y$, then $Y \subseteq K(T)$ ([8, Proposition 2]).

The subspace $H_0(T)$, defined by

$$H_0(T) = \{x \in X : \lim_{n\to\infty} ||T^n x||^{1/n} = 0\},$$

is called the *quasinilpotent part* of $T$.

We close the section with the following definition: an operator $T \in \mathcal{L}(X)$ is said to have the *single-valued extension property* (SVEP) in $\lambda_0 \in \mathbb{C}$ if for any holomorphic function $f : U \to X$, where $U$ is a neighbourhood of $\lambda_0$, with $(\lambda I - T)f(\lambda) \equiv 0$ for all $\lambda \in U$, the result is $f(\lambda) \equiv 0$. We say that $T$ has the SVEP if $T$ has the SVEP in each $\lambda \in \mathbb{C}$.

It is clear that each $T \in \mathcal{M}(X)$ has the SVEP. Furthermore, we have $\sigma(T)\backslash\{0\} \subseteq \sigma_p(T)$ if $T \in \mathcal{M}(X)$ (see Theorem 1).

## 2. Preliminary results

In this section we collect some results which we need in the sequel.

**Proposition 1.** *Let $T, S \in \mathcal{L}(X)$.*
(1) $T(K(T)) = K(T)$ *and* $T(H_0(T)) \subseteq H_0(T)$.
(2) $K(T) \subseteq T^n(X)$ *and* $N(T^n) \subseteq H_0(T)$ *for all* $n \in \mathbb{N}$.
(3) $N(\lambda I - T) \subseteq K(T)$ *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.
(4) $H_0(T) \subseteq (\lambda I - T)(X)$ *for all* $\lambda \in \mathbb{C}\backslash\{0\}$.
(5) *If* $TS = ST$, *then* $H_0(T) \subseteq H_0(TS)$.
(6) $0 \in \rho(T) \Longleftrightarrow K(T) = X$ *and* $H_0(T) = \{0\}$.

*Proof.* (1) is shown in [6];
(2) is clear;
(3) if $x \in N(\lambda I - T)$, put $c = |\lambda|^{-1}$ and $x_n = |\lambda|^{-n} x$ ($n \in \mathbb{N}$);
(4) is shown in [8, Proposition 1];
(5) is clear;
(6) follows from (2) and(5). □

**Proposition 2.** *Let $T \in \mathcal{L}(X)$, $\lambda_0 \in \sigma(T)$ and $K(\lambda_0 I - T) = \{0\}$. Then $\lambda_0$ is the only possible isolated point in $\sigma(T)$.*

*Proof.* Corollary 1.3 in [7]. □

**Proposition 3.** *Suppose that $T \in \mathcal{L}(X)$ has the SVEP in $\lambda_0 = 0$.*
(1) *If* $q(T) < \infty$, *then* $p(T) = q(T)$.
(2) $0$ *is a pole of* $R_\lambda(T)$ *if and only if* $0 < q(T) < \infty$.

*Proof.* (1) is shown in [8, Proposition 3].
(2) If 0 is a pole of $R_\lambda(T)$, then $0 < q(T) < \infty$ by Theorem 2. If $0 < q(T) < \infty$, it follows from (1) that $0 < p(T) = q(T) < \infty$: Theorem 2 shows now that 0 is a pole of $R_\lambda(T)$. □

**Proposition 4.** *Let $T \in \mathcal{L}(X)$. $0$ is an isolated point of $\sigma(T)$ if and only if $K(T)$ is closed, $X = K(T) + H_0(T)$ and $K(T) \cap H_0(T) = \{0\}$. In this case,*

$$P_0(X) = H_0(T) \text{ and } N(P_0) = K(T).$$

*Proof.* Proposition 4 and Theorem 4 in [8]. □

**Notation.** If $T \in \mathcal{L}(X)$ and if $Y$ is a $T$-invariant subspace of $X$, then $T|_Y$ means the restriction of $T$ to $Y$.

