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Generalized Fredholm theory in semisimple algebras

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RIASSUNTO: *Sia* $\mathcal{A}$ *una algebra complessa semisemplice con identità* $e \neq 0$. *Sia* $\Phi_g(\mathcal{A})$ *la sottoclasse formata dagli elementi* $x \in \mathcal{A}$ *che verificano la seguente condizione:*

$\exists\, y \in \mathcal{A}$: tale che $xyx = x$, e inoltre $e - xy - yx$ è un elemento di Fredholm.

Ogni elemento di Fredholm appartiene a $\Phi_g(\mathcal{A})$. *Si studia la classe* $\Phi_g(\mathcal{A})$ *i cui elementi sono detti* elementi di Fredholm generalizzati.

ABSTRACT: *Let* $\mathcal{A}$ *be a semisimple complex algebra with identity* $e \neq 0$. *We write* $\Phi_g(\mathcal{A})$ *for the following class of elements of* $\mathcal{A}$.

$$\Phi_g(\mathcal{A}) = \{x \in \mathcal{A} : \exists\, y \in \mathcal{A} \text{ such that } xyx = x \text{ and } e - xy - yx \text{ is Fredholm}\}.$$

Each Fredholm element of $\mathcal{A}$ *belongs to* $\Phi_g(\mathcal{A})$. *Elements in* $\Phi_g(\mathcal{A})$ *we call* generalized Fredholm elements. *In this paper we investigate the class* $\Phi_g(\mathcal{A})$.

1 – Introduction

In this paper we always assume that $\mathcal{A}$ *is a complex algebra with identity* $e \neq 0$. If X is a complex Banach space, then it is well known that $\mathcal{L}(X) = \{T : X \to X : T \text{ is linear and bounded}\}$ is a semisimple Banach algebra.

In [1] S. R. CARADUS has introduced the class of *generalized Fredholm operators.* $T \in \mathcal{L}(X)$ is called a generalized Fredholm operator, if

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there is some $S \in \mathcal{L}(X)$ with $TST = T$ and $I - TS - ST$ is a Fredholm operator. This class of operators is studied in [15], [16] and [17].

If $\mathcal{A}$ is semisimple, *generalized Fredholm elements* in $\mathcal{A}$ are introduced in [10] as follows: $x \in \mathcal{A}$ is called a generalized Fredholm element if there is some $y \in \mathcal{A}$ such that $xyx = x$ and $e - xy - yx$ is a Fredholm element in $\mathcal{A}$. Some of the results in [15] and [16] are generalized in [10].

The present paper is an improvement and a continuation of [10]. Furthermore we generalize some of the results in [17].

In Section 2 of this paper we collect some results concerning relatively regular elements in algebras. Section 3 contains a summary of Fredholm theory in semisimple algebras. In section 4 we investigate generalized invertible elements. This concept will be useful in the next sections, where we present the main results of this paper.

In Section 5 we study algebraic properties of generalized Fredholm elements. Section 6 contains a characterization of Riesz elements in complex semisimple Banach algebras and a result concerning the stability of generalized Fredholm elements under holomorphic functional calculus. Section 7 contains various results on ascent and descent and a "punctured neighbourhood theorem" for generalized Fredholm elements.

2 – Relatively regular elements

An element $x \in \mathcal{A}$ is called *relatively regular*, if $xyx = x$ for some $y \in \mathcal{A}$. In this case y is called a *pseudo-inverse* of x.

Proposition 2.1. *For $x \in \mathcal{A}$ the following assertions are equivalent:*

(1) *x is relatively regular.*

(2) *There is $y \in \mathcal{A}$ with $xyx = x$ and $yxy = y$.*

(3) *There is $p = p^2 \in \mathcal{A}$ with $x\mathcal{A} = p\mathcal{A}$.*

(4) *There is $q = q^2 \in \mathcal{A}$ with $\mathcal{A}x = \mathcal{A}q$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $xy_0x = x$. Put $y = y_0xy_0$. Then it is easy to see that $xyx = x$ and $yxy = y$.

(2) $\Rightarrow$ (1): Clear.

(1) $\Rightarrow$ (3): Take $y \in \mathcal{A}$ with $xyx = x$ and put $p = xy$. Then $x\mathcal{A} = xyx\mathcal{A} \subseteq p\mathcal{A} = xy\mathcal{A} \subseteq x\mathcal{A}$.

(3) $\Rightarrow$ (1): We have $p = xa$ for some $a \in \mathcal{A}$ and $x = px$, thus $x = px = (xa)x = xax$.

Similar arguments as above show that (1) and (4) are equivalent. ∎

PROPOSITION 2.2. *Suppose that $x, u \in \mathcal{A}$, $xux - x$ is relatively regular and that r is a pseudo-inverse of $xux - x$. Then x is relatively regular and*

$$y = u - r + uxr + rxu - uxrxu$$

is a pseudo-inverse of x.

PROOF. From $(xux - x)r(xux - x) = xux - x$, we get

$$\begin{aligned} x &= xux - xuxrxux + xuxrx + xrxux - xrx = \\ &= x(u - uxrxu + uxr + rxu - r)x = xyx \ . \end{aligned}$$ ∎

For $x \in \mathcal{A}$ we define

$$R(x) = \{a \in \mathcal{A} : \ xa = 0\} \text{ and } L(x) = \{a \in \mathcal{A} : \ ax = 0\} \ .$$

The proof of the next proposition is easy and left to the reader.

PROPOSITION 2.3. *Suppose that $x \in \mathcal{A}$ is relatively regular and y is a pseudo-inverse of x. Then xy, yx, $e - xy$ and $e - yx$ are idempotent and*

$$\begin{aligned} &xy\mathcal{A} = x\mathcal{A},\ \mathcal{A}yx = \mathcal{A}x, \\ &R(x) = (e - yx)\mathcal{A},\ L(x) = \mathcal{A}(e - xy) \ . \end{aligned}$$

A proof for the following result can be found in [6, p. 15].

PROPOSITION 2.4. *If $x \in \mathcal{A}$ is relatively regular, $xyx = x$ and $yxy = y$, then we have for $z \in \mathcal{A}$:*

z is a pseudo-inverse of x if and only if there is some $u \in \mathcal{A}$ with

$$z = y + u - yxuxy \ .$$

3 – Fredholm theory in semisimple algebras

Throughout this section we assume that $\mathcal{A}$ is semisimple. This means that $\operatorname{rad}(\mathcal{A}) = \{0\}$, where $\operatorname{rad}(\mathcal{A})$ denotes the radical of $\mathcal{A}$. For the convenience of the reader we shall summarize some concepts of the Fredholm theory in algebras. See [2]-[4], [11]-[14], [18]-[20] for details.

We call an element $e_0 \in \mathcal{A}$ *minimal idempotent*, if $e_0\mathcal{A}e_0$ is a division algebra and $e_0^2 = e_0$. $\operatorname{Min}(\mathcal{A})$ denotes the set of all minimal idempotents of $\mathcal{A}$.

Proposition 3.1. (1) *Suppose that $\mathcal{R} \subseteq \mathcal{A}[\mathcal{L} \subseteq \mathcal{A}]$ is a right [left] ideal in $\mathcal{A}$. Then $\mathcal{R}[\mathcal{L}]$ is a minimal right [left] ideal if and only if $\mathcal{R} = e_0\mathcal{A}\,[\mathcal{L} = \mathcal{A}e_0]$ for some $e_0 \in \operatorname{Min}(\mathcal{A})$.*

(2) *If $\operatorname{Min}(\mathcal{A}) \neq \emptyset$, then the sum of all minimal right ideals equals the sum of all minimal left ideals.*

Proof. (1) [4, B.A. 3.1], (2) [5, Prop. 30.10.]. □

The *socle* of $\mathcal{A}$, $\operatorname{soc}(\mathcal{A})$, is defined to be the sum of all minimal right ideals if $\operatorname{Min}(\mathcal{A}) \neq \emptyset$. If $\operatorname{Min}(\mathcal{A}) = \emptyset$, then we set $\operatorname{soc}(\mathcal{A}) = \{0\}$. Proposition 3.1 shows that

$$\operatorname{soc}(\mathcal{A}) \text{ is an ideal of } \mathcal{A}, \tag{3.2}$$

and

$$\operatorname{Min}(\mathcal{A}) \subseteq \operatorname{soc}(\mathcal{A}). \tag{3.3}$$

From now on we always assume in this section that $\operatorname{soc}(\mathcal{A}) \neq \{0\}$.

