CHARACTERIZATIONS OF SOME CLASSES OF RELATIVELY REGULAR OPERATORS

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Abstract. A bounded linear operator A on a Banach space is called relatively regular, if there is a bounded linear operator B such that $ABA = A$. In this case B is called a g_1-inverse of A. In this paper we characterize some classes of relatively regular operators A via the set $\{B_1 - B_2 : B_1$ and B_2 are g_1-inverses of $A\}$.

1. Terminology and introduction

Throughout this paper let X denote an infinite-dimensional complex Banach space and let $\mathscr{L}(X)$ denote the Banach algebra of all bounded linear operators on X.

If $A \in \mathscr{L}(X)$, we denote by $N(A)$ the kernel and by $A(X)$ the range of A. We write $\sigma(A)$ for the spectrum of A. The *ascent* $a(A)$ and the *descent* $d(A)$ of A are defined by

$$a(A) = \inf\{n \geq 0 : N(A^n) = N(A^{n+1})\}$$

and

$$d(A) = \inf\{n \geq 0 : A^n(X) = A^{n+1}(X)\},$$

respectively (where $\inf \emptyset = \infty$).

We call $A \in \mathscr{L}(X)$ *relatively regular* if $ABA = A$ for some $B \in \mathscr{L}(X)$. In this case AB is a projection onto $A(X)$ and $I - BA$ is a projection onto $N(A)$ and we say that B is a g_1*-inverse* of A.

The set of all relatively regular operators on X is denoted by $\mathscr{R}(X)$.

If $A \in \mathscr{R}(X)$ let

$$\mathscr{G}_1(A) = \{B \in \mathscr{L}(X) : B \text{ is a } g_1\text{-inverse of } A\}.$$

If $B \in \mathscr{G}_1(A)$, set $B_0 = BAB$. Then it is easy to see that

$$AB_0A = A \quad \text{and} \quad B_0AB_0 = B_0.$$

B_0 is called a g_2*-inverse* of A. Set

$$\mathscr{G}_2(A) = \{B \in \mathscr{L}(X) : B \text{ is a } g_2\text{-inverse of } A\}.$$

Then

$$\emptyset \neq \mathscr{G}_2(A) \subseteq \mathscr{G}_1(A).$$

For $A \in \mathscr{R}(X)$ let

$$\mathscr{D}(A) = \{B_1 - B_2 : B_1, B_2 \in \mathscr{G}_1(A)\}.$$

In this paper we characterize the following classes of relatively regular operators A via the sets $\mathscr{D}(A)$:

invertible operators, one-sided invertible operators, Fredholm operators, simply polar operators, orthogonal projections on Hilbert spaces and algebraic operators.

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The following proposition is shown in [1, Theorem 2.3.2].

1.1. Proposition. *If $A \in \mathscr{R}(X)$ and $B_0 \in \mathscr{G}_2(A)$, then*
$$\mathscr{G}_1(A) = \{B_0 + U - B_0 AUAB_0 : U \in \mathscr{L}(X)\}\,.$$

1.2. Proposition. *If $A \in \mathscr{R}(X)$ and $B \in \mathscr{G}_1(A)$, then*
$$\mathscr{D}(A) = \{U - BAUAB : U \in \mathscr{L}(X)\}\,.$$

Proof. Let $B_0 = BAB$, then $B_0 \in \mathscr{G}_2(A)$, $B_0 A = BABA = BA$ and $AB_0 = ABAB = AB$, hence
$$U - BAUAB = U - B_0 AUAB_0 \quad \text{for all } U \in \mathscr{L}(X)\,.$$
Now let $C \in \mathscr{D}(A)$, then $C = B_1 - B_2$ for some B_1, $B_2 \in \mathscr{G}_1(A)$. By Proposition 1.1, there are U_1, $U_2 \in \mathscr{L}(X)$ such that
$$B_j = B_0 + U_j - B_0 AU_j AB_0 = B_0 + U_j - BAU_j AB$$
$(j = 1, 2)$. Hence
$$C = B_1 - B_2 = U - BAUAB$$
with $U = U_1 - U_2$.

