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On the operator equation $e^A = e^B$

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Abstract

Suppose that A and B are bounded linear operators on a complex Hilbert space and that $e^A = e^B$. It is well-known that if the spectrum of A is incongruent (mod $2\pi i$) then $AB = BA$. In this note we show that if A is normal and $\|A\| \leqslant \pi$ then $e^A = e^B$ implies that $A^2B = BA^2$. If B is also normal, $\|B\| \leqslant \pi$ and $-i\pi$ is not an eigenvalue of A then we show that $e^A = e^B$ implies $AB = BA$ and $(A - B)^2 = 2\pi i(A - B)$.

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Throughout this paper let $\mathscr{H}$ denote a complex Hilbert space and $\mathscr{L}(\mathscr{H})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$. For $A \in \mathscr{L}(\mathscr{H})$ the spectrum, the set of eigenvalues, and the spectral radius of A are denoted by $\sigma(A)$, $\sigma_p(A)$, and $r(A)$, respectively. For the resolvent set of A we write $\rho(A)$. We use $N(A)$ and $A(\mathscr{H})$ to denote the kernel and the range of A, respectively.

We say that $\sigma(A)$ is *incongruent* (mod $2\pi i$), if

$$\sigma(A) \cap \sigma(A + 2j\pi i) = \emptyset \quad \text{for } j = \pm 1, \pm 2, \ldots .$$

If $A \in \mathscr{L}(\mathscr{H})$ is *normal* ($AA^* = A^*A$) and has the spectral resolution

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda), \tag{1}$$

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let $E(\Omega)$ denote the associated projection measure defined on the Borel subsets $\Omega \subseteq \sigma(A)$. It is convenient to think of E as being defined for all Borel subsets in $\mathbb{C}$: put $E(\Omega) = E(\Omega \cap \sigma(A))$. It is well-known that $\|A\| = r(A)$ if A is normal.

Let $T \in \mathscr{L}(\mathscr{H})$. The map $\delta_T : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$, defined by

$$\delta_T(C) = CT - TC \quad (C \in \mathscr{L}(\mathscr{H}))$$

is called the *inner derivation* determined by T. δ_T is a bounded linear operator on $\mathscr{L}(\mathscr{H})$ with $\|\delta_T\| \leqslant 2\|T\|$.

It is shown in [3] that

$$\sigma(\delta_T) = \{\lambda - \mu : \lambda, \mu \in \sigma(T)\}, \tag{2}$$

where $\sigma(\delta_T)$ denotes the spectrum of δ_T. From [4, Proposition 6.4.8] it follows that

$$e^{\delta_T}(C) = e^{-\mathrm{T}} C e^{\mathrm{T}} \quad \text{for all } C \in \mathscr{L}(\mathscr{H}). \tag{3}$$

Throughout this note let f denote the entire function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

From $zf(z) = f(z)z = e^z - 1$ and (3) we get

$$f(\delta_T)(\delta_T(c)) = e^{-\mathrm{T}} C e^{\mathrm{T}} - C \text{ for all } C \in \mathscr{L}(\mathscr{H}). \tag{4}$$

For a bounded linear opterator $\mathscr{T}$ on $\mathscr{L}(\mathscr{H})$ we denote the kernel of $\mathscr{T}$ by $N(\mathscr{T})$.

Proposition 1. *Let $A \in \mathscr{L}(\mathscr{H})$ such that $r(A) \leqslant \pi$. Let*

$$M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\}.$$

(a) *If $\sigma(A)$ is incongruent* (mod 2πi), *then $M_A = \emptyset$.*
(b) *If $M_A = \emptyset$, then $f(\delta_A)$ is an invertible operator on $\mathscr{L}(\mathscr{H})$.*
(c) $M_A \subseteq \{2\pi\mathrm{i}, -2\pi\mathrm{i}\}$.
(d) $N(f(\delta_A)) = N(\delta_A - 2\pi\mathrm{i}) \oplus N(\delta_A + 2\pi\mathrm{i})$.
(e) *If $C \in N(\delta_A - 2\pi\mathrm{i})$ or $C \in N(\delta_A + 2\pi\mathrm{i})$, then $C^2 = 0$.*

Proof. (a) and (b) are clear.

