

**On the norm-closure of the
class of hypercyclic operators**

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Abstract. Let T be a bounded linear operator acting on a complex, separable, infinite-dimensional Hilbert space and let $f : D \rightarrow \mathbb{C}$ be an analytic function defined on an open set $D \subseteq \mathbb{C}$ which contains the spectrum of T . If T is the limit of hypercyclic operators and if f is nonconstant on every connected component of D , then $f(T)$ is the limit of hypercyclic operators if and only if $f(\sigma_W(T)) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected, where $\sigma_W(T)$ denotes the Weyl spectrum of T .

1. Terminology and introduction. In this note X always denotes a complex, infinite-dimensional Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X . We write $\mathcal{K}(X)$ for the ideal of all compact operators on X . For $T \in \mathcal{L}(X)$ the spectrum of T is denoted by $\sigma(T)$. The reader is referred to [5] for the definitions and properties of Fredholm operators, semi-Fredholm operators and the index $\text{ind}(T)$ of a semi-Fredholm operator T in $\mathcal{L}(X)$. For $T \in \mathcal{L}(X)$ we will use the following notations:

$$\begin{aligned}\varrho_F(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is Fredholm}\}, \\ \varrho_{s-F}(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-Fredholm}\}, \\ \varrho_W(T) &= \{\lambda \in \varrho_F(T) : \text{ind}(\lambda I - T) = 0\}, \\ \sigma_0(T) &= \{\lambda \in \sigma(T) : \lambda \text{ is isolated in } \sigma(T), \text{ and } \lambda \in \varrho_F(T)\}, \\ \sigma_F(T) &= \mathbb{C} \setminus \varrho_F(T), \quad \sigma_{s-F}(T) = \mathbb{C} \setminus \varrho_{s-F}(T), \\ \sigma_W(T) &= \mathbb{C} \setminus \varrho_W(T) \quad (\text{Weyl spectrum}), \\ \text{Hol}(T) &= \{f : D(f) \rightarrow \mathbb{C} : D(f) \text{ is open, } \sigma(T) \subseteq D(f), \\ &\quad f \text{ is holomorphic}\}, \\ \widetilde{\text{Hol}}(T) &= \{f \in \text{Hol}(T) : f \text{ is nonconstant on every connected} \\ &\quad \text{component of } D(f)\}.\end{aligned}$$

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For $f \in \text{Hol}(T)$, the operator $f(T)$ is defined by the well known analytic calculus (see [5]).

If X is separable, then $T \in \mathcal{L}(X)$ is called *hypercyclic* if $\{x, Tx, T^2x, \dots\}$ is dense in X for some $x \in X$. We denote by $\mathcal{HC}(X)$ the class of all hypercyclic operators in $\mathcal{L}(X)$. The following simple spectral description of the norm-closure $\mathcal{HC}(X)^-$ is due to D. A. Herrero [3], Theorem 2.1:

THEOREM 1. *If X is a separable Hilbert space, then $A \in \mathcal{HC}(X)^-$ if and only if A satisfies the conditions*

- (1) $\sigma_W(A) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected,
- (2) $\sigma_0(A) = \emptyset$, and
- (3) $\text{ind}(\lambda I - A) \geq 0$ for all $\lambda \in \varrho_{s-F}(A)$.

Furthermore, $\mathcal{HC}(X)^- + \mathcal{K}(X) = \{A \in \mathcal{L}(X) : A \text{ satisfies (1) and (3)}\}$.

The main result of the present note reads as follows:

THEOREM 2. *Let X be a separable Hilbert space, $T \in \mathcal{HC}(X)^-$ and let $f \in \widetilde{\text{Hol}}(T)$. Then the following assertions are equivalent:*

- (1) $f(T) \in \mathcal{HC}(X)^-$.
- (2) $f(T) \in \mathcal{HC}(X)^- + \mathcal{K}(X)$.
- (3) $f(\sigma_W(T)) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected.