Now let $U \in \mathscr{L}(X)$ and define B_1, $B_2 \in \mathscr{L}(X)$ by
$$B_j = B_0 + jU - jB_0 AUAB_0 \quad (j = 1, 2)\,.$$
From Proposition 1.1 we know that $B_j \in \mathscr{G}_1(A)$, hence
$$U - BAUAB = U - B_0 AUAB_0 = B_1 - B_2 \in \mathscr{D}(A)\,.$$
□

1.3. Proposition. *Let $A \in \mathscr{R}(X)$ and $B \in \mathscr{G}_1(A)$. If $\Psi : \mathscr{L}(X) \to \mathscr{L}(X)$ is defined by*
$$\Psi(U) = U - BAUAB\,,$$
then
(1) $\Psi \in \mathscr{L}(\mathscr{L}(X))$;
(2) $\Psi^2 = \Psi$;
(3) $\mathscr{D}(A) = \Psi(\mathscr{L}(X))$.

Proof. (1) is clear; (2) is an easy computation; (3) follows from Proposition 1.2. □

2. Invertible operators

We denote the group of invertible operators in $\mathscr{L}(X)$ by $\mathscr{L}(X)^{-1}$.

2.1. Theorem. *If $A \in \mathscr{R}(X)$, then*
$$A \in \mathscr{L}(X)^{-1} \iff \mathscr{D}(A) = \{0\}\,.$$

Proof. "⇒": If $A \in \mathscr{L}(X)^{-1}$, then it is clear that $\mathscr{G}_1(A) = \mathscr{G}_2(A) = \{A^{-1}\}$, hence $\mathscr{D}(A) = \{0\}$.
"⇐": Take $B \in \mathscr{G}_1(A)$, then, by Proposition 1.2,
$$U = BAUAB \quad \text{for all } U \in \mathscr{L}(X)\,.$$
Therefore $I = BA^2B$, hence $B \in \mathscr{L}(X)^{-1}$,
$$A = ABA^2B = A^2B$$
and
$$A = BA^2BA = BA^2\,.$$

This gives

$$BA = BA^2B = I \quad \text{and} \quad AB = BA^2B = I\,,$$

thus $A \in \mathscr{L}(X)^{-1}$. $\square$

3. One-sided invertible operators

W say that $A \in \mathscr{L}(X)$ is *left invertible* [resp. *right invertible*], if there is $B \in \mathscr{L}(X)$ such that

$$BA = I\,[\text{resp. } AB = I]\,.$$

It is clear that left or right invertible operators are relatively regular.

3.1. Proposition. *Let $A \in \mathscr{R}(X)$ and $B \in \mathscr{G}_1(A)$.*
(1) *A is left invertible $\Longleftrightarrow BA = I \Longleftrightarrow N(A) = \{0\}$.*
(2) *A is right invertible $\Longleftrightarrow AB = I \Longleftrightarrow N(B) = \{0\}$.*

Proof. (1) If A is left invertible, then $CA = I$ for some $C \in \mathscr{L}(X)$. It follows that $C \in \mathscr{G}_2(A)$. By Proposition 1.1 there is $U \in \mathscr{L}(X)$ such that

$$B = C + U - CAUAC = C + U - UAC\,.$$

Hence $BA = CA + UA - UACA = CA = I$ and $N(A) = \{0\}$.

Now assume that $N(A) = \{0\}$. Hence $(I - BA)(X) = \{0\}$. Since $I - BA$ is a projection, we get $BA = I$.
(2) Similar. $\square$

3.2. Theorem. *Suppose that $A \in \mathscr{R}(X)$ and that $A \neq 0$.*

$$A \textit{ is right invertible } \Longleftrightarrow \mathscr{D}(A) = \{C \in \mathscr{L}(X) : AC = 0\}\,.$$

Proof. Take $B \in \mathscr{G}_1(A)$.
"$\Rightarrow$": Let $C \in \mathscr{L}(X)$ and $AC = 0$, then $C = C - BACAB$, hence $C \in \mathscr{D}(A)$, by Proposition 1.2.
Now let $C \in \mathscr{D}(A)$, hence $C = U - BAUAB$ for some $U \in \mathscr{L}(X)$. Since $AB = I$ (Proposition 3.1(2)), we have $C = U - BAU$, therefore $AC = AU - ABAU = AU - AU = 0$.

