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AN EXAMPLE ON ORDERED BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{B}$ be a complex unital Banach algebra. We consider the Banach algebra $\mathcal{A} = \mathcal{B} \times \mathbb{C}$ ordered by the algebra cone $K = \{(a, \xi) \in \mathcal{A} : \|a\| \leq \xi\}$, and investigate the connection between results on ordered Banach algebras and the right bound of the numerical range of elements in $\mathcal{B}$.

1. Ordered Banach algebras

The aim of this paper is to stress the aspect of the applicability of the ordered Banach algebra theory, within a wider scope of general Banach algebras. To this end we will embed a given (non-ordered) Banach algebra $\mathcal{B}$ into a certain ordered Banach algebra $\mathcal{A}$; see section 2. In particular Theorems 6 and 7 in section 5 are results strictly in terms of the originally given Banach algebra $\mathcal{B}$, while the proofs involve results on ordered Banach algebras applied to $\mathcal{A}$. We start with some notations and results on general ordered Banach algebras.

Let $\mathcal{A}$ be a complex unital Banach algebra with unit $\mathbf{1}$, and assume that $\mathcal{A}$ is ordered by an algebra cone K, that is, K is a closed convex subset of $\mathcal{A}$ with $\lambda K \subseteq K$ $(\lambda \geq 0)$, $K \cap (-K) = \{0\}$, $\mathbf{1} \in K$, $K \cdot K \subseteq K$, and $a \leq b :\iff b - a \in K$.

For a general survey on ordered Banach algebras we refer to [4], [9], and the references given there.

The spectrum of $a \in \mathcal{A}$ is denoted by $\sigma(a)$, and $r(a)$ denotes its spectral radius. Moreover let $\tau(a)$ denote the right spectral bound of a, that is,

$$\tau(a) := \max\{\text{Re } \lambda : \lambda \in \sigma(a)\}.$$

Next, we consider the sets

$$Q_+ := \{a \in \mathcal{A} : \exp(ta) \geq 0 \ (t \geq 0)\}, \quad Q_\pm = Q_+ \cap (-Q_+).$$

Amongst other things the following properties of these sets are known. Note that $\emptyset \neq W \subseteq \mathcal{A}$ is called a wedge if W is closed, convex and $\lambda W \subseteq W$ $(\lambda \geq 0)$.

Theorem 1. *Under the assumptions above*

1. Q_+ *is a wedge, and* $\overline{K + \mathbb{R}\mathbf{1}} = Q_+$*;*
2. $Q_\pm$ *is a closed real subspace of* $\mathcal{A}$*;*
3. $Q_\pm$ *is a real Lie algebra;*
4. $a \in Q_\pm \Longrightarrow a^2 \in Q_+$*;*

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5. $a \in Q_+$, $\tau(a) < 0 \Longrightarrow a^{-1} \le 0$;
6. $a \in Q_+$, $\tau(a) < 0 \Longrightarrow a(\lambda\mathbf{1} - a)^{-1} \in Q_+$ $(\lambda \ge 0)$.

Proof. Parts 1, 2, 4 and 5 of Theorem 1 have been proved in [4] for real Banach algebras (where then in part 5 $\tau(a)$ is obtained by complexification), but hold obviously in the complex case by considering $\mathcal{A}$ as a real Banach algebra (by restricting the scalar field to $\mathbb{R}$). Part 3 of Theorem 1 follows easily from the commutator formula for the exponential function; see [5, Lemma I.2.13]. To prove part 6 note that $\tau(a - \lambda\mathbf{1}) < 0$ $(\lambda \ge 0)$ and consider

$$a(\lambda\mathbf{1} - a)^{-1} = (\lambda\mathbf{1} + a - \lambda\mathbf{1})(\lambda\mathbf{1} - a)^{-1} = -\lambda(a - \lambda\mathbf{1})^{-1} - \mathbf{1}.$$

Thus $a(\lambda\mathbf{1} - a)^{-1} \in K + \mathbb{R}\mathbf{1}$ according to part 5; hence it is in Q_+ by 1. □

In particular part 6 of Theorem 1 can be used to find roots of elements of $-Q_+$ in Q_+ by using the formula of Balakrishnan [10, p.260].