**Proposition 5.** *Let $T \in \mathcal{L}(X)$ and $\lambda_0 \in \mathbb{C}\backslash\{0\}$. If $\lambda_0$ is an isolated point of $\sigma(T)$, then*

$$H_0(\lambda_0 I - T) \text{ is a closed } T\text{-invariant subspace and } \sigma(T|_{H_0(\lambda_0 I - T)}) = \{\lambda_0\}.$$

*Proof.* By Proposition 1(1) and Proposition 4, $T(H_0(\lambda_0 I - T)) \subseteq H_0(\lambda_0 I - T)$ and $H_0(\lambda_0 I - T) = P_{\lambda_0}(X)$, thus $H_0(\lambda_0 I - T)$ is closed and $T$-invariant. From [4, Satz 100.1] we get $\sigma(T|_{H_0(\lambda_0 I - T)}) = \{\lambda_0\}$. □

The next result generalizes Proposition 2.4 in [7].

**Proposition 6.** *Suppose that $T \in \mathcal{L}(X)$ has the SVEP, $\lambda_0 \in \mathbb{C}\backslash\{0\}$, $\lambda_0 \in \rho(T)$ or $\lambda_0$ is an isolated point of $\sigma(T)$ and that*

$$H_0(\lambda_0 I - T) + H_0(T) = X.$$

*Then $0 \in \rho(T)$ or $0$ is an isolated point of $\sigma(T)$.*

*Proof.* Put $Y = H_0(\lambda_0 I - T)$. If $\lambda_0 \in \rho(T)$, then $Y = \{0\}$ (by Proposition 1(6)); thus,

$$(\lambda I - T)(Y) = Y \text{ for all } \lambda \in \mathbb{C}.$$

If $\lambda_0 \in \sigma(T)$, then, by Proposition 5, there exists $\rho > 0$ such that

$$(\lambda I - T)(Y) = Y \text{ for } |\lambda| < \rho.$$

Therefore we have in both cases that there is some $\rho > 0$ with $(\lambda I - T)(Y) = Y$ for $|\lambda| < \rho$.

Now take $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \rho$. Then

$$H_0(\lambda I - T) = Y \subseteq (\lambda I - T)(X),$$

thus $X = H_0(\lambda_0 I - T) + H_0(T) \subseteq (\lambda I - T)(X) + H_0(T)$.

Since $H_0(T) \subseteq (\lambda I - T)(X)$ (by Proposition 1(4)),

$$X = (\lambda I - T)(X),$$

therefore $q(\lambda I - T) = 0$ for $0 < |\lambda| < \rho$. Since $T$ has the SVEP, we get from Proposition 3(1) that $p(\lambda I - T) = q(\lambda I - T) = 0$ for $0 < |\lambda| < \rho$. Hence $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \rho\} \subseteq \rho(T)$. □

**Corollary 1.** *Suppose that $T \in \mathcal{L}(X)$ has the SVEP, $\lambda_0 \in \mathbb{C}\backslash\{0\}$, $\lambda_0 \in \rho(T)$ or $\lambda_0$ is an isolated point of $\sigma(T)$ and that*

$$H_0(T) = K(\lambda_0 I - T).$$

*Then $0 \in \rho(T)$ or $0$ is an isolated point of $\sigma(T)$.*

*Proof.* If $\lambda_0 \in \rho(T)$, then $K(\lambda_0 I - T) = X$ and $H_0(\lambda_0 I - T) = \{0\}$ (Proposition 1(6)). Thus

$$X = K(\lambda_0 I - T) + H(\lambda_0 I - T),$$

hence

$$X = H_0(T) + H_0(\lambda_0 I - T).$$

If $\lambda_0 \in \sigma(T)$, then, by Proposition 4,

$$X = K(\lambda_0 I - T) + H(\lambda_0 I - T),$$

therefore

$$X = H_0(T) + H_0(\lambda_0 I - T).$$

Thus we have in both cases that $X = H_0(T) + H_0(\lambda_0 I - T)$. Now use Proposition 6. □

*Remark.* Corollary 1 generalizes [7, Corollary 2.5].

## 3. Meromorphic operators

In this section we present the main results of this paper. The first result deals with Riesz operators and generalizes Theorem 2.6 in [7].