"$\Leftarrow$": If $U \in \mathscr{L}(X)$, then $U - BAUAB \in \mathscr{D}(A)$, hence

$$AU = AUAB \quad \text{for all } U \in \mathscr{L}(X)\,.$$

A is right invertible if we have shown that $N(B) = \{0\}$ (Proposition 3.1(2)). To this end assume that $N(B) \neq \{0\}$. Then there is $x_0 \in N(B)$ with $x_0 \neq 0$. By the Hahn-Banach theorem, there is $x^* \in X^*$ (the dual space of X) such that $x^*(x_0) = 1$. We have

$$AUx_0 = AUABx_0 = 0 \quad \text{for all } U \in \mathscr{L}(X)\,.$$

Now let $z \in X$ and let $Ux := x^*(x)z\,(x \in X)$. Then $Ux_0 = z$, thus $Az = 0$. Since $z \in X$ was arbitrary, we get the contradiction $A = 0$. $\square$

3.3. Theorem. *Suppose that $A \in \mathscr{R}(X)$ and that $A \neq 0$.*

$$A \textit{ is left invertible } \Longleftrightarrow \mathscr{D}(A) = \{C \in \mathscr{L}(X) : CA = 0\}\,.$$

Proof. Take $B \in \mathscr{G}_1(A)$.
"$\Rightarrow$": As in the proof of Theorem 3.2.
"$\Leftarrow$": If $U \in \mathscr{L}(X)$, then $U - BAUAB \in \mathscr{D}(A)$, thus

$$UA = BAUA \quad \text{for all } U \in \mathscr{L}(X)\,.$$

Assume that there is $x_0 \in N(A)$ with $x_0 \neq 0$. Let $x^* \in X^*$ and define $U \in \mathscr{L}/X)$ by $Ux = x^*(x)x_0$. Then $UAx = x^*(Ax)x_0$ for all $x \in X$, hence

$$BAUAx = x^*(Ax)BAx_0 = 0 \quad \text{for all } x \in X\,,$$

thus $BAUA = 0$ and so $UA = 0$. Since $x_0 \neq 0$, we get $x^*(Ax) = 0$ for all $x \in X$ and all $x^* \in X^*$, therefore $A = 0$, a contradiction. Thus we have shown that $N(A) = \{0\}$. Now use Proposition 3.1(1). □

4. Fredholm operators

An operator $A \in \mathscr{L}(X)$ is called a *Fredholm operator* if $\alpha(A) := \dim N(A) < \infty$ and $\beta(A) := \operatorname{codim} A(X) < \infty$. In this case the *index* $\operatorname{ind}(A)$ of A is defined by

$$\operatorname{ind}(A) = \alpha(A) - \beta(A)\,.$$

We denote the set of all Fredholm operators by $\Phi(X)$. It is well-kown (see [4, Theorem 13.2]) that

$$\Phi(X) \subseteq \mathscr{R}(X)\,.$$

Notation.

$$\mathscr{F}(X) = \{F \in \mathscr{L}(X) : \dim F(X) < \infty\}\,;$$
$$\mathscr{K}(X) = \{K \in \mathscr{L}(X) : K \text{ is compact}\}\,.$$

$\mathscr{F}(X)$ is an ideal in $\mathscr{L}(X)$ and $\mathscr{K}(X)$ is a closed ideal in $\mathscr{L}(X)$. We consider the quotient algebras

$$\mathscr{L}(X)/\mathscr{F}(X) \quad \text{and} \quad \mathscr{L}(X)/\mathscr{K}(X)\,.$$

The elements in $\mathscr{L}(X)/\mathscr{F}(X)$ are denoted by $\widehat{A}$, hence $\widehat{A} = A + \mathscr{F}(X)$, and the elements in $\mathscr{L}(X)/\mathscr{K}(X)$ are denoted by $\widetilde{A}$, hence $\widetilde{A} = A + \mathscr{K}(X)$.