(c) Take $\lambda \in M_A$, then $0 \neq \lambda \in \sigma(\delta_A)$ and $e^\lambda = 1$. Thus $\lambda = 2j\pi\mathrm{i}$ for some $j \in \mathbb{Z}\backslash\{0\}$. It follows from (2) that there are $\alpha, \beta \in \sigma(A)$ such that $\lambda = \alpha - \beta$. Hence $2|j|\pi = |\lambda| \leqslant |\alpha| + |\beta| \leqslant 2r(A) \leqslant 2\pi$, therefore $|j| \leqslant 1$. This shows that $\lambda \in \{2\pi\mathrm{i}, -2\pi\mathrm{i}\}$.

(d) Since 2πi and -2πi are simple zeros of f, there is an entire function g such that

$$f(\lambda) = g(\lambda)(\lambda - 2\pi\mathrm{i})(\lambda + 2\pi\mathrm{i})$$

and $g(\lambda) \neq 0$ for all $\lambda \in \sigma(\delta_A)$, thus $g(\delta_A)$ is invertible on $\mathscr{L}(\mathscr{H})$ and

$$f(\delta_A) = g(\delta_A)(\delta_A - 2\pi\mathrm{i})(\delta_A + 2\pi\mathrm{i}).$$

Satz 80.3 in [1] gives now

$$N(f(\delta_A)) = N(\delta_A - 2\pi i) \oplus N(\delta_A + 2\pi i).$$

(e) Take $C \in N(\delta_A - 2\pi i)$, thus $CA - AC = 2\pi i C$. Therefore $2\pi i C^2 = C^2A - CAC = C^2A - (AC + 2\pi i C)C = C^2A - AC^2 - 2\pi i C^2$, hence

$$C^2A - AC^2 = 4\pi i C^2.$$

Assume that $C^2 \neq 0$, then $4\pi i \in \sigma(\delta_A)$, therefore, by (2), $4\pi \leqslant 2r(A) \leqslant 2\pi$, a contradiction. Thus $C^2 = 0$. □

Notation. Let $T \in \mathscr{L}(\mathscr{H})$, $\Omega \subseteq \mathbb{C}$ and $\Omega \neq \emptyset$. Let $S(T, \Omega)$ be the subset of $\mathscr{H}$ defined by

$$S(T, \Omega) = \bigcap_{\lambda \in \Omega} (T - \lambda)(\mathscr{H}).$$

Proposition 2. *Let $A \in \mathscr{L}(\mathscr{H})$ be normal.*
(a) *For $\mu \in \mathbb{C}$, $(A - \mu)(\mathscr{H}) = (A^* - \overline{\mu})(\mathscr{H})$.*
(b) *If A has the spectral resolution* (1) *and if $B \in \mathscr{L}(\mathscr{H})$, then*

$$S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(\mathscr{H}).$$

Proof. (a) Since A is normal, $A - \mu$ is normal. Use Exercise 12.36 in [6] to see that $(A - \mu)(\mathscr{H}) = (A^* - \overline{\mu})(\mathscr{H})$.
(b) follows from Theorem 1 in [5]. □

Proposition 3. *Let $A \in \mathscr{L}(\mathscr{H})$ be normal and suppose that A has the spectral resolution* (1). *For $\lambda_0 \in \mathbb{C}$, $C \in N(\delta_A - \lambda_0)$ and $D \in N(\delta_A + \lambda_0)$ we have*

$$C(\mathscr{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathscr{H})$$

and

$$D^*(\mathscr{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathscr{H}).$$