Suppose that $\mathcal{J} \subseteq \mathcal{A}$ is a right [left] ideal of $\mathcal{A}$. $\mathcal{J}$ has *finite order* if $\mathcal{J}$ can be written as the sum of a finite number of minimal right [left] ideals of $\mathcal{A}$. The *order* $\Theta(\mathcal{J})$ of $\mathcal{J}$ is defined to be the smallest number of minimal right [left] ideals which have sum $\mathcal{J}$. We define $\Theta(\{0\}) = 0$ and $\Theta(\mathcal{J}) = \infty$, if $\mathcal{J}$ does not have finite order.

Proposition 3.4. *Suppose that $\mathcal{J}$ and $\mathcal{K}$ are right [left] ideals of $\mathcal{A}$ and $n \in \mathbb{N}$.*

(1) $\Theta(\mathcal{J}) < \infty \;\Leftrightarrow\; \mathcal{J} \subseteq \operatorname{soc}(\mathcal{A})$.

(2) $\Theta(\mathcal{J}) = n$, *if and only if there are* $e_1, \dots, e_n \in \mathrm{Min}(\mathcal{A})$ *such that* $e_i e_j = 0$ *for* $i \neq j$ *and*

$$\mathcal{J} = (e_1 + \dots e_n)\mathcal{A} = e_1\mathcal{A} \oplus \cdots \oplus e_n\mathcal{A}$$
$$[\mathcal{J} = \mathcal{A}(e_1 + \cdots + e_n) = \mathcal{A}e_1 \oplus \cdots \oplus \mathcal{A}e_n] \ .$$

(3) *If* $\Theta(\mathcal{K}) < \infty$, $\mathcal{J} \subseteq \mathcal{K}$ *and* $\mathcal{J} \neq \mathcal{K}$ *then* $\Theta(\mathcal{K}) < \Theta(\mathcal{J})$.

(4) $\Theta(x\mathcal{A}) = \Theta(\mathcal{A}x)$ *for each* $x \in \mathcal{A}$.

(5) $\mathrm{soc}(\mathcal{A}) = \{x \in \mathcal{A} : \ \Theta(x\mathcal{A}) < \infty\}$.

PROOF. (1) Clear. (2) and (3): [2, §2]. (4) and (5): [9]. □

DEFINITIONS.

(1) For $x \in \mathcal{A}$ we define the *nullity* of x by

$$\mathrm{nul}(x) = \Theta(R(x))$$

and the *defect* of x by

$$\mathrm{def}(x) = \Theta(L(x)) \ .$$

(2) The group of the invertible elements of $\mathcal{A}$ is denoted by $\mathcal{A}^{-1}$.

(3) The quotient algebra $\mathcal{A}/\mathrm{soc}(\mathcal{A})$ is denoted by $\widehat{\mathcal{A}}$. For $x \in \mathcal{A}$ we write $\widehat{x} = x + \mathrm{soc}(\mathcal{A})$ for the coset of x in $\widehat{\mathcal{A}}$.

(4) The set of *Fredholm elements* of $\mathcal{A}$ is given by

$$\Phi(\mathcal{A}) = \{x \in \mathcal{A} : \ \widehat{x} \in \widehat{\mathcal{A}}^{-1}\} \ .$$

The next proposition contains some useful characterisations of Fredholm elements.

PROPOSITION 3.5. *For* $x \in \mathcal{A}$ *the following assertions are equivalent:*

(1) $x \in \Phi(\mathcal{A})$.

(2) *There are* $p, q \in \mathrm{soc}(\mathcal{A})$ *such that* $p = p^2$, $q = q^2$ *and*

$$\mathcal{A}x = \mathcal{A}(e - p), \ x\mathcal{A} = (e - q)\mathcal{A} \ .$$

(3) *x is relatively regular and* $R(x), L(x) \subseteq \mathrm{soc}(\mathcal{A})$*.*

(4) *x is relatively regular and* $\mathrm{nul}(x)$*,* $\mathrm{def}(x) < \infty$*.*

(5) *x is relatively regular and for each pseudo-inverse y of x we have* $\widehat{x}\widehat{y} = \widehat{e} = \widehat{y}\widehat{x}$*.*

(6) *x is relatively regular and there is a pseudo-inverse y of x such that* $\widehat{x}\widehat{y} = \widehat{e} = \widehat{y}\widehat{x}$*.*

Proof. (1) $\Leftrightarrow$ (2) [4, F.1.10].

(2) $\Rightarrow$ (3): It is easy to see that $R(x) = p\mathcal{A}$ and $L(x) = q\mathcal{A}$. Thus $R(x), L(x) \subseteq \mathrm{soc}(\mathcal{A})$.

(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5): Suppose that y is a pseudo-inverse of x. Proposition 2.3 gives

$$R(x) = (e - yx)\mathcal{A} \text{ and } L(x) = \mathcal{A}(e - xy) \ .$$

Therefore we get from Proposition 3.4 (1):

$$\begin{aligned} &R(x), L(x) \subseteq \mathrm{soc}(\mathcal{A}) \Leftrightarrow \Theta(R(x)), \Theta(L(x)) < \infty \Leftrightarrow \\ &\Leftrightarrow \mathrm{nul}(x), \mathrm{def}(x) < \infty \Leftrightarrow e - yx, e - xy \in \mathrm{soc}(\mathcal{A}) \ . \end{aligned}$$

(5) $\Rightarrow$ (6): Clear.

(6) $\Rightarrow$ (1): From $\widehat{x}\widehat{y} = \widehat{e} = \widehat{y}\widehat{x}$ we get $\widehat{x} \in \widehat{\mathcal{A}}^{-1}$, thus $x \in \Phi(\mathcal{A})$. □

The *index* of $x \in \Phi(\mathcal{A})$ is defined by

$$\mathrm{ind}(x) = \mathrm{nul}(x) - \mathrm{def}(x) \ .$$

A proof of the next result can be found in [19, Theorem 4.5 and Theorem 4.6].

Theorem 3.6. *If* $x, y \in \Phi(\mathcal{A})$ *and* $s \in \mathrm{soc}(\mathcal{A})$ *then*

(1) $xy \in \Phi(\mathcal{A})$ *and* $\mathrm{ind}(xy) = \mathrm{ind}(x) + \mathrm{ind}(y)$*;*

(2) $x + s \in \Phi(\mathcal{A})$ *and* $\mathrm{ind}(x + s) = \mathrm{ind}(x)$*;*

(3) *If* $\mathcal{A}$ *is a Banach algebra then there are* $\delta > 0$ *and* $\alpha, \beta \in \mathbb{N}_0$ *such that*
   - (i) $x + u \in \Phi(\mathcal{A})$*,* $\mathrm{ind}(x + u) = \mathrm{ind}(x)$*,* $\mathrm{nul}(x + u) \le \mathrm{nul}(x)$ *and* $\mathrm{def}(x + u) \le \mathrm{def}(x)$ *for all* $u \in \mathcal{A}$ *with* $\|u\| < \delta$*.*
   - (ii) $\mathrm{nul}(\lambda e - x) = \alpha \le \mathrm{nul}(x)$ *and* $\mathrm{def}(\lambda e - x) = \beta \le \mathrm{def}(x)$ *for* $\lambda \in \mathbb{C}$ *with* $0 < |\lambda| < \delta$*.*

The ideal of *inessential elements* of $\mathcal{A}$ is given by

$$I(\mathcal{A}) = \bigcap \{P : \ P \text{ is a primitive ideal of } \mathcal{A} \text{ with } \mathrm{soc}(\mathcal{A}) \subseteq \mathcal{P}\} \, .$$

We write $\widetilde{\mathcal{A}}$ for the quotient algebra $\mathcal{A}/I(\mathcal{A})$ and $\widetilde{x}$ for the coset $x + I(\mathcal{A})$ of $x \in \mathcal{A}$.