4.1. Proposition. *For $A \in \mathscr{L}(X)$ the following assertions are equivalent:*
(1) $A \in \Phi(X)$*;*
(2) $\widehat{A}$ *is invertible in* $\mathscr{L}(X)/\mathscr{F}(X)$*;*
(3) $\widetilde{A}$ *is invertible in* $\mathscr{L}(X)/\mathscr{K}(X)$*.*

Proof. [2, Satz 75.11]. □

4.2. Proposition. *If $A \in \Phi(X)$ and $B \in \mathscr{G}_1(A)$, then $B \in \Phi(X)$ and* $\operatorname{ind}(B) = -\operatorname{ind}(A)$*.*

Proof. First let $B_0 \in \mathscr{G}_2(A)$, then $AB_0A = A$, $B_0 = B_0AB_0$,

$$X = \underbrace{(AB_0)(X)}_{=A(X)} \oplus \underbrace{(I - AB_0)(X)}_{=N(B_0)}$$

and

$$X = \underbrace{(B_0A)(X)}_{=B_0(X)} \oplus \underbrace{(I - B_0A)(X)}_{=N(A)}\,.$$

This shows that $B_0 \in \Phi(X)$ and $\operatorname{ind}(B_0) = -\operatorname{ind}(A)$. Furthermore we have $I - AB_0$, $I - B_0A \in \mathscr{F}(X)$, thus

$$\widehat{A}\widehat{B_0} = \widehat{I} = \widehat{B_0}\widehat{A}\,.$$

Proposition 1.1 shows that $B = B_0 + U - B_0AUAB_0$ for some $U \in \mathscr{L}(X)$. Hence $\widehat{B} = \widehat{B_0}$, thus $\widehat{B}$ is invertible in $\mathscr{L}(X)/\mathscr{F}(X)$, therefore $B \in \Phi(X)$.
From $ABA = A$ and the index-theorem ([4, Theorem 13.1]) it follows that

$$\operatorname{ind}(A) = 2\operatorname{ind}(A) + \operatorname{ind}(B),$$

hence $\operatorname{ind}(B) = -\operatorname{ind}(A)$. □

4.3. Corollary. *Suppose that A is relatively regular. Then*

$$A \in \Phi(X) \Longleftrightarrow B \in \Phi(X) \quad \textit{for all } B \in \mathscr{G}_1(A)\,.$$

Proof. "⇒": Proposition 4.2.
"⇐": If $B_0 \in \mathscr{G}_2(A)$, then $B_0 \in \Phi(X)$ and $A \in \mathscr{G}_2(B_0)$. Proposition 4.2 gives $A \in \Phi(X)$. □

4.4. Theorem. *The following assertions are equivalent:*
(1) $A \in \Phi(X)$*;*
(2) $\mathscr{D}(A) \subseteq \mathscr{F}(X)$*;*
(3) $\mathscr{D}(A) \subseteq \mathscr{K}(X)$*.*

Proof. (1) ⇒ (2): Take $B_0 \in \mathscr{G}_2(A)$ and $B_1, B_2 \in \mathscr{G}_1(A)$. The proof of Proposition 4.2 shows that $B_1 - B_0, B_2 - B_0 \in \mathscr{F}(X)$, hence $B_1 - B_2 = B_1 - B_0 - (B_2 - B_0) \in \mathscr{F}(X)$.
(2) ⇒ (3): Clear.
(3) ⇒ (1): Let $B \in \mathscr{G}_1(A)$. Then $\mathscr{D}(A) = \{U - BAUAB : U \in \mathscr{L}(X)\}$. Hence

$$\widetilde{U} = \widetilde{B}\widetilde{A}\widetilde{U}\widetilde{A}\widetilde{B} \quad \text{for all } U \in \mathscr{L}(X)\,.$$

With $U = I$, we get $\widetilde{I} = \widetilde{B}(\widetilde{A})^2\widetilde{B}$, hence $\widetilde{B}$ is invertible in $\mathscr{L}(X)/\mathscr{K}(X)$. By Proposition 4.1, $B \in \Phi(X)$. Now use Corollary 4.3 to get $A \in \Phi(X)$. □

5. Simply polar operators

In this section we assume that $A \in \mathscr{L}(X)$ is relatively regular, $A \notin \mathscr{L}(X)^{-1}$ and that $A \neq 0$.

We say that A is *simply polar* if $AB = BA$ for some $B \in \mathscr{G}_2(A)$.

