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ON NORMAL OPERATOR EXPONENTIALS

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Abstract. Suppose that A and B are bounded normal operators on a complex Hilbert space and that $e^A e^B = e^B e^A$. In this paper some conditions implying $AB = BA$ are given.

1. Terminology and results

Throughout this paper let $\mathcal{H}$ denote a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the Banach algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$ the spectrum and the spectral radius of A are denoted by $\sigma(A)$ and $r(A)$, respectively. The set of eigenvalues of A is denoted by $\sigma_p(A)$. For the resolvent set of A we write $\rho(A)$. We use $N(A)$ and $A(\mathcal{H})$ to denote the kernel and the range of A, respectively.

We say that $\sigma(A)$ is *$2\pi i$-congruence-free* if

$$\sigma(A) \cap \sigma(A + 2j\pi i) = \emptyset \text{ for } j = \pm 1, \pm 2, \dots .$$

If $A \in \mathcal{L}(\mathcal{H})$ is normal and has the spectral resolution

$$A = \int_{\sigma(A)} \lambda dE(\lambda) , \tag{1.1}$$

let $E(\Omega)$ denote the associated projection measure defined on the Borel subsets $\Omega \subseteq \sigma(A)$. It is convenient to think of E as being defined for all Borel sets in $\mathbb{C}$: put $E(\Omega) = E(\Omega \cap \sigma(A))$.

Definition. Let $A \in \mathcal{L}(\mathcal{H})$ be normal with the spectral resolution (1.1). We say that $\sigma(A)$ is *generalized $2\pi i$-congruence-free* if

$$E(\sigma(A) \cap \sigma(A + 2j\pi i)) = 0 \text{ for } j = 1, 2, \dots . \tag{1.2}$$

Remarks. Let $A \in \mathcal{L}(\mathcal{H})$ be normal with the spectral resolution (1.1).

(1) Since $E(\emptyset) = 0$, it is clear that if $\sigma(A)$ is $2\pi i$-congruence-free, then $\sigma(A)$ is generalized $2\pi i$-congruence-free.

(2) Let n denote the smallest integer with $n > \frac{r(A)}{\pi}$. Then $\sigma(A) \cap \sigma(A + 2j\pi i) = \emptyset$ for $j \geq n$. Thus (1.2) holds if and only if

$$E(\sigma(A) \cap \sigma(A + 2j\pi i)) = 0 \text{ for } j = 1, 2, \dots, n - 1 .$$

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Examples. Let $A \in \mathcal{L}(\mathcal{H})$ be normal with the spectral resolution (1.1).

(1) If $r(A) < \pi$, then (1.2) holds.

(2) Let $\pi \le r(A) < 2\pi$. By Remark (2), (1.2) holds if and only if

$$E(\sigma(A) \cap \sigma(A + 2\pi i)) = 0 \, .$$

(3) Let $2\pi \le r(A) < 3\pi$. By Remark (2), (1.2) holds if and only if

$$E(\sigma(A) \cap \sigma(A + 2\pi i)) = E(\sigma(A) \cap \sigma(A + 4\pi i)) = 0 \, .$$

(4) Let $j \in \mathbb{N}$ and suppose that there is some $\mu \in \mathbb{C}$ with $\sigma(A) \cap \sigma(A + 2j\pi i) \subseteq \{\mu\}$ and $\mu \notin \sigma_p(A)$. Since $E(\{\mu\}) = 0$ ([4, Theorem 12.29]), it follows that

$$E(\sigma(A) \cap \sigma(A + 2j\pi i)) = 0.$$

The following result is due to E. M. E. Wermuth ([8]). See [6] for a very short proof.

Theorem 1.1. *Let $A, B \in \mathcal{L}(\mathcal{H})$ and suppose that $\sigma(A)$, $\sigma(B)$ are $2\pi i$-congruence-free. If $e^A e^B = e^B e^A$, then $AB = BA$.*

If $\sigma(A)$ or $\sigma(B)$ is not $2\pi i$-congruence-free, then, in general, $e^A e^B = e^B e^A$ does not imply that $AB = BA$. For examples see [6], [7] and [8].

But for normal operators we can say more. Now we state the main results of this paper. Proofs will be given in Section 3 of this paper.