Proposition 3.7.
(1) $\mathrm{soc}(\mathcal{A}) \subseteq I(\mathcal{A})$.
(2) $x \in \Phi(\mathcal{A}) \Leftrightarrow \widetilde{x} \in \widetilde{\mathcal{A}}^{-1}$.
(3) *If* $\mathcal{A}$ *is a Banach algebra, then* $I(\mathcal{A})$ *is closed.*

Proof. (1) Clear. (2) [4, F.3.2]. (3) Each primitive ideal of a Banach algebra is closed. ∎

Proposition 3.8. *Let* $s \in \mathrm{soc}(\mathcal{A})$. *Then* s *is relatively regular and there is* $b \in \mathrm{soc}(\mathcal{A})$ *such that*

$$sbs = s \quad \text{and} \quad bsb = b \, .$$

Proof. From Proposition 3.4 we get $e_1, \ldots, e_n \in \mathrm{Min}(\mathcal{A})$ with $e_i e_j = \delta_{ij} e_i$ and

$$s\mathcal{A} = (e_1 + \cdots + e_n)\mathcal{A} = e_1\mathcal{A} \oplus \cdots \oplus e_n\mathcal{A} \, .$$

Put $p = e_1 + \cdots + e_n$. Then $s\mathcal{A} = p\mathcal{A}$ and $p^2 = p$. Proposition 2.1 shows that s is relatively regular, hence there is $a \in \mathcal{A}$ with $sas = s$. Put $b = asa$. Then $sbs = s$ and $bsb = b$. ∎

Now we are ready to introduce the class of generalized Fredholm elements. First we give some examples.

Examples 3.9. (1) Let $s \in \mathrm{soc}(\mathcal{A})$. By Proposition 3.8 there is $b \in \mathrm{soc}(\mathcal{A})$ such that $sbs = s$. Hence

$$(e - sb - bs) + \mathrm{soc}(\mathcal{A}) = \widehat{e} \in \widehat{\mathcal{A}}^{-1}$$

thus

$$e - sb - bs \in \Phi(\mathcal{A}) \, .$$

(2) Let $x \in \Phi(\mathcal{A})$. Proposition 3.5 gives $\widehat{x}\widehat{y} = \widehat{e} = \widehat{y}\widehat{x}$ for each pseudo-inverse y of x. Thus

$$(e - xy - yx) + \mathrm{soc}(\mathcal{A}) = -\widehat{e} \in \widehat{\mathcal{A}}^{-1}$$

hence

$$e - xy - yx \in \Phi(\mathcal{A}) .$$

(3) If $x \in \mathcal{A}^{-1}$ and $y = x^{-1}$, then $xyx = x$ and

$$e - xy - yx = -e \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) .$$

(4) Let $x \in \mathcal{A}$ with $\mathcal{A} = x\mathcal{A} \oplus R(x)$ or $\mathcal{A} = \mathcal{A}x \oplus L(x)$. Theorem 3.3 in [15] shows that there exists $y \in \mathcal{A}$ such that $xyx = x$ and $xy = yx$. Therefore $e - xy - yx = e - 2xy$ and $(e - 2xy)^2 = e - 4xy + 4xyxy = e$. Thus

$$e - xy - yx \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) .$$

(5) Let $x \in \mathcal{A}$ with $x^2 = x$. Put $y = x$. Then $xyx = x$ and $e - xy - yx = e - 2x$. From $(e - 2x)^2 = e$ we get

$$e - xy - yx \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) .$$

In each of the above examples the elements $x \in \mathcal{A}$ has the following property: there is a pseudo-inverse y of x such that $e - xy - yx \in \Phi(\mathcal{A})$.

Therefore we call an element $x \in \mathcal{A}$ a *generalized Fredholm element* if x is relatively regular and there is a pseudo-inverse y of x with $e - xy - yx \in \Phi(\mathcal{A})$. By $\Phi_g(\mathcal{A})$ we denote the set of all generalized Fredholm elements of $\mathcal{A}$.

Before we state our first results concerning the class $\Phi_g(\mathcal{A})$ we need the following lemma.

LEMMA 3.10. *Suppose that $x, u \in \mathcal{A}$.*

(1) *If $xux - x \in \mathcal{A}^{-1}$ then $x \in \mathcal{A}^{-1}$.*

(2) *If $xux - x \in \Phi(\mathcal{A})$ then $x \in \Phi(\mathcal{A})$.*

Proof. (1) Put $v = (xux - x)^{-1}$, $x_1 = v(xu - e)$ and $x_2 = (ux - e)v$. Then $x_1 x = v(xu - e)x = v(xux - x) = e$ and $xx_2 = x(ux - e)v = (xux - x)v = e$.

(2) Since $\widehat{x}\widehat{u}\widehat{x} - \widehat{x} \in \widehat{\mathcal{A}}^{-1}$, it follows from (1) that $\widehat{x} \in \widehat{\mathcal{A}}^{-1}$, thus $x \in \Phi(\mathcal{A})$. □

Theorem 3.11.

(1) $\mathrm{soc}(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

(2) $\Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

(3) *If* $x \in \Phi(\mathcal{A})$ *and if* y *is a pseudo-inverse of* x, *then* $e - xy - yx \in \Phi(\mathcal{A})$ *and* $\mathrm{ind}(e - xy - yx) = 0$.

(4) *If* $\mathcal{A}$ *is a Banach algebra and* $x \in \Phi_g(\mathcal{A})$, *then there is* $\delta > 0$ *such that*

$$\lambda e - x \in \Phi_g(\mathcal{A}) \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| < \delta$$

and

$$\lambda e - x \in \Phi(\mathcal{A}) \text{ for all } \lambda \in \mathbb{C} \text{ with } 0 < |\lambda| < \delta\ .$$

Proof. (1) follows from Example 3.9 (1).
(2) follows from Example 3.9 (2).
(3) From Example 3.9 (2) we get $s \in \mathrm{soc}(\mathcal{A})$ such that $e - xy - yx = -e + s$. Use Theorem 3.6 (2) to derive

$$\mathrm{ind}(e - xy - yx) = \mathrm{ind}(-e + s) = \mathrm{ind}(-e) = 0\ .$$

(4) Take $y \in \mathcal{A}$ such that $xyx = x$ and $v = e - xy - yx \in \Phi(\mathcal{A})$. For $\lambda \in \mathbb{C}$ put

$$w(\lambda) = (\lambda e - x)(y + \lambda^2 e)(\lambda e - x) + (\lambda e - x)\ .$$

An easy computation gives

$$w(\lambda) = \lambda(\lambda(\lambda e - x)^2 + \lambda y + v)\ .$$

Since $\Phi(\mathcal{A})$ is open (Theorem 3.6 (3)), there is $\gamma > 0$ such that $v + u \in \Phi(\mathcal{A})$ for all $u \in \mathcal{A}$ with $\|u\| < \gamma$. There is $\delta > 0$ such that

$$\|\lambda(\lambda e - x)^2 + \lambda y\| < \gamma \quad \text{for all} \quad \lambda \in \mathbb{C} \text{ with } |\lambda| < \delta\ .$$

Thus $\frac{1}{\lambda}w(\lambda) \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. This gives $w(\lambda) \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. From Lemma 3.10 we get $\lambda e - x \in \Phi(\mathcal{A})$ if $0 < |\lambda| < \delta$.

THEOREM 3.12. *For $x \in \mathcal{A}$ the following assertions are equivalent:*
(1) $x \in \Phi_g(\mathcal{A})$.
(2) *There is $y \in \mathcal{A}$ such that $\widehat{x}\widehat{y}\widehat{x} = \widehat{x}$ and $\widehat{e} - \widehat{x}\widehat{y} - \widehat{y}\widehat{x} \in \widehat{\mathcal{A}}^{-1}$.*

PROOF. (1) $\Rightarrow$ (2) Clear.