Proof. Take $C \in \mathscr{L}(\mathscr{H})$ with $CA - AC = \lambda_0 C$, thus $AC = C(A - \lambda_0)$. Put $B = A - \lambda_0$. For $\mu \in \rho(B)$ we get

$$\begin{aligned}(A - \mu)C(B - \mu)^{-1} &= AC(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \\ &= CB(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \\ &= C(B - \mu)(B - \mu)^{-1} = C.\end{aligned}$$

This shows that $C(\mathscr{H}) \subseteq S(A, \rho(B))$. From Proposition 2(b) we get

$$S(A, \rho(B)) = E(\sigma(A) \cap \sigma(B))(\mathscr{H}), \tag{5}$$

thus $C(\mathscr{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathscr{H})$.

Now take $D \in N(\delta_A + \lambda_0)$, hence $DA - AD = -\lambda_0 D$ thus $DA = (A - \lambda_0)D$. As above let $B = A - \lambda_0$. Then $A^*D^* = D^*B^*$. A similar computation as above gives

$$(A^* - \mu)D^*(B^* - \mu)^{-1} = D^* \text{ for all } \mu \in \rho(B^*).$$

Thus $D^*(\mathscr{H}) \subseteq S(A^*, \rho(B^*))$. Since $\rho(B^*) = \{\mu \in \mathbb{C} : \overline{\mu} \in \rho(B)\}$, we get from Proposition 2(a) that

$$S(A^*, \rho(B^*)) = S(A, \rho(B)),$$

hence, by (5), $D^*(\mathscr{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathscr{H})$. □

Proposition 4. *Let $A \in \mathscr{L}(\mathscr{H})$ be normal and $r(A) \leqslant \pi$.*
(a) *If $C \in N(\delta_A + 2\pi\mathrm{i})$ then $AC = \mathrm{i}\pi C$.*
(b) *If $D \in N(\delta_A - 2\pi\mathrm{i})$ then $AD^* = \mathrm{i}\pi D^*$ and $DA = \mathrm{i}\pi D$.*

Proof. Suppose that A has the spectral resolution (1). It is clear that

$$\sigma(A) \cap \sigma(A + 2\pi\mathrm{i}) \subseteq \{\mathrm{i}\pi\}.$$

From Theorem 12.29 in [6] it follows that $E(\{\mathrm{i}\pi\}) = N(A - \mathrm{i}\pi)$. Thus

$$E(\sigma(A) \cap \sigma(A + 2\pi\mathrm{i}))(\mathscr{H}) \subseteq N(A - \mathrm{i}\pi) \ . \tag{6}$$

(a) Take $C \in N(\delta_A + 2\pi\mathrm{i})$ and put $\lambda_0 = -2\pi\mathrm{i}$. Then $C \in N(\delta_A - \lambda_0)$. Proposition 3 and (6) give

$$C(\mathscr{H}) \subseteq N(A - \mathrm{i}\pi),$$

hence $AC = \mathrm{i}\pi C$.

(b) Take $D \in N(\delta_A - 2\pi\mathrm{i})$ and put $\lambda_0 = -2\pi\mathrm{i}$. Then $D \in N(\delta_A + \lambda_0)$. Proposition 3 and (6) give

$$D^*(\mathscr{H}) \subseteq N(A - \mathrm{i}\pi),$$

hence $AD^* = \mathrm{i}\pi D^*$. Therefore we have $AD^*x = \mathrm{i}\pi D^*x$ for each $x \in \mathscr{H}$. The normality of A gives $A^*D^*x = -\mathrm{i}\pi D^*x$ for all $x \in \mathscr{H}$, thus $A^*D^* = -\mathrm{i}\pi D^*$, hence $DA = \mathrm{i}\pi D$. □

We now are in a position to state the main results of this paper.

The following theorem is due to Hille [2]. For the convenience of the reader we shall include a proof.