As an immediate consequence we have:

COROLLARY. *Let X, T and f be as in Theorem 2. If $\sigma_W(T)$ is connected and $|f(\lambda_0)| = 1$ for some $\lambda_0 \in \sigma_W(T)$, then $f(T) \in \mathcal{HC}(X)^-$.*

A result closely related to the above corollary can be found in [4], Theorem 2.

The proof of Theorem 2 will be given in Section 3 of this paper. For this proof we need some preliminary results, which we collect in Section 2. Many of these preliminary results can be found in [1], Section 3, in the Hilbert space case.

2. Preliminary results. In this section X will denote an arbitrary complex Banach space.

PROPOSITION 1. *Let $T \in \mathcal{L}(X)$ and $f \in \text{Hol}(T)$.*

- (1) $f(\sigma_F(T)) = \sigma_F(f(T))$.
- (2) $f(\sigma_{s-F}(T)) \subseteq \sigma_{s-F}(f(T))$ (if f is univalent, we have equality).
- (3) If $f \in \text{Hol}(T)$, then $\sigma_0(f(T)) \subseteq f(\sigma_0(T))$.
- (4) If $\text{ind}(\lambda I - T) \geq 0$ for all $\lambda \in \varrho_F(T)$ or $\text{ind}(\lambda I - T) \leq 0$ for all $\lambda \in \varrho_F(T)$, then

$$\sigma_W(f(T)) = f(\sigma_W(T)).$$

Proof. (1) $\sigma_F(T)$ is the spectrum of $T + \mathcal{K}(X)$ in the Banach algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Hence the spectral mapping theorem holds for $\sigma_F(T)$.

(2) See [6], Corollary 1, or [2], Theorem 1.

(3) Let $\mu_0 \in \sigma_0(f(T))$; thus μ_0 is an isolated point in $\sigma(f(T)) = f(\sigma(T))$ and $\mu_0 \in \rho_F(f(T))$. We have $\mu_0 = f(\lambda_0)$ for some $\lambda_0 \in \sigma(T)$. By (1), $\lambda_0 \in \rho_F(T)$. Let C denote the connected component of $D(f)$ which contains λ_0 . Assume that λ_0 is not isolated in $\sigma(T)$, thus there is a sequence (λ_n) in $C \cap \sigma(T)$ such that $\lambda_n \rightarrow \lambda_0$ and $\lambda_n \neq \lambda_0$ for all $n \in \mathbb{N}$. This gives $f(\lambda_n) \rightarrow f(\lambda_0) = \mu_0$ ($n \rightarrow \infty$). Since $f(\lambda_n) \in f(\sigma(T)) = \sigma(f(T))$ and μ_0 is isolated in $\sigma(f(T))$, we derive $f(\lambda_n) = \mu_0$ for all n . By the uniqueness theorem for analytic functions, it follows that $f(\lambda) = \mu_0$ for all $\lambda \in C$, a contradiction. Thus λ_0 is an isolated point in $\sigma(T)$. Since $\lambda_0 \in \rho_F(T)$, we get $\lambda_0 \in \sigma_0(T)$, hence $\mu_0 = f(\lambda_0) \in f(\sigma_0(T))$.

(4) follows from [8], Theorem 3.6. ■

Remark. In general, the spectral mapping theorem for the Weyl spectrum $\sigma_W(T)$ does not hold (see [2], p. 23, or [8], Example 3.3).

NOTATIONS. For $T \in \mathcal{L}(X)$, we write $\alpha(T)$ for the dimension of the kernel of T and $\beta(T)$ for the co-dimension of the range of T . Thus, if T is semi-Fredholm,

$$\text{ind}(T) = \alpha(T) - \beta(T) \in \mathbb{Z} \cup \{-\infty, +\infty\}.$$

According to C. Pearcy [7], the next proposition has already appeared in the preprint *Fredholm operators* by P. R. Halmos in 1967. For the convenience of the reader we shall include a proof.