(2) $\Leftarrow$ (1): Since $xyx - x \in \mathrm{soc}(\mathcal{A})$, $xyx - x$ is relatively regular (Proposition 3.8). Proposition 2.2 shows that x is relatively regular and that

$$y_0 = y - r + yxr + rxy - yxrxy$$

is a pseudo-inverse of x, where r is a pseudo-inverse of $xyx - x$. Then we get

$$\begin{aligned}\widehat{x}\widehat{y}_0 &= \widehat{x}\widehat{y} - \widehat{x}\widehat{r} + \underbrace{\widehat{x}\widehat{y}\widehat{x}}_{=\widehat{x}}\widehat{r} + \widehat{x}\widehat{r}\widehat{x}\widehat{y} - \underbrace{\widehat{x}\widehat{y}\widehat{x}}_{=\widehat{x}}\widehat{r}\widehat{x}\widehat{y} = \\ &= \widehat{x}\widehat{y}\ .\end{aligned}$$

A similar argument shows that $\widehat{y}_0\widehat{x} = \widehat{y}\widehat{x}$. We summarize: $xy_0x = x$ and

$$\widehat{e} - \widehat{x}\widehat{y}_0 - \widehat{y}_0\widehat{x} = \widehat{e} - \widehat{x}\widehat{y} - \widehat{y}\widehat{x} \in \widehat{\mathcal{A}}^{-1}\ .$$

Thus $x \in \Phi_g(\mathcal{A})$. ∎

Let $\mathcal{B}$ be a complex algebra with identity $e \neq 0$. In view of Theorem 3.12 it seems to be useful to consider elements $t \in \mathcal{B}$ with the following property:

(3.13) t is relatively regular and for some pseudo-inverse s of t the element $e - ts - st$ belongs to $\mathcal{B}^{-1}$.

Therefore we define

$$\mathcal{B}^g = \{t \in \mathcal{B} :\ t \text{ has the property (3.13)}\}\ .$$

Elements in $\mathcal{B}^g$ can be called *generalized invertible*, since $\mathcal{B}^{-1} \subseteq \mathcal{B}^g$. Observe that $0 \in \mathcal{B}^g$, thus $\mathcal{B}^{-1}$ is a proper subset of $\mathcal{B}^g$. With these notations we have

$$x \in \Phi(\mathcal{A}) \Leftrightarrow \widehat{x} \in \widehat{\mathcal{A}}^{-1}$$

and, by Theorem 3.12,

$$x \in \Phi_g(\mathcal{A}) \Leftrightarrow \widehat{x} \in \widehat{\mathcal{A}}^g . \tag{3.14}$$

In the next section we shall investigate the class $\mathcal{B}^g$.

4 – Properties of $\mathcal{B}^g$

In this section $\mathcal{B}$ always denotes a complex algebra with identity e.

Let V be a vector space and $T: V \to V$ linear. For the definitions of the *ascent* $p(T)$ and the *descent* $q(T)$ of T we refer the reader to [8, §72].

Let $t \in \mathcal{B}$ and let the linear operators $L_t,\ R_t : \mathcal{B} \to \mathcal{B}$ be defined by

$$L_t(b) = tb \text{ and } R_t(b) = bt \ (b \in \mathcal{B}) .$$

Then we have $L_t(\mathcal{B}) = t\mathcal{B},\ R_t(\mathcal{B}) = \mathcal{B}t$, $\mathrm{kern} L_t = R(t)$ and $\mathrm{kern} R_t = L(t)$.

Put

$$\begin{aligned} p_l(t) &= p(L_t) , \qquad & q_l(t) &= q(L_t) \\ p_r(t) &= p(R_t) , \qquad & q_r(t) &= q(R_t) . \end{aligned}$$

PROPOSITION 4.1. *Let $t \in \mathcal{B}$.*

(1) *If $p_l(t)$ and $q_l(t)$ $[p_r(t)$ and $q_r(t)]$ are both finite, then they are equal and for $n = p_l(t)$ $[n = p_r(t)]$ we have*

$$\mathcal{B} = R(t^n) \oplus t^n\mathcal{B} \ [\mathcal{B} = L(t^n) \oplus \mathcal{B}t^n] .$$

(2) *$p_r(t) \le q_l(t),\ p_l(t) \le q_r(t)$.*

PROOF. (1) [8, §72].

(2) We only show that $p_r(t) \le q_l(t)$. If $n = q_l(t) < \infty$ then $t^n\mathcal{B} = t^{n+1}\mathcal{B}$. Take $b \in \mathcal{B}$ with $t^n = t^{n+1}b$. If $c \in L(t^{n+1})$ then $ct^n = ct^{n+1}b = 0$, thus $c \in L(t^n)$, therefore $L(t^{n+1}) \subseteq L(t^n)$, thus $p_r(t) \le n$. ∎

From [15, Theorem 3.3] we get the following characterization of elements of $\mathcal{B}^g$.

PROPOSITION 4.2. *For $t \in \mathcal{B}$ the following assertions are equivalent.*
(1) $t \in \mathcal{B}^g$.

(2) $p_l(t) = q_l(t) \leq 1$.

(3) $p_r(t) = q_r(t) \leq 1$.

(4) *There is $u \in \mathcal{B}$ with $tut = t$ and $tu = ut$.*

COROLLARY 4.3. *Let $\mathcal{A}$ be a complex semisimple algebra with identity. Suppose that $0 < p = p_l(x) = q_l(x) < \infty$. Then $x^p \in \Phi_g(\mathcal{A})$.*

PROOF. Since $p_l(x^p) = q_l(x^p) \leq 1$, it follows from Proposition 4.2, that $x^p u x^p = x^p$ and $x^p u = u x^p$ for some $u \in \mathcal{A}$. Then we get $e - x^p u - u x^p = e - 2ux^p$ and $(e - 2ux^p)^2 = e$. Thus

$$e - x^p u - u x^p \in \mathcal{A}^{-1} \subseteq \Phi(\mathcal{A}) \ . \qquad \square$$

PROPOSITION 4.4. *Suppose that $t, u, t_1, t_2 \in \mathcal{B}$.*

(1) *If $t_1, t_2 \in \mathcal{B}^g$, $t_1 t_2 = t_1 t_2$ then $t_1 t_2 \in \mathcal{B}^g$.*

(2) *If $t \in \mathcal{B}^g$ and $n \in \mathbb{N}$ then $t^n \in \mathcal{B}^g$.*

(3) *If $t \in \mathcal{B}^g$ then there is a unique $s \in \mathcal{B}$ with*

$$tst = t, \ \ sts = s \text{ and } ts = st \ .$$

Furthermore we have $s \in \mathcal{B}^g$ and if $ta = at$ for some $a \in \mathcal{B}$, then $sa = as$.

(4) *If $t, u \in \mathcal{B}^g$ and $tu = ut = 0$, then $t + u \in \mathcal{B}^g$.*

(5) $e - t_1 t_2 \in \mathcal{B}^g \iff e - t_2 t_1 \in \mathcal{B}^g$.

PROOF. (1) follows from Proposition 3.4 in [15] and 4.2.

(2) follows from (1).

(3) Proposition 3.9 in [15] shows that there is a unique $s \in \mathcal{B}$ such that $tst = t$, $sts = s$ and $ts = st$. From Proposition 4.2 (4) we get $s \in \mathcal{B}^g$. If $ta = at$, then

$$sta = sat = satst = sat^2s = st^2as = tas \ ,$$

thus $s^2ta = tas^2$ and therefore

$$sa = stsa = s^2ta = tas^2 = ats^2 = as \ .$$

(4) From (3) we get $s, v \in \mathcal{B}$ such that $tst = t$, $sts = s$, $ts = st$, $uvu = u$, $vuv = v$ and $uv = vu$. Then

$$\begin{aligned}(t+u)(s+v) = ts + tv + us + uv = ts + tvuv + usts + uv = \\ = ts + tuv^2 + uts^2 + uv = ts + uv \ .\end{aligned}$$

A similar computation gives $(s+v)(t+u) = st + vu$. Thus $(t+u)(s+v) = (s+v)(t+u)$. From Proposition 4.2 we get $t + u \in \mathcal{B}^g$ since

$$\begin{aligned}(t+u)(s+v)(t+u) = (ts+uv)(t+u) = tst + tsu + uvt + uvu = \\ = t + stu + vut + u = t + u \ .\end{aligned}$$

(5) We only have to show that $e - t_1t_2 \in \mathcal{B}^g$ implies $e - t_2t_1 \in \mathcal{B}^g$. By Proposition 4.2 (4) there is a pseudo-inverse s of $e - t_1t_2$ which commutes with $e - t_1t_2$. Put $r = e + t_2st_1$. A simple computation shows that r is a pseudo-inverse of $e - t_2t_1$ which commutes with $e - t_2t_1$. ∎

NOTATIONS. Let $\mathcal{B}$ be a Banach algebra and $t \in \mathcal{B}$. By $\sigma(t)$ and $r(t)$ we denote the spectrum and the spectral radius of t, respectively. If $D \subseteq \mathbb{C}$ is open, $\sigma(t) \subseteq D$ and $f : D \to \mathbb{C}$ holomorphic, then $f(t)$ is defined by the well-known operational calculus.