Theorem 1. *Let $A, B \in \mathscr{L}(\mathscr{H})$ and $e^A = e^B$. If $\sigma(A)$ is incongruent* (mod $2\pi\mathrm{i}$) *then $AB = BA$.*

Proof. From (4) we get

$$f(\delta_A)(\delta_A(B)) = e^{-A}Be^A - B = 0,$$

thus $AB - BA \in N(f(\delta_A))$. Use Proposition 1(a) and (b) to see that $AB = BA$. □

The restriction concerning the spectrum of A in Theorem 1 cannot be dispensed with, as is seen by the following two-dimensional example.

Example. Let

$$A = \pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \pi \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

By induction we see that

$$A^{2n} = (-1)^n \pi^{2n} I = B^{2n}, \quad A^{2n+1} = (-1)^n \pi^{2n} A,$$

and

$$B^{2n+1} = (-1)^n \pi^{2n} B \quad \text{for } n = 0, 1, 2, \ldots.$$

This shows that $e^A = -I = e^B$. We have

$$AB = \pi^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \pi^2 \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = BA.$$

It is easily seen that A is normal and $\sigma(A) = \{\mathrm{i}\pi, -\mathrm{i}\pi\}$, thus $r(A) = \pi$ and $\sigma(A)$ is not incongruent (mod $2\pi\mathrm{i}$). But we have $A^2B = BA^2$, which will be the case in general as the following theorem shows.

Theorem 2. *Suppose that* $A \in \mathscr{L}(\mathscr{H})$ *is normal,* $B \in \mathscr{L}(\mathscr{H})$ *and* $e^A = e^B$.
(a) *If* $r(A) < \pi$ *then* $AB = BA$.
(b) *If* $r(A) = \pi$ *then* $A^2B = BA^2$.
(c) *If* $r(A) \leqslant \pi$ *and* $\mathrm{i}\pi \notin \sigma_p(A)$ *then* $AB = BA$.
(d) *If* $r(A) \leqslant \pi$ *and* $-\mathrm{i}\pi \notin \sigma_p(A)$ *then* $AB = BA$.

Proof. As in the proof of Theorem 1 we get $AB - BA \in N(f(\delta_A))$.

(a) If $r(A) < \pi$, then $\sigma(A)$ is incongruent (mod $2\pi\mathrm{i}$), hence $AB = BA$, by Theorem 1.

(b) From Proposition 1 we see that there are operators $C \in N(\delta_A + 2\pi\mathrm{i})$ and $D \in N(\delta_A - 2\pi\mathrm{i})$ such that $AB - BA = C + D$. From $CA - AC = -2\pi\mathrm{i}C$ and Proposition 4(a) we derive $AC = \mathrm{i}\pi C$, hence

$$CA = AC - 2\pi\mathrm{i}C = -\mathrm{i}\pi C = -AC,$$

thus

$$AC + CA = 0. \tag{7}$$

Use Proposition 4(b) and $DA - AD = 2\pi\mathrm{i}D$ to get $DA = \mathrm{i}\pi D$ and

$$AD = DA - 2\pi\mathrm{i}D = -\mathrm{i}\pi D = -DA,$$

hence

$$AD + DA = 0 . \tag{8}$$

From $AB - BA = C + D$ it follows that

$$A^2B - ABA = AC + AD \tag{9}$$

and

$$ABA - BA^2 = CA + DA. \tag{10}$$

The addition of (9) and (10) shows

$$A^2B - BA^2 = AC + CA + AD + DA.$$

Now use (7) and (8) to get $A^2B = BA^2$.

(c) As in (b) we have $AB - BA = C + D$ with $C \in N(\delta_A + 2\pi\mathrm{i})$ and $D \in N(\delta_A - 2\pi\mathrm{i})$. Since $N(A - \mathrm{i}\pi) = \{0\}$ we conclude from Proposition 4 that $C = 0 = D^*$, thus $C = D = 0$, hence $AB = BA$.