**Theorem 3.** *Let $T \in \mathcal{R}(X)$. The following assertions are equivalent:*
(1) $0$ *is a pole of* $R_\lambda(T)$*;*
(2) *there exists* $q \in \mathbb{N}$ *such that* $T^q \in \mathcal{F}(X)$*;*
(3) *there exists* $n \in \mathbb{N}$ *with* $K(T) = T^n(X)$*;*
(4) $q(T) < \infty$.

*Proof.* (1) $\Leftrightarrow$ (2): [4, Aufgabe 105.2].

(2) $\Rightarrow$ (3): Since $T^{q+k}(X) \subseteq T^q(X)$ for $k \geq 0$ and $\dim T^q(X) < \infty$, we get $q \leq q(T) < \infty$. Put $n = q(T)$. Then $\dim T^n(X) < \infty$, hence $T^n(X)$ is closed. Furthermore $T(T^n(X)) = T^{n+1}(X) = T^n(X)$. Proposition 2 in [8] implies now that $T^n(X) \subseteq K(T)$. Therefore $K(T) = T^n(X)$, by Proposition 1(2).

(3) $\Rightarrow$ (4): From $T^{n+1}(X) = T(T^n(X)) = T(K(T))$ and $T(K(T)) = K(T)$ (Proposition 1(1)) we derive $T^{n+1}(X) = T^n(X)$, thus $q(T) \leq n < \infty$.

(4) $\Rightarrow$ (1): Since $T$ has the SVEP, it follows from Proposition 3(2) that 0 is a pole of $R_\lambda(T)$. □

*Remark.* The above proof shows that if $T \in \mathcal{L}(X)$ has the SVEP in $\lambda_0 = 0$ and if $0 \in \sigma(T)$, then the assertions (1), (3) and (4) in Theorem 3 are equivalent (for the implication (1) $\Rightarrow$ (3) use Theorem 1 and Proposition 4).

Our next result generalizes Theorem 2.1 in [7].

**Theorem 4.** *Let $T \in \mathcal{M}(X)$. Then:*

$$0 \in \rho(T) \text{ or } 0 \text{ is an isolated point of } \sigma(T) \Leftrightarrow K(T) \text{ is closed.}$$

*Proof.* "$\Rightarrow$": Proposition 1(6) and Proposition 4 show that $K(T)$ is closed if $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$.

"$\Leftarrow$": *Case 1:* $K(T) = \{0\}$. Proposition 1(6) shows that $0 \in \sigma(T)$. Proposition 2 implies then that 0 is the only possible isolated point of $\sigma(T)$. Since $T \in \mathcal{M}(X)$ we get $\sigma(T) = \{0\}$ (hence $T \in \mathcal{Q}(X)$).

*Case 2:* $K(T) \neq \{0\}$. Since $K(T)$ is closed, $K(T)$ is a Banach space. Put $T_0 := T|_{K(T)}$ and $I_0 = I|_{K(T)}$. Use Proposition 1(1) to get

$$T_0 \in \mathcal{L}(K(T)) \text{ and } q(T_0) = 0.$$

Since $T$ has the SVEP, $T_0$ has the SVEP.

From Proposition 3(1) we therefore derive $p(T_0) = q(T_0) = 0$, hence $0 \in \rho(T_0)$. Thus there is $\rho > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda| < \rho\} \subseteq \rho(T_0)$. Now take $\lambda \in \mathcal{C}$ with $0 < |\lambda| < \rho$. Then $N(\lambda I - T) \subseteq K(T)$ (Proposition 1(3)), thus

$$N(\lambda I - T) = N(\lambda I_0 - T_0) = \{0\},$$

hence $\lambda \notin \sigma_p(T)$. Since $\lambda \neq 0$ and $T \in \mathcal{M}(X)$, $\lambda \notin \sigma(T)$. Therefore $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \rho\} \subseteq \rho(T)$. □

We proceed with a corollary that generalizes Corollary 2.2 in [7].