PROPOSITION 4.5. *Suppose that $\mathcal{B}$ is a Banach algebra and $t \in \mathcal{B}^g$.*

(1) *t is quasinilpotent if and only if $t = 0$.*

(2) *If* $t \neq 0$ *and if* s *is the unique pseudo-inverse of* t *with the properties in part* (3) *of Proposition* 4.4*, then*

(i) s *is not quasinilpotent;*

(ii) $\sigma(s)\backslash\{0\} = \{\lambda \in \mathbb{C}\backslash\{0\} : \frac{1}{\lambda} \in \sigma(t)\}$ *;*

(iii) $t \in \mathcal{B}^{-1}$ *or* 0 *is a pole of order* 1 *of* $(\lambda e - t)^{-1}$*;*

(iv) $\mathrm{dist}(0, \sigma(t)\backslash\{0\}) = r(s)^{-1}$.

(3) *Suppose that* $0 \in \sigma(t)$, $D \subseteq \mathbb{C}$ *is a region,* $\sigma(t) \subseteq D$, $f : D \to \mathbb{C}$ *holomorphic, injective and* $f(0) = 0$*. Then* $f(t) \in \mathcal{B}^g$*.*

PROOF. [16, Propositions 2.6, 2.7 and 2.13]. □

PROPOSITION 4.6. *Let* $\lambda_1, \ldots, \lambda_m$ *pairwise distinct complex numbers. If* $t \in \mathcal{B}$ *and*

$$\prod_{j=1}^{m}(t - \lambda_j e) = 0 \ ,$$

then $t - \lambda e \in \mathcal{B}^g$ *for each* $\lambda \in \mathbb{C}$.

PROOF. [16, Proposition 2.4]. □

5 – Algebraic properties of $\Phi_g(\mathcal{A})$

As in Section 3 we denote by $\mathcal{A}$ a complex semisimple algebra with identity. Furthermore we assume $\{0\} \neq \mathrm{soc}(\mathcal{A})$.

THEOREM 5.1. $\Phi_g(\mathcal{A}) + \mathrm{soc}(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

PROOF. Let $x \in \Phi_g(\mathcal{A})$ and $s \in \mathrm{soc}(\mathcal{A})$. Then $\widehat{x+s} = \widehat{x} + \widehat{s} = \widehat{x}$. By (3.14), $\widehat{x} \in \widehat{\mathcal{A}}^g$, thus $\widehat{x+s} \in \widehat{\mathcal{A}}^g$, hence $x + s \in \Phi_g(\mathcal{A})$. □

REMARK. From Proposition 3 (2) we know that

$$\Phi(\mathcal{A}) + I(\mathcal{A}) \subseteq \Phi(\mathcal{A}) \ .$$

In Section 6 of this paper we shall see that in general

$$\Phi_g(\mathcal{A}) + I(\mathcal{A}) \not\subseteq \Phi_g(\mathcal{A}) \ .$$

THEOREM 5.2. *(1) If $x_1, x_2 \in \Phi_g(\mathcal{A})$ and $x_1x_2 - x_2x_1 \in \mathrm{soc}(\mathcal{A})$, then $x_1x_2 \in \Phi_g(\mathcal{A})$.*

(2) If $x \in \Phi_g(\mathcal{A})$ and $n \in \mathbb{N}$, then $x^n \in \Phi_g(\mathcal{A})$.

PROOF. (1) By (3.14), $\widehat{x}_1, \widehat{x}_2 \in \widehat{\mathcal{A}}^g$. Since $\widehat{x}_1\widehat{x}_2 = \widehat{x}_2\widehat{x}_1$, we get from Proposition 4.4 (1), that $\widehat{x_1x_2} = \widehat{x}_1\widehat{x}_2 \in \widehat{\mathcal{A}}^g$. By (3.14), $x_1x_2 \in \Phi_g(\mathcal{A})$.

(2) follows from (1). ∎

REMARKS. (1) In [15, 1.7 (d)] it is shown by an example, that if $x_1, x_2 \in \Phi_g(\mathcal{A})$ it does not follow that $x_1x_2 \in \Phi_g(\mathcal{A})$.

(2) In [15, 1.7 (b)] it is shown by an example, that if $x^n \in \Phi_g(\mathcal{A})$ for some $n \in \mathbb{N}$ it does not follow that $x \in \Phi_g(\mathcal{A})$.

THEOREM 5.3. *For $x_1, x_2 \in \mathcal{A}$ we have:*

$$e - x_1x_2 \in \Phi_g(\mathcal{A}) \iff e - x_2x_1 \in \Phi_g(\mathcal{A}) .$$

PROOF. Proposition 4.4 (5) and (3.14) give

$$\begin{aligned} e - x_1x_2 \in \Phi_g(\mathcal{A}) &\iff \widehat{e} - \widehat{x}_1\widehat{x}_2 \in \widehat{\mathcal{A}}^g \iff \\ &\iff e - x_2x_1 \in \Phi_g(\mathcal{A}) . \end{aligned}$$

∎

THEOREM 5.4. *For $x \in \mathcal{A}$ the following assertions are equivalent:*

(1) $x \in \Phi_g(\mathcal{A})$;

(2) there is $y \in \mathcal{A}$ such that $xyx = x$ and $\widehat{x}\widehat{y} = \widehat{y}\widehat{x}$.

PROOF. (1) ⇒ (2): From $\widehat{x} \in \widehat{\mathcal{A}}^g$ it follows that there exists some $u \in \mathcal{A}$ with $\widehat{x}\widehat{u}\widehat{x} = \widehat{x}$ and $\widehat{x}\widehat{u} = \widehat{u}\widehat{x}$ (Proposition 4.2 (4)). Then we have $xux - x \in \mathrm{soc}(\mathcal{A})$. Proposition 3.8 shows that $xux - x$ is relatively regular. Let r be a pseudo-inverse of $xux - x$. Then

$$y = u - r + uxr + rxu - uxrxu$$

is a pseudo-inverse of x (Proposition 2.2). Then it is easy to see that $\widehat{x}\widehat{y} = \widehat{x}\widehat{u} = \widehat{u}\widehat{x} = \widehat{y}\widehat{x}$.

(2) ⇒ (1) Proposition 4.2 (4) gives $\widehat{x} \in \widehat{\mathcal{A}}^g$, thus $x \in \Phi_g(\mathcal{A})$. ∎

THEOREM 5.5. *For $x \in \Phi_g(\mathcal{A})$ the following assertions are equivalent:*

(1) $x \in \Phi(\mathcal{A})$;

(2) $\mathrm{nul}(x) < \infty$;

(3) $\mathrm{def}(x) < \infty$.

PROOF. It is clear that (1) implies (2) and (3).

(2) $\Rightarrow$ (1): Take a pseudo-inverse y of x with $\widehat{y}\widehat{x} = \widehat{x}\widehat{y}$ (Theorem 5.4). Since

$$\mathrm{nul}(x) = \Theta(R(x)) = \Theta((e - yx)\mathcal{A}) < \infty \ ,$$

we get from Proposition 3.4 (5), that $e - yx \in \mathrm{soc}(\mathcal{A})$, hence $\widehat{e} = \widehat{y}\widehat{x} = \widehat{x}\widehat{y}$, thus $\widehat{x} \in \widehat{\mathcal{A}}^{-1}$.