Theorem 1.2. *Suppose that A and B are normal operators in $\mathcal{L}(\mathcal{H})$ and suppose that $e^A e^B = e^B e^A$.*

(a) *If $\sigma(A)$ is generalized $2\pi i$-congruence-free, then $Ae^B = e^B A$.*

(b) *If $\sigma(A)$ and $\sigma(B)$ are generalized $2\pi i$-congruence-free, then $AB = BA$.*

Theorem 1.3. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If $A + B$ is normal, if $\sigma(A + B)$ is generalized $2\pi i$-congruence-free and if*

$$e^A e^B = e^{A+B} = e^B e^A,$$

then $AB = BA$.

We introduce the following notation: for $B \in \mathcal{L}(\mathcal{H})$ write $\{B\}^c$ for the set

$$\{T \in \mathcal{L}(\mathcal{H}) : TB = BT\} \, .$$

The set $\{B\}^{cc}$ is defined by

$$\{B\}^{cc} = \{S \in \mathcal{L}(\mathcal{H}) : ST = TS \text{ for all } T \in \{B\}^c\} \, .$$

Theorem 1.4. *Let $A \in \mathcal{L}(\mathcal{H})$ be normal and suppose that $\sigma(A)$ is generalized $2\pi i$-congruence-free. If $B \in \mathcal{L}(\mathcal{H})$ and $e^A = e^B$, then $A \in \{B\}^{cc}$ (and so $AB = BA$).*

Theorem 1.5. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If A is selfadjoint, $\sigma(A) \subseteq [-\pi, \pi]$ and $e^{iA} = e^B$, then we have*

(a) *$B^* = -B$ if B is normal,*

(b) *$A \in \{B\}^{cc}$ (and so $AB = BA$), if $-\pi \notin \sigma_p(A)$ or $\pi \notin \sigma_p(A)$.*

Observe that part (b) of Theorem 1.5 is a generalization of [5, Satz 4 (b)].

2. Preparations

In order to prove the theorems in Section 1 we need some supplementary results. Let $A \in \mathcal{L}(\mathcal{H})$. The map $\delta_A : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, defined by

$$\delta_A(C) = CA - AC \ (C \in \mathcal{L}(\mathcal{H})),$$

is called the *inner derivation* determined by A. It is clear that δ_A is a bounded linear operator on $\mathcal{L}(\mathcal{H})$ with $\|\delta_A\| \le 2\|A\|$. It is shown in [1] that

$$\sigma(\delta_A) = \{\lambda - \mu : \lambda, \mu \in \sigma(A)\} \ . \tag{2.1}$$

From [2, Proposition 6.4.8] it follows that

$$e^{\delta_A}(C) = e^{-A} C e^{A} \text{ for all } C \in \mathcal{L}(\mathcal{H}) \ . \tag{2.2}$$

Throughout this paper let f denote the entire function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0 \ , \\ 1, & \text{if } z = 0 \ . \end{cases}$$

From $zf(z) = f(z)z = e^z - 1$ and (2.2) we get

$$f(\delta_A)(\delta_A(C)) = e^{-A} C e^{A} - C \text{ for all } C \in \mathcal{L}(\mathcal{H}) \ . \tag{2.3}$$

Proposition 2.1. *Let $A \in \mathcal{L}(\mathcal{H})$, let n denote the smallest integer with $n > \frac{r(A)}{\pi}$ and let*

$$M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\} \ .$$

(a) *If $\sigma(A)$ is $2\pi i$-congruence-free, then $M_A = \emptyset$.*
(b) *If $M_A = \emptyset$, then $f(\delta_A)$ is an invertible operator on $\mathcal{L}(\mathcal{H})$.*
(c) *If $\lambda \in M_A$, then λ is a simple zero of f and there is $j \in \mathbb{Z}\setminus\{0\}$ with $\lambda = 2j\pi i$.*
(d) *M_A has at most $2(n-1)$ elements,*

$$M_A \subseteq \{\pm 2\pi i, \ \pm 4\pi i, \dots, \pm 2(n-1)\pi i\} \ .$$

(e) *If $M_A \neq \emptyset$ and $M_A = \{\lambda_1, \dots, \lambda_k\}$ with $k \le 2(n-1)$ and $\lambda_j \neq \lambda_l$ for $j \neq l$, then*

$$N(f(\delta_A)) = N(\delta_A - \lambda_1) \oplus \cdots \oplus N(\delta_A - \lambda_k)$$

and $C^n = 0$ for each $C \in N(\delta_A - \lambda_j)$.