5.1. Proposition. *The following assertions are equivalent:*
(1) *A is simply polar;*
(2) $a(A) = d(A) = 1$*.*

Proof. [2, Satz 72.4 and Satz 101.2], [1, Theorem 5.2] (observe that $A \notin \mathscr{L}(X)^{-1}$ and $A \neq 0$). □

5.2. Theorem. *The following assertions are equivalent:*
(1) *A is simply polar;*
(2) *there is $B \in \mathscr{G}_1(A)$ such that $\mathscr{D}(A) = \mathscr{D}(B)$.*

Proof. (1) ⇒ (2): There is $B \in \mathscr{G}_2(A)$ such that $AB = BA$, hence

$$\underbrace{U - BAUAB}_{\in \mathscr{D}(A)} = \underbrace{U - ABUBA}_{\in \mathscr{D}(B)} \quad \text{for all } U \in \mathscr{L}(X)\,.$$

(2) ⇒ (1): From Proposition 1.2 we get that for each $U \in \mathscr{L}(X)$ there is $V \in \mathscr{L}(X)$ such that

$$V - ABVBA = U - BAUAB\,.$$

Hence there is $U \in \mathscr{L}(X)$ with

$$B - AB^3A = U - BAUAB\,.$$

It follows that

$$0 = A(U - BAUAB)A = ABA - A^2B^3A^2\,,$$

hence $A = A^2B^3A^2$. Consequently $N(A^2) = N(A)$ and $A(X) = A^2(X)$, which gives $a(A) = d(A) = 1$ (observe that $A \notin \mathscr{L}(X)^{-1}$). By Proposition 5.1, we derive that A is simply polar. $\square$

6. Orthogonal projections

In this section X is a complex Hilbert space with the inner product $(\cdot|\cdot)$. A projection $P \in \mathscr{L}(X)$ is called an *orthogonal projection* if $P = P^*$.

6.1. Theorem. *If $P \in \mathscr{L}(X)$ and $P^2 = P$, then*

$$P = P^* \iff \mathscr{D}(P)^* = \mathscr{D}(P)\,,$$

where $\mathscr{D}(P)^* = \{C^* : C \in \mathscr{D}(P)\}$.

Proof. We can assume that $0 \neq P \neq I$. Since $P = P^3$, $P \in \mathscr{G}_2(P)$ and so $\mathscr{D}(P) = \{U - PUP : U \in \mathscr{L}(X)\}$. Hence

$$\mathscr{D}(P)^* = \{U^* - P^*U^*P^* : U \in \mathscr{L}(X)\}\,,$$

and so $\mathscr{D}(P)^* = \mathscr{D}(P)$ if $P = P^*$.
Now assume that $\mathscr{D}(P)^* = \mathscr{D}(P)$. With $U = I$ we have $I - P \in \mathscr{D}(P)$, thus there is $V \in \mathscr{L}(X)$ such that

$$I - P = V - P^*VP^*\,,$$

thus $0 = P^*(I - P)P^* = P^* - P^*PP^*$, therefore $P^* = P^*PP^*$. Let $Q = P^*P$. Then we have $0 \neq Q$, $Q^* = Q$ and $Q^2 = Q$, hence Q is an orthogonal projection. It follows that $\|Q\| = 1$ ([5, Satz 5.9]). Now let $x \in X$, then

$$\begin{aligned}\|Px\|^2 &= (Px|Px) = (P^*Px) = (Qx|x)\\ &= |(Qx|x)| \leq \|Qx\|\|x\| \leq \|Q\|\|x\|^2 = \|x\|^2\,,\end{aligned}$$

therefore $\|P\| \leq 1$. Since $0 \neq P = P^2$, we also have $\|P\| \geq 1$. Consequently, $\|P\| = 1$. Use again Satz V.5.9 in [5] to see that $P = P^*$. $\square$

7. Algebraic operators

An operator $A \in \mathscr{L}(X)$ is called *algebraic* if $p(A) = 0$ for some non-zero polynomical p.