**Corollary 2.** *Let $T \in \mathcal{M}(X)$. Then*
(1) $K(T) = \{0\} \Leftrightarrow T \in Q(X)$*;*
(2) $K(T)$ *is closed and* $K(T) \neq \{0\} \Leftrightarrow T \in \mathcal{M}_0(X) \backslash \mathcal{Q}(X)$*;*
(3) $K(T)$ *is not closed* $\Leftrightarrow T \notin \mathcal{M}_0(X)$.

*Proof.* (1) We have seen in the proof of Theorem 4 that $K(T) = \{0\}$ implies $T \in \mathcal{Q}(X)$.

Now let $T \in \mathcal{Q}(X)$. Remarque 1.1 in [6] shows that $H_0(T) = X$. Since $H_0(T) \cap K(T) = \{0\}$, by Proposition 4, we derive $K(T) = \{0\}$.

(2) "$\Rightarrow$": (1) gives $T \notin \mathcal{Q}(X)$. Theorem 4 shows that $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$. Therefore, since $T \in \mathcal{M}(X)$, $\sigma(T)$ is finite, hence $T \in \mathcal{M}_0(X)$.

"$\Leftarrow$": From (1) we get $K(T) \neq \{0\}$. Since $\sigma(T)$ is finite, we see that $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$. Theorem 4 implies then that $K(T)$ is closed.

(3) "$\Rightarrow$": By Theorem 4, $0 \in \sigma(T)$ and 0 is not an isolated point of $\sigma(T)$, thus $T \notin \mathcal{M}_0(X)$.

"$\Leftarrow$": Since $T \in \mathcal{M}(X)\backslash\mathcal{M}_0(X)$, 0 is a point of accumulation of $\sigma(T)$, thus $K(T)$ is not closed by Theorem 4. □

We denote by $X^*$ the dual space of $X$ and by $T^*$ the adjoint of $T \in \mathcal{L}(X)$.

**Proposition 7.** *Let $T \in \mathcal{L}(X)$, and suppose that $T$ and $T^*$ have the SVEP in* 0. *Then:*

$$T \in \Phi(X) \;\Leftrightarrow\; 0 \textit{ is a Riesz point of } T.$$

*Proof.* The implication "$\Leftarrow$" follows from the definition of a Riesz point.

Now suppose that $T \in \Phi(X)$. Since $T$ has the SVEP in 0, it follows from [3, Theorem 15] that $p(T) < \infty$. Satz 82.1 in [4] gives $T^* \in \Phi(X^*)$. Since $T^*$ has the SVEP in 0, we have $q(T) < \infty$ by [3, Corollary 16]. Hence $p(T) = q(T) < \infty$. Satz 72.5 in [4] implies now that $\alpha(T) = \beta(T)$. □

**Corollary 3.** *For $T \in \mathcal{M}(X)$ the following assertions are equivalent:*

(1) $K(T)$ *is closed and* $\operatorname{codim} K(T) < \infty$*;*

(2) $K(T)$ *is closed and* $\dim H_0(T) < \infty$*;*

(3) 0 *is a Riesz point of $T$;*

(4) $T \in \Phi(X)$.

*Proof.* Since $T \in \mathcal{M}(X)$ and $\sigma(T^*) = \sigma(T)$, $T$ and $T^*$ have the SVEP. Proposition 7 shows then that (3) and (4) are equivalent.

Now suppose that $K(T)$ is closed. By Theorem 4, $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$. Now use Proposition 1(6) and Proposition 4 to derive

$$X = K(T) + H_0(T) \text{ and } K(T) \cap H_0(T) = \{0\}.$$

Hence $\dim H_0(T) = \operatorname{codim} K(T)$. Therefore (1) and (2) are equivalent.

Now we show that (2) implies (3): By Theorem 4, $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$. If $0 \in \rho(T)$, then 0 is a Riesz point of $T$. Hence suppose that $0 \in \sigma(T)$. By Proposition 4, $P_0(X) = H_0(T)$, thus $P_0 \in \mathcal{F}(X)$. [4, Satz 105.2] shows now that 0 is a Riesz point of $T$.