Theorem 2. *Let $a \in Q_+$ with $\tau(a) < 0$, and let $\alpha \in (0, 1)$. Then*

$$-(-a)^{\alpha} := \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \lambda^{\alpha-1} a(\lambda\mathbf{1} - a)^{-1} d\lambda \in Q_+.$$

Proof. By means of part 6 of Theorem 1 we see that the integrand is in Q_+ for each $\lambda \in (0, \infty)$. Since Q_+ is a wedge we conclude that $-(-a)^{\alpha} \in Q_+$. □

For our third theorem, which is on majorization, let the unital Banach algebra $\mathcal{A} \times \mathcal{A}$ (endowed with coordinatewise multiplication and the norm $\|(a, b)\| = \max\{\|a\|, \|b\|\}$) be ordered by the algebra cone

$$\widetilde{K} = \{(a, b) \in \mathcal{A} \times \mathcal{A} : -b \le a \le b\},$$

and let $\widetilde{Q}_+$ denote the corresponding set of exponentially nonnegative elements. The following theorem is proved in [3] for operators on ordered Banach spaces and holds with an analog proof for Banach algebras in the following version:

Theorem 3. *Let $a, b \in \mathcal{A}$ be such that $a \le b$ and $a + b \in Q_+$. Then $(a, b) \in \widetilde{Q}_+$.*

2. Special ordered Banach algebras

Now, let $\mathcal{B}$ be any complex unital Banach algebra with unit $\mathbf{1}$ and let $\mathcal{A}$ denote the Banach algebra $\mathcal{B} \times \mathbb{C}$, endowed with multiplication and norm

$$(a, \xi) \cdot (b, \eta) = (ab, \xi\eta), \quad \|(a, \xi)\| = \max\{\|a\|, |\xi|\},$$

respectively. Then $\mathcal{A}$ is a unital Banach algebra with unit $(\mathbf{1}, 1)$, and we consider the ordering defined by the cone

$$K = \{(a, \xi) \in \mathcal{A} : \|a\| \le \xi\}.$$

We will characterize Q_+ and $Q_{\pm}$ in this case, and we will obtain results on the right bound of the numerical range of elements of $\mathcal{B}$, by means of Theorem 1.

3. Bounds of the numerical range

Let $\mathcal{B}$ be a complex unital Banach algebra and consider the one-sided directional derivatives of the norm at $\mathbf{1}$:

$$n_+[a] = \lim_{h\to 0+} \frac{\|\mathbf{1}+ha\|-1}{h}, \quad n_-[a] = \lim_{h\to 0-} \frac{\|\mathbf{1}+ha\|-1}{h} \quad (a \in \mathcal{B}).$$

For each $a \in \mathcal{B}$ let $V(a)$ denote the numerical range of a, that is,

$$V(a) = \{\varphi(a) : \varphi \in \mathcal{B}^*, \|\varphi\| = \varphi(\mathbf{1}) = 1\}.$$

The set $V(a)$ is bounded closed and convex, and $\sigma(a) \subseteq V(a)$ for each $a \in \mathcal{B}$; see [2]. The number $n_+[a]$ is the right bound of the numerical range of a, that is,

$$n_+[a] = \max\{\text{Re } \lambda : \lambda \in V(a)\},$$

according to Mazur's representation of directional derivatives of sublinear functionals [8]; see also [2], [7]. In particular $\tau(a) \leq n_+[a]$.

Note the following properties of n_+ and n_-: The function $a \mapsto n_+[a]$ is sublinear, and

$$n_-[a] \leq n_+[a], \quad n_-[-a] = -n_+[a], \quad n_\pm[a + \lambda\mathbf{1}] = n_\pm[a] + \lambda$$

for all $a \in \mathcal{B}$ and $\lambda \in \mathbb{R}$.

Next, let $H(\mathcal{B})$ denote the set of all hermitian elements of $\mathcal{B}$, that is,

$$H(\mathcal{B}) = \{a \in \mathcal{B} : \|\exp(ita)\| = 1 \ (t \in \mathbb{R})\}.$$

For the following properties of hermitian elements see [2, 5. Lemma 2] for part 1 and [1] for part 2.

Proposition 1. *We have*

1. $a \in H(\mathcal{B}) \iff V(a) \subseteq \mathbb{R} \iff n_+[ia] = n_-[ia] = 0$;
2. $a \in H(\mathcal{B}) \iff \sigma(a) \subseteq \mathbb{R}$ *and* $r(\alpha\mathbf{1} + \beta a) = \|\alpha\mathbf{1} + \beta a\| \ (\alpha, \beta \in \mathbb{C})$.

Now, in the sequel let $\mathcal{A} = \mathcal{B} \times \mathbb{C}$ be as in section 2.

4. Representation of Q_+ and $Q_\pm$

Note that

$$Q_+ = \{(a, \xi) \in \mathcal{A} : \|\exp(ta)\| \leq \exp(t\xi) \ (t \geq 0)\}$$

in our case.