7.1. Proposition. *Let $A \in \mathscr{L}(X)$. Then A is algebraic if and only if $\sigma(A)$ consists of a finite number of poles of $(\lambda I - A)^{-1}$.*

Proof. [4, Chapter V.11]. $\square$

7.2. Proposition. *Suppose that A is algebraic and simply polar. Then there is a polynomial p such that $p(A) \in \mathscr{G}_2(A)$.*

Proof. [3, Corollary 1.4 and Theorem 2.1]. $\square$

7.3. Theorem. *Let $A \in \mathscr{R}(X)$, let p be a polynomical and suppose that $p(0) = 0$.
Then*

$$A = \pm Ap(A) \iff \mathscr{D}(A) = \{U - p(A)Up(A) : U \in \mathscr{L}(X)\}\,.$$

Proof. "$\Rightarrow$": Since $p(0)=0$, $p(\lambda)=\lambda q(\lambda)$ for a polynomial q, hence $p(A)=Aq(A)$. Let

$$B=\begin{cases} q(A), & \text{if } A=Ap(A)\\ -q(A), & \text{if } A=-Ap(A)\,. \end{cases}$$

Then $ABA=A$, thus $B\in\mathscr{G}_1(A)$ and

$$\begin{aligned}\mathscr{D}(A)&=\{U-BAUAB : U\in\mathscr{L}(X)\}\\ &=\{U-p(A)Up(A) : U\in\mathscr{L}(X)\}\,.\end{aligned}$$

"$\Leftarrow$": We have

$$(*)\qquad AUA=Ap(A)Up(A)A \quad \text{for all } U\in\mathscr{L}(X)\,.$$

If $B\in\mathscr{G}_1(A)$ it follows that

$$A=ABA=Ap(A)Bp(A)A=p(A)ABAp(A)=Ap(A)^2\,,$$

thus $A(I-p(A)^2)=0$. The spectral mapping theorem gives

$$\sigma(A)\subseteq\{0\}\cup\{\lambda\in\mathbb{C} : p(\lambda)^2=1\}\,.$$

Since A is algebraic, each element of $\sigma(A)$ is an eigenvalue of A (see [2, Satz 101.2]). Now take $\lambda\in\sigma(A)\setminus\{0\}$. Then $Ax_0=\lambda x_0$ for some $x_0\neq 0$, thus $p(A)x_0=p(\lambda)x_0=\pm x_0$.

From $(*)$ we get

$$\begin{aligned}\lambda AUx_0&=AUAx_0=Ap(A)Up(A)Ax_0\\ &=\pm\lambda Ap(A)Ux_0\end{aligned}$$

for all $U\in\mathscr{L}(X)$. Since $\lambda\neq 0$, we derive

$$AUx_0=\pm Ap(A)Ux_0 \quad \text{for all } U\in\mathscr{L}(X)\,.$$

Now let $x\in X$. There is $U\in\mathscr{L}(X)$ such that $Ux_0=x$, hence

$$Ax=\pm Ap(A)x\,.$$

Therefore $A=\pm Ap(A)$. □

The proof just given shows:

7.4. Corollary. *Let A and p as in Theorem 7.3. Then:*

(1) $A=Ap(A)\iff\mathscr{D}(A)=\{U-p(A)Up(A) : U\in\mathscr{L}(X)\}$ *and* $p(\lambda)\neq -1$ *for all* $\lambda\in\sigma(A)$*;*
(2) $A=-Ap(A)\iff\mathscr{D}(A)=\{U-p(A)Up(A) : U\in\mathscr{L}(X)\}$ *and* $p(\lambda)\neq 1$ *for all* $\lambda\in\sigma(A)$*.*

7.5. Examples. *Let $A\in\mathscr{R}(X)$. Then:*

(1) $A=\pm A^2\iff\mathscr{D}(A)=\{U-AUA : U\in\mathscr{L}(X)\}$*;*
(2) $A^2=A\iff\mathscr{D}(A)=\{U-AUA : U\in\mathscr{L}(X)\}$ *and* $-1\notin\sigma(A)$*;*
(3) $A^2=-A\iff\mathscr{D}(A)=\{U-AUA : U\in\mathscr{L}(X)\}$ *and* $1\notin\sigma(A)$*;*
(4) $A=\pm A^3\iff\mathscr{D}(A)=\{U-A^2UA^2 : U\in\mathscr{L}(X)\}$*;*
(5) $A=A^3\iff\mathscr{D}(A)=\{U-A^2UA^2 : U\in\mathscr{L}(X)\}$ *and* $i,-i\notin\sigma(A)$*;*
(6) $A=-A^3\iff\mathscr{D}(A)=\{U-A^2UA^2 : U\in\mathscr{L}(X)\}$ *and* $1,-1\notin\sigma(A)$*.*

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