(d) If $-\mathrm{i}\pi \notin \sigma_p(A)$ then $\mathrm{i}\pi \notin \sigma_p(-A)$. From $e^{-A} = e^{-B}$ we get then $AB = BA$ as in the proof of (c). □

Now suppose that the spectrum $\sigma(A)$ of $A \in \mathscr{L}(\mathscr{H})$ satisfies

$$\sigma(A) \subseteq \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leqslant \pi\} \tag{11}$$

and

$$\sigma(A) \cap \sigma(A + 2\pi\mathrm{i}) \subseteq \{\mathrm{i}\pi\}. \tag{12}$$

Then it is easily seen that $M_A \subseteq \{2\pi\mathrm{i}, -2\pi\mathrm{i}\}$ and

$$N(f(\delta_A)) = N(\delta_A - 2\pi\mathrm{i}) \oplus N(\delta_A + 2\pi\mathrm{i}).$$

Let A be normal and suppose that (11) and (12) hold. Then it is easy to see that the statements of Proposition 4 remain valid. Thus an inspection of the proof of Theorem 2 shows the following result:

Theorem 3. *Suppose that $A \in \mathscr{L}(\mathscr{H})$ is normal, $B \in \mathscr{L}(\mathscr{H})$, $e^A = e^B$ and $\sigma(A)$ satisfies* (11) *and* (12). *Then $A^2B = BA^2$. If $\mathrm{i}\pi \notin \sigma_p(A)$ or $-\mathrm{i}\pi \notin \sigma_p(A)$ then $AB = BA$.*

Theorem 4. *Let $A, B \in \mathscr{L}(\mathscr{H})$, let $e^A = e^B$ and let $\sigma(A)$ and $\sigma(A - B)$ be incongruent* (mod $2\pi\mathrm{i}$). *Then there is some $k \in \mathbb{Z}$ with*

$$A - B = (2k\pi\mathrm{i})I.$$

Proof. From Theorem 1 we get $AB = BA$, thus $e^{A-B} = I$. Put $C = A - B$ and let $g(z) = e^z - 1$ $(z \in \mathbb{C})$. Hence $g(C) = 0$. Take $\lambda, \mu \in \sigma(C)$, then $e^\lambda = e^\mu = 1$, thus $\lambda - \mu = 2j\pi\mathrm{i}$ for some $j \in \mathbb{Z}$. Since $\sigma(C)$ is incongruent (mod $2\pi\mathrm{i}$), we get $\lambda = \mu$.This shows that there is $k \in \mathbb{Z}$ such that $\sigma(C) = \{2k\pi i\}$. Since $2k\pi i$ is a simple zero of g, there is an entire function h with $g(\lambda) = h(\lambda)(\lambda - 2k\pi i)$ and $h(2k\pi i) \neq 0$.

This gives

$$0 = g(C) = h(C)(C - 2k\pi i).$$

Since $h(C)$ is invertible, $C = 2k\pi iI$. □

As an immediate consequence of Theorem 4 we have the following well-known result:

Corollary 1. *If $A, B \in \mathscr{L}(\mathscr{H})$ are selfadjoint and if $e^A = e^B$ then $A = B$.*

Proof. From $\sigma(A)$, $\sigma(A - B) \subseteq \mathbb{R}$ we see that $\sigma(A)$ and $\sigma(A - B)$ are incongruent (mod $2\pi i$). Theorem 4 gives $A - B = 2k\pi iI$ for some $k \in \mathbb{Z}$. Since $A - B = (A - B)^* = -2k\pi iI = B - A$, $A = B$. □

Corollary 2. *If $A, B \in \mathscr{L}(\mathscr{H})$ are normal and if $e^A = e^B$ then $A + A^* = B + B^*$.*

Proof. Since A and B are normal, we see that

$$\begin{aligned} e^{A+A^*} &= e^A e^{A^*} = e^A (e^A)^* = e^B (e^B)^* \\ &= e^B e^{B^*} = e^{B+B^*}. \end{aligned}$$

We now use Corollary 1. □

For our next result we need the following proposition.