Theorem 4. *We have*

1. $(a, \xi) \in Q_+$ *if and only if* $n_+[a] \leq \xi$.
2. $(a, \xi) \in Q_\pm$ *if and only if one of the following equivalent conditions is valid:*
   (a) $\|\exp(ta)\| = \exp(t\xi) \ (t \in \mathbb{R})$;
   (b) $n_+[a] = n_-[a] = \xi$;
   (c) $\xi \in \mathbb{R}$ *and* $i(a - \xi\mathbf{1}) \in H(\mathcal{B})$;
   (d) $\text{Re } \lambda = \xi \ (\lambda \in \sigma(a))$ *and* $r(\alpha\mathbf{1} + \beta a) = \|\alpha\mathbf{1} + \beta a\| \ (\alpha, \beta \in \mathbb{C})$.

Proof. 1. First note that for each $a \in \mathcal{B}$

$$\|\exp(ta)\| \leq \exp(tn_+[a]) \quad (t \geq 0);$$

see [2, 3. Theorem 4]. Thus, $n_+[a] \leq \xi$ implies $(a, \xi) \in Q_+$. On the other hand, $(a, \xi) \in Q_+$ implies

$$\frac{\|\exp(ta)\| - 1}{t} \leq \frac{\exp(t\xi) - 1}{t} \quad (t > 0)$$

and as $t \to 0+$ we obtain $n_+[a] \le \xi$, since $\exp(ta) = 1 + ta + O(t^2)$ $(t \to 0+)$.

2. Let $(a, \xi) \in Q_\pm$. Then $\|\exp(ta)\| \le \exp(t\xi)$ $(t \in \mathbb{R})$. From

$$\|\exp(ta)\| \ge \frac{1}{\|(\exp(ta))^{-1}\|} = \frac{1}{\|\exp(-ta)\|} \ge \frac{1}{\exp(-t\xi)} = \exp(t\xi)$$

we obtain $\|\exp(ta)\| = \exp(t\xi)$ $(t \in \mathbb{R})$.

Next, by means of part 1 we have

$$n_-[a] \le n_+[a] \le \xi, \; -n_-[a] = n_+[-a] \le -\xi,$$

thus

$$\xi \le n_-[a] \le n_+[a] \le \xi \Rightarrow n_-[a] = n_+[a] = \xi.$$

In particular, $\xi \in \mathbb{R}$ and

$$n_\pm[a - \xi\mathbf{1}] = n_\pm[a] - \xi = 0.$$

Thus

$$\lim_{h \to 0} \frac{\|1 + h(a - \xi\mathbf{1})\| - 1}{h} = 0,$$

and according to part 1 of Proposition 1, $i(a - \xi\mathbf{1}) \in H(\mathcal{B})$.

Since clearly each element (a, ξ) with the property in (a) or (b) is in $Q_\pm$ and since (c) implies (b) according to part 1 of Proposition 1, we have proved that $(a, \xi) \in Q_\pm$ if and only if (a, ξ) satisfies (a), (b) or (c), respectively

To prove the equivalence of (c) and (d) first let (a, ξ) be such that $\xi \in \mathbb{R}$ and $b := i(a - \xi\mathbf{1}) \in H(\mathcal{B})$. According to part 2 of Proposition 1

$$\sigma(b) \subseteq \mathbb{R} \Rightarrow \sigma(a) \subseteq \{\xi + it : t \in \mathbb{R}\}.$$

For $\alpha, \beta \in \mathbb{C}$ we obtain

$$r(\alpha\mathbf{1} + \beta a) = r((\alpha + \xi\beta)\mathbf{1} - i\beta b) = \|(\alpha + \xi\beta)\mathbf{1} - i\beta b\| = \|\alpha\mathbf{1} + \beta a\|.$$

Vice versa, let (a, ξ) satisfy (d), and set $b := i(a - \xi\mathbf{1})$. Then $\xi \in \mathbb{R}$, and $\sigma(b) \subseteq \mathbb{R}$. For $\alpha, \beta \in \mathbb{C}$ we have

$$\alpha\mathbf{1} + \beta b = (\alpha - i\xi\beta)\mathbf{1} + i\beta a,$$

thus

$$r(\alpha\mathbf{1} + \beta b) = \|\alpha\mathbf{1} + \beta b\|.$$

Once more using part 2 of Proposition 1 we conclude $b \in H(\mathcal{B})$. □

5. Applications

By application of Theorem 1 to the algebra $\mathcal{A}$ we obtain:

Theorem 5. *Let $a, b \in \mathcal{B}$.*

1. *If $a, b \in H(\mathcal{B})$, then $i(ab - ba) \in H(\mathcal{B})$; compare* [2, 5. Lemma 4].
2. *If $a \in H(\mathcal{B})$, then $n_+[-a^2] \le 0$.*
3. *$\tau((a, n_+[a])) = n_+[a]$.*
4. *If $n_+[a] < 0$, then $\|a^{-1}\| \le -(n_+[a])^{-1}$; compare* [6, Proposition1.1(3)].
5. *If $n_+[a] < 0$, then*

$$n_+[a(\lambda\mathbf{1} - a)^{-1}] \le \frac{n_+[a]}{\lambda - n_+[a]} \quad (\lambda \ge 0).$$

Proof. 1. According to Theorem 4 we have $(ia, 0), (ib, 0) \in Q_\pm$. Hence $(-ab + ba, 0) \in Q_\pm$, and again Theorem 4 proves $i(ab - ba) \in H(\mathcal{B})$.