A similar proof shows that (3) implies (1). ∎

THEOREM 5.6. *Suppose that $x, u \in \Phi_g(\mathcal{A})$ and that $xu, ux \in \mathrm{soc}(\mathcal{A})$. Then $x + u \in \Phi_g(\mathcal{A})$.*

PROOF. Since $\widehat{x}, \widehat{u} \in \widehat{\mathcal{A}}^g$ and $\widehat{x}\widehat{u} = \widehat{0} = \widehat{u}\widehat{x}$, Proposition 4.4 (4) gives $\widehat{x + u} = \widehat{x} + \widehat{u} \in \widehat{\mathcal{A}}^g$, thus $x + u \in \Phi_g(\mathcal{A})$. ∎

6 – Topological properties of $\Phi_g(\mathcal{A})$

In this section we assume that $\mathcal{A}$ is complex semisimple Banach algebra with identity $e \neq 0$. From [4, R. 3.6] it follows that

$$\mathcal{A} \neq I(\mathcal{A}) \iff \dim \mathcal{A} = \infty \ .$$

Hence, if $\dim \mathcal{A} = \infty$, $\widetilde{\mathcal{A}} = \mathcal{A}/I(\mathcal{A})$ is a complex Banach algebra with identity $\widetilde{e} \neq \widetilde{o}$.

The first result in this section is an improvement of Theorem 3.11 (4).

THEOREM 6.1. *Suppose that $x \in \Phi_g(\mathcal{A})$, $z \in \Phi(\mathcal{A})$ and $xz - zx \in \mathrm{soc}(\mathcal{A})$. Then there is $\delta > 0$ such that*

$$x - \lambda z \in \Phi(\mathcal{A}) \text{ for } 0 < |\lambda| < \delta \ .$$

Proof. Take $u \in \mathcal{A}$ such that $\widehat{z}\widehat{u} = \widehat{u}\widehat{z} = \widehat{e}$. From $\widehat{x}\widehat{z} = \widehat{z}\widehat{x}$ we get $\widehat{x} = \widehat{x}(\widehat{u}\widehat{z}) = \widehat{x}\widehat{z}\widehat{u} = \widehat{z}\widehat{x}\widehat{u}$, thus $\widehat{u}\widehat{x} = \widehat{u}\widehat{z}\widehat{x}\widehat{u} = \widehat{x}\widehat{u}$, hence $xu - ux \in \mathrm{soc}(\mathcal{A})$. Since $u \in \Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$, we derive from Theorem 5.2 (1) that $ux \in \Phi_g(\mathcal{A})$. Theorem 3.11 (4) shows that there is $\delta > 0$ such that $ux - \lambda e \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. This implies, since $z \in \Phi(\mathcal{A})$, that $zux - \lambda z = z(ux - \lambda e) \in \Phi(\mathcal{A})$ $(0 < |\lambda| < \delta)$. Then we have for $0 < |\lambda| < \delta$ that

$$\widehat{x} - \lambda\widehat{z} = \widehat{z}\widehat{u}\widehat{x} - \lambda\widehat{z} \in \widehat{\mathcal{A}}^{-1} ,$$

thus $x - \lambda z \in \Phi(\mathcal{A})$. □

Definition. Let $x \in \mathcal{A}$. The set

$$\sigma_\Phi(x) = \{\lambda \in \mathbb{C} : \ \lambda e - x \notin \Phi(\mathcal{A})\}$$

is called the *Fredholm spectrum* of x. If $\sigma_\Phi(x) = \{0\}$, then x is called a *Riesz element* of $\mathcal{A}$.

If $\dim \mathcal{A} = \infty$, then by Proposition 3.7 (2)

$$\sigma_\Phi(x) = \sigma(\widetilde{x}) ;$$

and (see Theorem 6.1)

$$\mathrm{dist}(0, \sigma_\Phi(x)\backslash\{0\}) > 0 \text{ if } x \in \Phi_g(\mathcal{A}) .$$

Suppose that $x \in \Phi_g(\mathcal{A})$, then there is $y \in \mathcal{A}$ such that

$$xyx = x, \quad yxy = y, \quad \text{and} \quad xy - yx \in \mathrm{soc}(\mathcal{A}). \tag{6.2}$$

In fact, by Theorem 5.4, $xy_0x = x$ and $xy_0 - y_0x \in \mathrm{soc}(\mathcal{A})$ for some $y_0 \in \mathcal{A}$. Then $y = y_0xy_0$ satisfies (6.2).

Theorem 6.3. *Suppose that* $\dim \mathcal{A} = \infty$, $x \in \Phi_g(\mathcal{A})$ *and* y *is a pseudo-inverse of* x *which satisfies* (6.2).

(1) *x is a Riesz element $\Longleftrightarrow$ y is a Riesz element. In this case* $\sigma_\Phi(x) = \{0\}$ *and* $\mathrm{dist}(0, \sigma_\Phi(x)\backslash\{0\}) = \infty$.

(2) *If x is not a Riesz element, then*

$$\sigma_\Phi(y)\backslash\{0\} = \Big\{\lambda \in \mathbb{C}\backslash\{0\} : \ \frac{1}{\lambda} \in \sigma_\Phi(x)\Big\}$$

and

$$\operatorname{dist}(0, \sigma_\Phi(x)\backslash\{0\}) = r(\widetilde{y})^{-1} .$$

Proof. (1) (6.2) and Proposition 4.2 (4) show that $\widetilde{x}, \widetilde{y} \in \widetilde{\mathcal{A}}^g$. Therefore, by Proposition 4.5 (1) and (6.2)

$$x \text{ is Riesz } \iff r(\widetilde{x}) = 0 \iff \widetilde{x} = \widetilde{0} \iff \widetilde{y} = \widetilde{0} \iff$$
$$\iff r(\widetilde{y}) = 0 \iff y \text{ is Riesz} .$$

(2) From (1) we see that $r(\widetilde{x}), r(\widetilde{y}) > 0$. Proposition 4.5 (2) (ii) shows that

$$\sigma_\Phi(y)\backslash\{0\} = \sigma(\widetilde{y})\backslash\{0\} = \Big\{\lambda \in \mathbb{C}\backslash\{0\} : \ \frac{1}{\lambda} \in \sigma(\widetilde{x})\Big\} =$$
$$= \Big\{\lambda \in \mathbb{C}\backslash\{0\} : \ \frac{1}{\lambda} \in \sigma_\Phi(x)\Big\} .$$

From Proposition 4.5 (2) (iv) we conclude that

$$\operatorname{dist}(0, \sigma_\Phi(x)\backslash\{0\}) = \operatorname{dist}(0, \sigma(\widetilde{x})\backslash\{0\}) = r(\widetilde{y})^{-1} . \qquad \square$$

Theorem 6.4. *Suppose that* $\dim\mathcal{A} = \infty$, $x \in \Phi_g(\mathcal{A})$, $0 \in \sigma(x)$, $D \subseteq \mathbb{C}$ *is a region,* $\sigma(x) \subseteq D$, $f : D \to \mathbb{C}$ *is holomorphic and injective and* $f(0) = 0$. *Then* $f(x) \in \Phi_g(\mathcal{A})$.

Proof. It is clear that $\sigma(\widetilde{x}) \subseteq \sigma(x)$.

Case 1. $x \in \Phi(\mathcal{A})$. Hence $0 \notin \sigma(\widetilde{x})$. Since f is injective and $f(0) = 0$, we get

$$0 \notin f(\sigma(\widetilde{x})) = \sigma(f(\widetilde{x})) = \sigma(\widetilde{f(x)}) = \sigma_\Phi(f(x)) ,$$

thus $f(x) \in \Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$.

Case 2. $x \notin \Phi(\mathcal{A})$. Then $0 \in \sigma(\widetilde{x})$. Let y be a pseudo-inverse of x such that (6.2) holds. Since f' is injective, there is a holomorphic function

$g: f(D) \to \mathbb{C}$ such that $g(f(\lambda)) = \lambda$. Without loss of generality we can assume that $f'(0) = g'(0) = -1$, thus there is $h : f(D) \to \mathbb{C}$ holomorphic such that

$$g(\lambda) = -\lambda + \lambda^2 h(\lambda) .$$

Put $\varphi = h \circ f$. From

$$\lambda = g(f(\lambda)) = -f(\lambda) + f(\lambda)^2 h(f(\lambda)) = f(\lambda)\varphi(\lambda)f(\lambda) - f(\lambda)$$

we derive

$$x = f(x)\varphi(x)f(x) - f(x) . \tag{6.5}$$

Since x is relatively regular, (6.5) and Proposition 2.2 show that $f(x)$ is relatively regular and that

$$r = \varphi(x) - [\varphi(x)f(x) - e]y[\varphi(x)f(x) - e]$$

is a pseudo-inverse of $f(x)$ (observe that $f(x)\varphi(x) = \varphi(x)f(x)$). From $\widetilde{x}\widetilde{y} = \widetilde{y}\widetilde{x}$ we get $\widetilde{y}\widetilde{f(x)} = \widetilde{f(x)}\widetilde{y}$ and $\widetilde{y}\widetilde{\varphi(x)} = \widetilde{\varphi(x)}\widetilde{y}$, thus $\widetilde{r}\widetilde{f(x)} = \widetilde{f(x)}\widetilde{r}$. Hence

$$\begin{aligned}(\widetilde{e} - \widetilde{r}\widetilde{f(x)} - \widetilde{f(x)}\widetilde{r})^2 &= (\widetilde{e} - 2\widetilde{r}\widetilde{f(x)})^2 = \\ &= \widetilde{e} - 4\widetilde{r}\widetilde{f(x)} + 4\widetilde{r}\underbrace{\widetilde{f(x)}\widetilde{r}\widetilde{f(x)}}_{=\widetilde{f(x)}} = \widetilde{e} .\end{aligned}$$

This shows that $e - rf(x) - f(x)r \in \Phi(\mathcal{A})$. □

For our next results we denote by $\overline{\mathcal{M}}$ the closure of a subset $\mathcal{M} \subseteq \mathcal{A}$.