Proposition 5. *Suppose that A and B are normal operators in $\mathscr{L}(\mathscr{H})$, $r(A) \leqslant \pi$, $r(B) \leqslant \pi$ and $AB = BA$. Then*

(a) *$A - B$ is normal*;

(b) *$N(A - B - 2\pi i) = N(A - i\pi) \cap N(B + i\pi)$.*

Proof. (a) From $AB = BA$ we get $AB^* = B^*A$ and $A^*B = BA^*$ by the Fuglede–Putnam–Rosenblum Theorem [6, Theorem 12.16]. A simple computation gives then that $(A - B)(A - B)^* = (A - B)^*(A - B)$.

(b) It is clear that $N(A - i\pi) \cap N(B + i\pi) \subseteq N(A - B - 2\pi i)$. Put $C = A - B$ and take $x_0 \in N(A - B - 2\pi i)$. We can assume that $\|x_0\| = 1$. For the following computations let $(\cdot|\cdot)$ denote the inner product $\mathscr{H}$. From $Cx_0 = 2\pi i x_0$ and (a) we get $C^* x_0 = -2\pi i x_0$, thus

$$Ax_0 = Bx_0 + 2\pi i x_0 \quad \text{and} \quad A^* x_0 = B^* x_0 - 2\pi i x_0. \tag{13}$$

Put $D = \mathrm{i}(B - B^*)$. Then $D^* = D$ and

$$\begin{aligned}(Dx_0|x_0) &= \mathrm{i}((Bx_0|x_0) - \overline{(Bx_0|x_0)})\\ &= -2\,\mathrm{Im}(Bx_0|x_0).\end{aligned}$$

From $|\mathrm{Im}(Bx_0|x_0)| \leqslant |(Bx_0|x_0)| \leqslant \|B\| = r(B) \leqslant \pi$, we see that $-\pi \leqslant \mathrm{Im}(Bx_0|x_0)$. Thus

$$(Dx_0|x_0) \leqslant 2\pi. \tag{14}$$

Now use (13) and (14) to derive

$$\begin{aligned}\|(A - \mathrm{i}\pi)x_0\|^2 &= ((A^* + \mathrm{i}\pi)(A - \mathrm{i}\pi)x_0|x_0)\\ &= \|Ax_0\|^2 + \mathrm{i}\pi((A - A^*)x_0|x_0) + \pi^2\\ &\leqslant 2\pi^2 + \mathrm{i}\pi((A - A^*)x_0|x_0)\\ &= 2\pi^2 + \mathrm{i}\pi((B - B^* + 4\pi\mathrm{i})x_0|x_0)\\ &= -2\pi^2 + \pi(Dx_0|x_0)\\ &\leqslant -2\pi^2 + 2\pi^2 = 0.\end{aligned}$$

Thus $x_0 \in N(A - \mathrm{i}\pi)$, hence, by (13), $x_0 \in N(B + \mathrm{i}\pi)$. □

Theorem 5. *Suppose that A and B are normal operators in $\mathscr{L}(\mathscr{H})$, $r(A) \leqslant \pi$, $r(B) \leqslant \pi$ and $e^A = e^B$.*

(a) *If $\mathrm{i}\pi \notin \sigma_p(A)$ then $AB = BA$ and $-(1/2\pi\mathrm{i})(A - B)$ is an orthogonal projection.*
(b) *If $-\mathrm{i}\pi \notin \sigma_p(A)$ then $AB = BA$ and $(1/2\pi\mathrm{i})(A - B)$ is an orthogonal projection.*
(c) *If $-\mathrm{i}\pi \notin \sigma_p(A)$ and $\mathrm{i}\pi \notin \sigma_p(A)$ then $A = B$.*
(d) *If $-\mathrm{i}\pi \notin \sigma_p(A)$ and $-\mathrm{i}\pi \notin \sigma_p(B)$ then $A = B$.*
(e) *If $\mathrm{i}\pi \notin \sigma_p(A)$ and $\mathrm{i}\pi \notin \sigma_p(B)$ then $A = B$.*

Proof. Let $C = A - B$.