Proof. (a) clear. (b) is valid, since $f(\lambda) \neq 0$ for all $\lambda \in \sigma(\delta_A)$. (c), (d) and (e) are shown in [7]. □

Notation. Let $T \in \mathcal{L}(\mathcal{H})$ and $\Omega \subseteq \mathbb{C}, \ \Omega \neq \emptyset$. Let $S(T;\Omega)$ be the subset of $\mathcal{H}$ defined by

$$S(T;\Omega) = \bigcap_{\lambda \in \Omega} (T - \lambda)(\mathcal{H}) \ .$$

The following propositions are of central importance for our investigation.

Proposition 2.2. *If $A \in \mathcal{L}(\mathcal{H})$ is normal and has the spectral resolution* (1.1) *and if $B \in \mathcal{L}(\mathcal{H})$, then*

$$S(A;\rho(B)) = E(\sigma(A) \cap \sigma(B))(\mathcal{H}) \ .$$

Proof. Theorem 1 in [3]. □

Proposition 2.3. *Let $A \in \mathcal{L}(\mathcal{H})$ be normal and let $\mu \in \mathbb{C}$. Then*

$$(A-\mu)(\mathcal{H}) = (A^* - \overline{\mu})(\mathcal{H}) \; . \tag{2.4}$$

Proof. Since A is normal, $A-\mu$ is normal. Exercise 12.36 in [4] shows that (2.4) holds. □

Proposition 2.4. *Let $A \in \mathcal{L}(\mathcal{H})$ be normal and suppose that A has the spectral resolution* (1.1).

(a) *If $\lambda_0 \in \mathbb{C}$, $C \in N(\delta_A - \lambda_0)$ and $D \in N(\delta_A + \lambda_0)$, then*

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A-\lambda_0))(\mathcal{H})$$

and

$$D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A-\lambda_0))(\mathcal{H}) \; .$$

(b) *If $\lambda_0 \in \mathbb{C}$ and $E(\sigma(A) \cap \sigma(A-\lambda_0)) = 0$, then λ_0 and $-\lambda_0$ are not eigenvalues of δ_A.*

(c) *If $\sigma(A)$ is generalized $2\pi i$-congruence-free, then $N(f(\delta_A)) = \{0\}$.*

Proof. (a) Take $C \in \mathcal{L}(\mathcal{H})$ with $CA - AC = \lambda_0 C$, thus $AC = C(A-\lambda_0)$. Put $B = A - \lambda_0$. For $\mu \in \rho(B)$ we get

$$\begin{aligned} (A-\mu)C(B-\mu)^{-1} &= AC(B-\mu)^{-1} - \mu C(B-\mu)^{-1} \\ &= CB(B-\mu)^{-1} - \mu C(B-\mu)^{-1} \\ &= C(B-\mu)(B-\mu)^{-1} = C \; . \end{aligned}$$

This shows that $C(\mathcal{H}) \subseteq S(A, \rho(B))$. From Proposition 2.2 we get

$$S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(\mathcal{H}) \; , \tag{2.5}$$

thus $C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A-\lambda_0))(\mathcal{H})$.

Now take $D \in N(\delta_A + \lambda_0)$, hence $DA - AD = -\lambda_0 D$ thus $DA = (A-\lambda_0)D$. As above let $B = A - \lambda_0$. Then $A^*D^* = D^*B^*$. A similar computation as above gives

$$(A^* - \mu)D^*(B^*-\mu)^{-1} = D^* \text{ for all } \mu \in \rho(B^*) \; .$$

Thus $D^*(\mathcal{H}) \subseteq S(A^*, \rho(B^*))$. Since $\rho(B^*) = \{\mu \in \mathbb{C} : \overline{\mu} \in \rho(B)\}$, we get from (2.4) that

$$S(A^*, \rho(B^*)) = S(A, \rho(B)) \; ,$$

hence, by (2.5), $D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A-\lambda_0))(\mathcal{H})$.

(b) follows from (a).