THEOREM 6.6. *Let* $x \in \mathcal{A}$.

(1) *If* x *is relatively regular then* $x\mathcal{A}x$ *is closed.*

(2) $x \in \operatorname{soc}(\mathcal{A}) \;\Leftrightarrow\; \dim x\mathcal{A}x < \infty$.

(3) *If* x *is relatively regular then*

$$x \in \overline{\operatorname{soc}(\mathcal{A})} \;\Leftrightarrow\; x \in \operatorname{soc}(\mathcal{A}) .$$

Proof. (1) Let y be a pseudo-inverse of x. Then $x\mathcal{A} = xy\mathcal{A}$, $\mathcal{A}x = \mathcal{A}yx$, xy and yx are idempotent. It follows that $x\mathcal{A}$ and $\mathcal{A}x$ are closed, thus $x\mathcal{A} \cap \mathcal{A}x$ is closed. To complete the proof we show that

$$x\mathcal{A}x = x\mathcal{A} \cap \mathcal{A}x \ .$$

The inclusion "$\subseteq$" is clear. Take $z \in x\mathcal{A} \cap \mathcal{A}x$, thus $z \in xy\mathcal{A}$ and $z \in \mathcal{A}yx$, hence $z = xyz$ and $z = zyx$. This gives $z = xy(zyx) = x(yzy)x \in x\mathcal{A}x$.

(2) is shown in [1].

(3) We only have to show the implication "$\Rightarrow$". There is a sequence (x_n) in $\mathrm{soc}(\mathcal{A})$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$. Thus there is $\gamma \geq 0$ with $\|x_n\| \leq \gamma$ for all $n \in \mathbb{N}$. Define the bounded linear operators $F_n,\ K : \mathcal{A} \to \mathcal{A}$ by

$$F_n a = x_n a x_n \quad \text{and} \quad Ka = xax \ (a \in \mathcal{A},\ n \in \mathbb{N}) \ .$$

For each $a \in \mathcal{A}$ we get

$$\begin{aligned} \|Ka - F_n a\| &= \|(x - x_n)ax + x_n a(x - x_n)\| \leq \\ &\leq \|a\|(\|x\| + \gamma)\|x - x_n\| \ , \end{aligned}$$

hence $\|K - F_n\| \leq (\|x\| + \gamma)\|x - x_n\|$.

This shows that (F_n) converges uniformly to K. From (2) we get that each F_n is a finite-dimensional operator, hence K is compact. Since x is relatively regular, K has closed range, by (1), hence K has a finite-dimensional range. Now use (2) to get $x \in \mathrm{soc}(\mathcal{A})$. ∎

Remark. From Proposition 3.7 (2) we know that

$$\Phi(\mathcal{A}) + I(\mathcal{A}) \subseteq \Phi(\mathcal{A}) \ .$$

Now take $x \in \overline{\mathrm{soc}(\mathcal{A})}$. Since $0 \in \Phi_g(\mathcal{A})$, we have $x \in \Phi_g(\mathcal{A}) + \overline{\mathrm{soc}(\mathcal{A})}$. Theorem 6.6 (3) shows that

$$x \in \Phi_g(\mathcal{A}) \ \Leftrightarrow \ x \in \mathrm{soc}(\mathcal{A}) \ .$$

Thus, in general,

$$\Phi_g(\mathcal{A}) + \overline{\mathrm{soc}(\mathcal{A})} \not\subseteq \Phi_g(\mathcal{A}) \ ,$$

hence

$$\Phi_g(\mathcal{A}) + I(\mathcal{A}) \not\subseteq \Phi_g(\mathcal{A}) \ .$$

THEOREM 6.7. *Let* $\dim \mathcal{A} = \infty$ *and let* $x \in \Phi_g(\mathcal{A})$. *The following assertions are equivalent:*

(1) x *is a Riesz element of* $\mathcal{A}$.

(2) $x \in I(\mathcal{A})$.

(3) $x \in \operatorname{soc}(\mathcal{A})$.

PROOF. (1) $\Rightarrow$ (2): If x is a Riesz element then $\widetilde{x}$ is quasinilpotent. As in the proof of Theorem 6.3 we have $\widetilde{x} \in \widetilde{\mathcal{A}}^g$. Proposition 4.5 (1) shows then that $\widetilde{x} = \widetilde{0}$, hence $x \in I(\mathcal{A})$.

(2) $\Rightarrow$ (3): By $\check{\mathcal{A}}$ we denote the quotient algebra $\mathcal{A}/\overline{\operatorname{soc}(\mathcal{A})}$. Since $\dim \mathcal{A} = \infty$ ($\Leftrightarrow \mathcal{A} \neq I(\mathcal{A})$), $\check{\mathcal{A}}$ is a Banach algebra with identity $\check{e} \neq \check{0}$, where $\check{z} = z + \overline{\operatorname{soc}(\mathcal{A})}$ denotes the coset of z in $\check{\mathcal{A}}$. Take $\lambda \in \mathbb{C}\backslash\{0\}$. Then $\lambda\widetilde{e} - \widetilde{x} = \lambda\widetilde{e} \in \widetilde{\mathcal{A}}^{-1}$. From Proposition 3.7 (2) we get $\lambda e - x \in \Phi(\mathcal{A})$, thus $\lambda\widehat{e} - \widehat{x} \in \widehat{\mathcal{A}}^{-1}$, hence, for some $z \in \mathcal{A}$,

$$z(\lambda e - x) - e,\ (\lambda e - x)z - e \in \operatorname{soc}(\mathcal{A}) \ ,$$

therefore $\check{z}(\lambda\check{e} - \check{x}) = \check{e} = (\lambda\check{e} - \check{x})\check{z}$. This gives $\lambda\check{e} - \check{x} \in \check{\mathcal{A}}^{-1}$. Thus, since $\lambda \in \mathbb{C}\backslash\{0\}$ was arbitrary, $\check{x}$ is quasinilpotent. Let y be a pseudo-inverse of x such that (6.2) holds. Proposition 4.2 (4) shows that $\check{x} \in \check{\mathcal{A}}^g$. Now use Proposition 4.5 (1) to derive $\check{x} = \check{0}$. Therefore $x \in \overline{\operatorname{soc}(\mathcal{A})}$. Theorem 6.6 (3) gives $x \in \operatorname{soc}(\mathcal{A})$.

(3) $\Rightarrow$ (1). For each $\lambda \in \mathbb{C}\backslash\{0\}$ we have $\lambda e - x \in \Phi(\mathcal{A}) + \operatorname{soc}(\mathcal{A}) \subseteq \Phi(\mathcal{A})$. ∎

Let $\mathcal{J}$ be an ideal in $\mathcal{A}$. $\mathcal{J}$ is called a Φ-*ideal*, if $\operatorname{soc}(\mathcal{A}) \subseteq \mathcal{J} \subseteq I(\mathcal{A})$. It is clear that $\operatorname{soc}(\mathcal{A})$, $\overline{\operatorname{soc}(\mathcal{A})}$ and $I(\mathcal{A})$ are Φ-ideals and that

$$\Phi(\mathcal{A}) + \mathcal{J} \subseteq \Phi(\mathcal{A})$$

for each Φ-ideal $\mathcal{J}$.