2. Since $(ia, 0) \in Q_\pm$ we have $(-a^2, 0) \in Q_+$. Hence $n_+[-a^2] \leq 0$.

3. If Re $\lambda > n_+[a]$, then $(a, n_+[a]) - \lambda(\mathbf{1}, 1)$ is invertible, since $\tau(a) \leq n_+[a]$ and $n_+[a] - \lambda \neq 0$. Moreover $(a, n_+[a]) - n_+[a](\mathbf{1}, 1) = (a - n_+[a]\mathbf{1}, 0)$ is not invertible. Thus $n_+[a] \in \sigma((a, n_+[a]))$.

4. We have $(a, n_+[a]) \in Q_+$, and $\tau((a, n_+[a])) = n_+[a] < 0$. Thus

$$-(a, n_+[a])^{-1} = (-a^{-1}, (-n_+[a])^{-1}) \geq 0,$$

which means $\|a^{-1}\| \leq -(n_+[a])^{-1}$.

5. Since $(a, n_+[a]) \in Q_+$ and since $\tau((a, n_+[a])) = n_+[a] < 0$ we have

$$(a, n_+[a])(\lambda(\mathbf{1}, 1) - (a, n_+[a]))^{-1}$$
$$= (a(\lambda\mathbf{1} - a)^{-1}, n_+[a](\lambda - n_+[a])^{-1}) \in Q_+$$

for each $\lambda \geq 0$. According to Theorem 4

$$n_+[a(\lambda\mathbf{1} - a)^{-1}] \leq \frac{n_+[a]}{\lambda - n_+[a]} \quad (\lambda \geq 0).$$

□

Analogously we can apply Theorem 2 and derive one sided estimates for fractional powers:

Theorem 6. *Let $a \in \mathcal{B}$ with $n_+[a] < 0$, and $\alpha \in (0, 1)$. Then*

$$n_+[-(-a)^\alpha] \leq -(-n_+[a])^\alpha.$$

Proof. Since $(a, n_+[a]) \in Q_+$ we have

$$-(-(a, n_+[a]))^\alpha = (-(-a)^\alpha, -(-n_+[a])^\alpha) \in Q_+.$$

□

Remark. For results on the numerical range of roots in Banach algebras see [6] and the references given there.

Next, Theorem 3 leads to the following inequalities for the exponential function in Banach algebras:

Theorem 7. *Let $a, b \in \mathcal{B}$. Then*

$$\|\exp(b) - \exp(a)\| \leq 2\sinh\left(\frac{\|b - a\|}{2}\right)\exp\left(\frac{n_+[a + b]}{2}\right)$$

and

$$\|\exp(b) + \exp(a)\| \leq 2\cosh\left(\frac{\|b - a\|}{2}\right)\exp\left(\frac{n_+[a + b]}{2}\right).$$

Proof. We set

$$\alpha = \frac{n_+[a + b] - \|b - a\|}{2}, \quad \beta = \frac{n_+[a + b] + \|b - a\|}{2}.$$

Then

$$\|b - a\| = \beta - \alpha \Rightarrow (a, \alpha) \leq (b, \beta)$$

and

$$n_+[a + b] = \alpha + \beta \Rightarrow (a, \alpha) + (b, \beta) \in Q_+.$$

According to Theorem 3 we have $((a,\alpha),(b,\beta)) \in \widetilde{Q}_+$. Hence

$$-(\exp(tb),\exp(t\beta)) \leq (\exp(ta),\exp(t\alpha)) \leq (\exp(tb),\exp(t\beta)) \quad (t \geq 0).$$

For $t = 1$ we obtain

$$\|\exp(b) - \exp(a)\| \leq \exp(\beta) - \exp(\alpha) = 2\sinh\left(\frac{\|b-a\|}{2}\right)\exp\left(\frac{n_+[a+b]}{2}\right),$$

and the second inequality in an analog way. □

Remark. The inequalities in Theorem 7 are quite obvious if a and b commute. Then, for example,

$$\|\exp(b) - \exp(a)\| = \|2\sinh\left(\frac{b-a}{2}\right)\exp\left(\frac{a+b}{2}\right)\|$$

$$\leq 2\sinh\left(\frac{\|b-a\|}{2}\right)\|\exp\left(\frac{a+b}{2}\right)\| \leq 2\sinh\left(\frac{\|b-a\|}{2}\right)\exp\left(\frac{n_+[a+b]}{2}\right).$$

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