PROPOSITION 2. *If T and S are semi-Fredholm operators with $\alpha(T)$ and $\alpha(S)$ finite [resp. $\beta(T)$ and $\beta(S)$ finite], then TS is a semi-Fredholm operator with $\alpha(TS) < \infty$ [resp. $\beta(TS) < \infty$] and*

$$\text{ind}(TS) = \text{ind}(T) + \text{ind}(S).$$

Proof. It suffices to consider the case where $\alpha(T), \alpha(S) < \infty$.

Case 1: T and S are Fredholm operators. Then it is well known that TS is Fredholm and $\text{ind}(TS) = \text{ind}(T) + \text{ind}(S)$ (see [5], §71).

Case 2: T or S is not Fredholm. Thus $\beta(T) = \infty$ or $\beta(S) = \infty$. Use [5], §82, Aufgaben 2, 4, to get: TS is semi-Fredholm, $\alpha(TS) < \infty$, $\beta(TS) = \infty$. Hence $\text{ind}(TS) = -\infty = \text{ind}(T) + \text{ind}(S)$. ■

PROPOSITION 3. *Let $T \in \mathcal{L}(X)$ satisfy*

$$\sigma_0(T) = \emptyset \quad \text{and} \quad \text{ind}(\lambda I - T) \geq 0 \quad \text{for all } \lambda \in \rho_{s-F}(T).$$

If $f \in \widetilde{\text{Hol}}(T)$ then we have

- (1) $\sigma_0(f(T)) = \emptyset$,
 (2) $\text{ind}(\mu I - f(T)) \geq 0$ for all $\mu \in \varrho_{s-F}(f(T))$.

Proof. (1) follows from Proposition 1(3).

(2) Take $\mu_0 \in \varrho_{s-F}(f(T))$ and put $g(\lambda) = \mu_0 - f(\lambda)$. If g has no zeroes in $\sigma(T)$, then $g(T) = \mu_0 I - f(T)$ is invertible in $\mathcal{L}(X)$, thus $\text{ind}(\mu_0 I - f(T)) = 0$. If g has zeroes in $\sigma(T)$, then g has only a finite number of zeroes in $\sigma(T)$, since $f \in \widetilde{\text{Hol}}(T)$. Let $\lambda_1, \dots, \lambda_k$ be those zeroes and ν_1, \dots, ν_k their respective orders. Then we have

$$g(\lambda) = h(\lambda) \prod_{j=1}^k (\lambda_j - \lambda)^{\nu_j}$$

with $h \in \text{Hol}(T)$ and $h(\lambda) \neq 0$ for all $\lambda \in \sigma(T)$. Therefore $h(T)$ is invertible and

$$g(T) = h(T) \prod_{j=1}^k (\lambda_j I - T)^{\nu_j}.$$

Since $0 \in \varrho_{s-F}(g(T))$, we get, by Proposition 1(2),

$$\lambda_1, \dots, \lambda_k \in \varrho_{s-F}(T).$$

Since $\text{ind}(\lambda_j I - T) \geq 0$, we have

$$\beta(\lambda_j I - T) < \infty \quad \text{for } j = 1, \dots, k.$$

Thus by Proposition 2 (recall that $\beta(h(T)) = 0 < \infty$),

$$\begin{aligned} \text{ind}(\mu_0 I - f(T)) &= \text{ind}(g(T)) \\ &= \underbrace{\text{ind}(h(T))}_{=0} + \sum_{j=1}^k \nu_j \underbrace{\text{ind}(\lambda_j I - T)}_{\geq 0} \geq 0. \quad \blacksquare \end{aligned}$$

Remark. The description of the index in [1], Theorem 3.7, sheds more light on claim (4) of Proposition 1 and on claim (2) of Proposition 3 in the Hilbert space case.

3. Proof of Theorem 2. (1) \Rightarrow (2). Clear.

(3) \Rightarrow (1). Use Proposition 1(4), Proposition 3 and Theorem 1.

(2) \Rightarrow (3). By Theorem 1, $\sigma_w(f(T)) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected. Use again Proposition 1(4) to derive (3). \blacksquare

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