It remains to show that (3) implies (2):

*Case 1:* $0 \in \rho(T)$. By Proposition 1(6), $K(T) = X$ and $H_0(T) = \{0\}$. Hence $K(T)$ is closed and $\dim H_0(T) < \infty$.

*Case 2:* $0 \in \sigma(T)$. Since 0 is a Riesz point of $T$, 0 is an isolated point of $\sigma(T)$ and $P_0 \in \mathcal{F}(X)$, by Satz 105.2 in [4]. From Proposition 4 and Theorem 4 it follows that $\dim H_0(T) = \dim P_0(X) < \infty$ and that $K(T)$ is closed. □

**Corollary 4.** *For $T \in \mathcal{M}(X)$ the following assertions are equivalent:*

(1) $\dim K(T) < \infty$*;*

(2) $T \in \mathcal{R}(X) \cap \mathcal{M}_0(X)$.

*Proof.* (1) ⇒ (2): Since $\dim X = \infty$ and $\dim K(T) < \infty$, it follows from Proposition 1(6) that $0 \in \sigma(T)$. Corollary 2 shows that $T \in \mathcal{M}_0(X)$.

Now take $\lambda \in \mathbb{C}\backslash\{0\}$. Since $T \in \mathcal{M}(X)$, $\lambda \in \rho(T)$ or $\lambda$ is a pole of $R_\lambda(T)$, thus $p(\lambda I - T) = q(\lambda I - T) < \infty$. By Proposition 1(3), $N(\lambda I - T) \subseteq K(T)$, thus $\alpha(\lambda I - T) < \infty$. Satz 72.5 in [4] implies now that

$$\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty.$$

Therefore $\lambda$ is a Riesz point of $T$. Since $\lambda \in \mathcal{C}\backslash\{0\}$ was arbitrary, $T \in \mathcal{R}(X)$.

(2) ⇒ (1): We can assume that $K(T) \neq \{0\}$. Since $T \in \mathcal{M}_0(X)$ and $0 \in \sigma(T)$ (see [4, Aufgabe 105.2]), 0 is an isolated point of $\sigma(T)$. Hence $K(T)$ is closed (Theorem 4). Put $T_0 = T|_{K(T)}$. By Proposition 1(1), $T(K(T)) = K(T)$, thus $T_0 \in \mathcal{L}(K(T))$. From Proposition 4 we get $K(T) = N(P_0)$. Now use Satz 100.1 in [4] to derive

$$\sigma(T_0) = \sigma(T)\backslash\{0\},$$

thus $0 \notin \sigma(T_0)$. Since $T \in \mathcal{R}(X)$ it follows from [4, Satz 105.6] that $T_0 \in \mathcal{R}(K(T))$. Now assume that $\dim K(T) = \infty$. Thus, by [4, Aufgabe 105.2] $0 \in \sigma(T_0)$, a contradiction. Hence $\dim K(T) < \infty$. □

## 4. Final remarks

**1.** The proof of Theorem 4 shows that the following result is valid.

**Theorem 5.** *Suppose that $T \in \mathcal{L}(X)$ has the SVEP in 0 and that there is a sequence $(\lambda_n)$ in $\sigma_p(T)$ with $\lambda_n \neq 0$ for all $n \in \mathbb{N}$ and $\lambda_n \to 0$ as $n \to \infty$. Then $K(T)$ is not closed.*

That the condition "$T$ has the SVEP in 0" cannot be dropped in Theorem 5 shows the example of the unilateral left shift on $l^2(\mathbb{N})$:

*Example.* Let $X = l^2(\mathbb{N})$, and define the operator $T \in \mathcal{L}(X)$ by

$$T(\xi_1, \xi_2, \xi_3, \dots) = (\xi_2, \xi_3, \dots).$$

It is well known that $\sigma_p(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Since $T(X) = X$, we have $K(T) = X$, by [8, Proposition 2]. Thus $K(T)$ is closed. Example 1.7 in [2] shows that $T$ does not have the SVEP in 0.

**2.** In [1] W. Bouamama proves independently some of the results of our paper.

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