(a) Theorem 2 gives $AB = BA$, hence $e^C = I$. Use Proposition 5 to see that C is normal. Define the polynomial p by $p(z) = z(z + 2\pi\mathrm{i})(z - 2\pi\mathrm{i})$. Then $p(C)$ is normal and therefore

$$\|p(C)\| = r(p(C)). \tag{15}$$

Now take $\lambda \in \sigma(C)$. Since $e^C = I$, $\lambda = 2j\pi\mathrm{i}$ for some $j \in \mathbb{Z}$. Thus

$$2|j|\pi = |\lambda| \leqslant \|C\| \leqslant \|A\| + \|B\| = r(A) + r(B) \leqslant 2\pi,$$

hence $j \in \{0, 1, -1\}$, thus $\sigma(C) \subseteq \{0, 2\pi\mathrm{i}, -2\pi\mathrm{i}\}$. The spectral mapping theorem gives now

$$\sigma(p(C)) = p(\sigma(C)) = \{0\},$$

hence, by (15),

$$(C - 2\pi i)(C + 2\pi i)C = 0. \tag{16}$$

Since $N(A - i\pi) = \{0\}$, we see from Proposition 5(b) and (16) that $C^2 = -2\pi i C$. Put $P = -(1/2\pi i)C$. Then $P^2 = P$, thus P is a projection. It remains to show that $P = P^*$. But this follows from Corollary 2, since

$$P^* = \frac{1}{2\pi i}(A^* - B^*) = -\frac{1}{2\pi i}(A - B) = P.$$

(b) If $-i\pi \notin \sigma_p(A)$ then $i\pi \notin \sigma_p(-A)$. Since $e^{-A} = e^{-B}$, (a) shows that $-(1/2\pi i)(-A - (-B)) = (1/2\pi i)(A - B)$ is an orthogonal projection.

(c) From (a) and (b) we conclude that $-2\pi i C = C^2 = 2\pi i C$, hence $C = 0$.

(d) From (b) we get $C^2 = 2\pi i C$. Use (b) with B instead of A to derive $C^2 = (B - A)^2 = (1/2\pi i)(B - A) = -2\pi i C$. Hence $C = 0$.

(e) is now clear.

Remark. Theorem 5 generalizes Satz 4 in [7].

For our final result in this paper we return to the situation of Theorem 3.

Theorem 6. *Suppose that* $A \in \mathscr{L}(\mathscr{H})$ *is normal*, $B \in \mathscr{L}(\mathscr{H})$, $e^A = e^B$ *and* $\sigma(A)$ *satisfies* (11) *and* (12). *Then we have*:
(a) $A^{2n}B = BA^{2n}$ *for all* $n \in \mathbb{N}$.
(b) $A^{2n+1}B - BA^{2n+1} = A^{2n}(AB - BA) = (-1)^n\pi^{2n}(AB - BA)$ *for all* $n \in \mathbb{N}$.
(c) $e^{2A}(AB - BA) = e^{2B}(AB - BA) = AB - BA$.
(d) *If* $AB \neq BA$ *then there is some* $k \in \mathbb{Z}$ *such that* $k\pi i \in \sigma_p(B)$.

Proof. (a) Follows from Theorem 3.