(c) Let n be the smallest integer with $n > \frac{r(A)}{\pi}$. Take $\lambda_0 \in M_A$. Thus $\lambda_0 = 2j\pi i$ with $j \in \mathbb{Z} \setminus \{0\}$ and $|j| \leq n-1$.

Case 1. $j > 0$. Since $\sigma(A)$ is generalized $2\pi i$-congruence-free,

$$E(\sigma(A) \cap \sigma(A+\lambda_0)) = 0 \; .$$

From (b) we then derive $N(\delta_A - \lambda_0) = \{0\}$.

Case 2. $j < 0$. Then we have

$$E(\sigma(A) \cap \sigma(A-\lambda_0)) = 0 \; .$$

Again use (b) to see that $N(\delta_A - \lambda_0) = \{0\}$.

Since $\lambda_0 \in M_A$ was arbitrary, we conclude from Proposition 2.1 (e) that $N(f(\delta_A)) = \{0\}$. □

3. Proofs

Proof of Theorem 1.2. (a) From (2.3) we get

$$f(\delta_A)(\delta_A(e^B)) = e^{-A}e^Be^A - e^B = 0\ ,$$

thus $\delta_A(e^B) = e^BA - Ae^B \in N(f(\delta_A))$. Proposition 2.4 (c) gives $Ae^B = e^BA$.

(b) From (a) and (2.3) we get

$$f(\delta_B)(\delta_B(A)) = e^{-B}Ae^B - A = 0\ ,$$

therefore, $AB - BA \in N(f(\delta_B))$. Hence, by Proposition 2.4 (c), $AB = BA$. □

Proof of Theorem 1.3. Use (2.3) to derive

$$\begin{aligned} f(\delta_{A+B})(\delta_{A+B}(e^A)) &= e^{-(A+B)}e^Ae^{A+B} - e^A \\ &= e^{-B}e^{-A}e^Ae^Be^A - e^A \\ &= 0\ . \end{aligned}$$

Proposition 2.4 (c) implies that $e^A(A+B) = (A+B)e^A$, therefore, $Be^A = e^AB$. Using (2.3), this gives

$$\begin{aligned} f(\delta_{A+B})(\delta_{A+B}(B)) &= e^{-(A+B)}Be^{A+B} - B \\ &= e^{-B}e^{-A}Be^Ae^B - B \\ &= 0\ . \end{aligned}$$

Since $N(f(\delta_{A+B})) = \{0\}$, $(A+B)B = B(A+B)$. This shows that $AB = BA$. □

Proof of Theorem 1.4. Let $T \in \{B\}^c$. (2.3) implies that

$$f(\delta_A)(\delta_A(T)) = e^{-A}Te^A - T = e^{-B}Te^B - T = 0\ .$$

Since $N(f(\delta_A)) = \{0\}$ (Proposition 2.4 (c)), we have $AT = TA$. Since $T \in \{B\}^c$ was arbitrary, it follows that $A \in \{B\}^{cc}$. □

Proof of Theorem 1.5. (a) From $e^{iA} = e^B$ we get $e^{-iA} = (e^{iA})^{-1} = (e^B)^{-1} = e^{-B}$ and $e^{-iA} = (e^{iA})^* = (e^B)^* = e^{B^*}$, hence $e^{B^*} = e^{-B}$, thus $e^{B+B^*} = I = e^0$, since B is normal. Since $B + B^*$ is selfadjoint, Corollary 2 in [7] gives $B + B^* = 0$.

(b) Observe that iA is normal and $r(iA) \leq \pi$. It is easy to see that

$$\sigma(iA) \cap \sigma(iA + 2\pi i) \subseteq \{i\pi\}\ .$$

Case 1. $\pi \notin \sigma_p(A)$. Then $i\pi \notin \sigma_p(iA)$. Example (4) in Section 1 gives

$$E(\sigma(iA) \cap \sigma(iA + 2\pi i)) = 0\ .$$

Example (2) in Section 1 shows that $\sigma(iA)$ is generalized $2\pi i$-congruence-free. Now use Theorem 1.4 to get $A \in \{B\}^{cc}$.

Case 2. $-\pi \notin \sigma_p(A)$. Then $\pi \notin \sigma_p(-A)$. Since $e^{-iA} = e^{-B}$, we get $A \in \{B\}^{cc}$ as in Case 1. □

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