(b) As in the proof of Theorem 2 there are operators $C \in N(\delta_A + 2\pi i)$ and $D \in N(\delta_A - 2\pi i)$ such that $AB - BA = C + D$, $AC = i\pi C$ and $DA = i\pi D$. It follows that

$$\begin{aligned} i\pi(AB - BA) &= i\pi C + i\pi D = AC + DA \\ &= AC + AD + 2\pi i D \\ &= A(AB - BA) + 2\pi i D, \end{aligned}$$

thus

$$(A - i\pi)(AB - BA) = -2\pi i D.$$

Since $D(A - i\pi) = 0$,

$$(A - i\pi)(AB - BA)(A - i\pi) = 0.$$

This and

$$\begin{aligned} A(AB - BA) &= A^2B - ABA = BA^2 - ABA \\ &= (BA - AB)A = -(AB - BA)A \end{aligned}$$

show that

$$0 = (A - \mathrm{i}\pi)((-A - \mathrm{i}\pi)(AB - BA)) = -(A^2 + \pi^2)(AB - BA),$$

hence

$$A^2(AB - BA) = -\pi^2(AB - BA). \tag{17}$$

Take $n \in \mathbb{N}$. Then, by (a) and (17),

$$\begin{aligned} A^{2n+1}B - BA^{2n+1} &= A^{2n+1}B - A^{2n}BA = A^{2n}(AB - BA) \\ &= (-1)^n \pi^{2n}(AB - BA). \end{aligned}$$

(c) Since (b) holds we get

$$\begin{aligned} \frac{1}{2}(e^A - e^{-A})(AB - BA) &= \sinh(A)(AB - BA) \\ &= \sum_{n=0}^{\infty} \frac{A^{2n+1}(AB - BA)}{(2n+1)!} \\ &= A \sum_{n=0}^{\infty} \frac{A^{2n}(AB - BA)}{(2n+1)!} \\ &= A \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}(AB - BA)}{(2n+1)!} \\ &= \frac{A}{\pi} \left(\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} \right) (AB - BA) \\ &= \frac{\sin \pi}{\pi} A(AB - BA) = 0. \end{aligned}$$

Thus $e^A(AB - BA) = e^{-A}(AB - BA)$. From $e^A = e^B$ we then derive

$$e^{2B}(AB - BA) = e^{2A}(AB - BA) = AB - BA.$$

(d) Suppose that $AB \neq BA$. From (c) we see that $1 \in \sigma_p(e^{2B})$. The spectral mapping theorem for the point spectrum ([6, Theorem 10.33]) shows that there is $\lambda \in \sigma_p(B)$ such that $e^{2\lambda} = 1$, hence $2\lambda = 2k\pi\mathrm{i}$ for some $k \in \mathbb{Z}$. This gives $k\pi\mathrm{i} \in \sigma_p(B)$. $\square$

Corollary 3. *Suppose that $A \in \mathscr{L}(\mathscr{H})$ is normal, $B \in \mathscr{L}(\mathscr{H})$, $e^A = e^B$, $\sigma(A)$ satisfies* (11) *and* (12) *and $\sigma(B)$ satisfies*

$$\sigma_p(B) \cap \{k\pi\mathrm{i} : k \in \mathbb{Z}\} = \emptyset$$

then $AB = BA$.

References

[1] H. Heuser, Funktionalanalysis, Teubner, Stuttgart, 1991.
[2] E. Hille, On roots and logarithms of elements of a complex Banach algebra, Amer. Math. Ann. 136 (1958) 46–57.
[3] G. Lumer, M. Rosenblum, Linear operator equations, Proc. Amer. Math. Soc. 10 (1959) 32–41.
[4] T.W. Palmer, Banach Algebras and the General Theory of ∗-Algebras, vol. 1, Cambridge, 1994.
[5] C.R. Putnam, Ranges of normal and subnormal operators, Michigan Math. J. 18 (1971) 33–36.
[6] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1991.
[7] Ch. Schmoeger, Über die Eindeutigkeit des Logarithmus eines unitären Operators, Nieuw Arch. Wiskd., IV, Ser. 15, No. 1–2 (1997) 57–61.