COROLLARY 6.8. *Suppose that* $\dim \mathcal{A} = \infty$, $\mathcal{J}$ *is a* Φ-*ideal and that*

$$\Phi_g(\mathcal{A}) + \mathcal{J} \subseteq \Phi_g(\mathcal{A}) \ .$$

Then $\mathcal{J} = \mathrm{soc}(\mathcal{A})$.

PROOF. Take $a \in \mathcal{J}$. Then $a = 0 + a \in \Phi_g(\mathcal{A}) + \mathcal{J} \subseteq \Phi_g(\mathcal{A})$. Since $a \in I(\mathcal{A})$, $\lambda\widetilde{e} - \widetilde{a} = \lambda\widetilde{e}$, thus we have that $\lambda e - a \in \Phi(\mathcal{A})$ for $\lambda \in \mathbb{C}\backslash\{0\}$. Therefore a is a Riesz element and $a \in \Phi_g(\mathcal{A})$. Theorem 6.7 gives $a \in \mathrm{soc}(\mathcal{A})$. □

THEOREM 6.9. $\Phi_g(\mathcal{A}) \subseteq \overline{\Phi(\mathcal{A})}$.

PROOF. Use Theorem 3.11 (4). □

7 – Ascent and descent of elements in $\Phi_g(\mathcal{A})$

In this section we assume that $\mathcal{A}$ is a complex semisimple Banach algebra with identity e and that $\mathrm{soc}(\mathcal{A}) \neq \{0\}$.

For $x \in \mathcal{A}$ we define

$$\Delta_l(x) = \{\alpha \in \mathbb{N}_0 : \ R(x) \cap x^\alpha \mathcal{A} = R(x) \cap x^{\alpha+k}\mathcal{A} \quad \text{for all} \quad k \geq 0\}$$

and

$$\Delta_r(x) = \{\beta \in \mathbb{N}_0 : \ L(x) \cap \mathcal{A}x^\beta = L(x) \cap \mathcal{A}x^{\beta+k} \quad \text{for all} \quad k \geq 0\} \ .$$

PROPOSITION 7.1. *If $x \in \mathcal{A}$ and $n \in \mathbb{N}_0$, then*

(1) $p_l(x) \leq n \ \Leftrightarrow \ R(x) \cap x^n\mathcal{A} = \{0\}$;

(2) $q_l(x) \leq n \ \Leftrightarrow \ R(x^n) + x\mathcal{A} = \mathcal{A}$;

(3) $q_r(x) \leq n \ \Leftrightarrow \ L(x^n) + \mathcal{A}x = \mathcal{A}$;

(4) $p_r(x) \leq n \Leftrightarrow L(x) \cap \mathcal{A}x^n = \{0\}$;

(5) $\Delta_l(x) = \{\alpha \in \mathbb{N}_0 : R(x^\alpha) + x\mathcal{A} = R(x^{\alpha+k}) + x\mathcal{A} \quad \text{for all} \quad k \geq 0\}$;

(6) $\Delta_r(x) = \{\beta \in \mathbb{N}_0 : L(x^\beta) + \mathcal{A}x = L(x^{\beta+k}) + \mathcal{A}x \quad \text{for all} \quad k \geq 0\}$.

PROOF. We only show (1), (2) and (5). The proofs for (3), (4) and (6) are similar.

(1) follows from [8, Satz 72.1].

(2) "$\Rightarrow$" By [8, Satz 72.2], there is a subspace $\mathcal{U}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{U} \oplus x\mathcal{A}$ and $\mathcal{U} \subseteq R(x^n)$. Thus $R(x^n) + x\mathcal{A} = \mathcal{A}$.

"$\Leftarrow$": Take $y \in x^n\mathcal{A}$. Then $y = x^n a$ for some $a \in \mathcal{A}$. There are u, v with $a = u + v$, $u \in R(x^n)$ and $v \in x\mathcal{A}$. It follows that $y = x^n(u + v) = x^n v \in x^{n+1}\mathcal{A}$. Hence $q_l(x) \leq n$.

(5) Denote by M the set on the right side in (5). Let $\alpha \in \Delta_l(x)$ and take $z \in R(x^{\alpha+1}) + x\mathcal{A}$, hence $z = u + xv$ with $u \in R(x^{\alpha+1})$ and $v \in \mathcal{A}$. Then $x^\alpha u \in R(x) \cap x^\alpha\mathcal{A} = R(x) \cap x^{\alpha+1}\mathcal{A}$, thus $x^\alpha u = x^{\alpha+1}w$ for some $w \in \mathcal{A}$, hence $u - xw \in R(x^\alpha)$. It follows that $z = u + xv = (u - xw) + x(w+v) \in R(x^\alpha)+x\mathcal{A}$. We have shown that $R(x^{\alpha+1})+x\mathcal{A} = R(x^\alpha)+x\mathcal{A}$. By induction we see that $\alpha \in M$.

Now let $\alpha \in M$ and take $z \in R(x) \cap x^\alpha\mathcal{A}$. Then there is $y \in \mathcal{A}$ with $z = x^\alpha y$ and $x^{\alpha+1}y = 0$. Thus $y \in R(x^{\alpha+1}) \subseteq R(x^{\alpha+1}) + x\mathcal{A} = R(x^\alpha) + x\mathcal{A}$. Therefore $y = y_1 + y_2$ with $y_1 \in R(x^\alpha)$, $y_2 \in x\mathcal{A}$. Then $z = x^\alpha(y_1 + y_2) = x^\alpha y_2 \in x^{\alpha+1}\mathcal{A}$, thus $z \in R(x) \cap x^{\alpha+1}\mathcal{A}$. We have shown that $R(x) \cap x^\alpha\mathcal{A} = R(x) \cap x^{\alpha+1}\mathcal{A}$. By induction we see that $\alpha \in \Delta_l(x)$. □

PROPOSITION 7.2. *Let $x \in \Phi_g(\mathcal{A})$. Then*

(1) $\Theta(R(x) \cap x\mathcal{A}) < \infty$;

(2) $\Theta(L(x) \cap \mathcal{A}x) < \infty$;

(3) $\Delta_l(x) \neq \emptyset$ *and* $\Delta_r(x) \neq \emptyset$.

PROOF. We only show that $\Theta(R(x) \cap x\mathcal{A}) < \infty$ and $\Delta_l(x) \neq \emptyset$. Take a pseudo-inverse y of x such that $v = e - xy - yx \in \Phi(\mathcal{A})$. If $z \in R(x) \cap x\mathcal{A}$, then $z = (e - yx)z = xyz$, hence $vz = z - xyz - yxz = 0$, thus $z \in R(v)$. Therefore $R(x) \cap x\mathcal{A} \subseteq R(v)$. It follows from Proposition 3.4 (3) and 3.5 (4) that $\Theta(R(x) \cap x\mathcal{A}) \leq \Theta(R(v)) = \mathrm{nul}(v) < \infty$. For $n \in \mathbb{N}$ put $\Theta_n = \Theta(R(x) \cap x^n\mathcal{A})$. Since $R(x) \cap x^{n+1}\mathcal{A} \subseteq R(x) \cap x^n\mathcal{A}$ we derive from Proposition 3.4 (3) that

$$0 \leq \cdots \leq \Theta_{n+1} \leq \Theta_n \leq \cdots \leq \Theta_1 < \infty .$$

Since $\Theta_n \in \mathbb{N}_0$ for $n \in \mathbb{N}$, there is some $\alpha \in \mathbb{N}$ such that $\Theta_{\alpha+k} = \Theta_\alpha$ for all $k \geq 0$. Use Proposition 3.4 (3) to see that $R(x) \cap x^\alpha \mathcal{A} = R(x) \cap x^{\alpha+k}\mathcal{A}$ for all $k \geq 0$. Hence $\alpha \in \Delta_l(x)$. ∎

In view of Proposition 7.2 (3) we define for $x \in \Phi_g(\mathcal{A})$:

$$\delta_l(x) = \min \Delta_l(x) \quad \text{and} \quad \delta_r(x) = \min \Delta_r(x) \; .$$

Proposition 7.3. *For $x \in \Phi_g(\mathcal{A})$ we have*

(1) *$p_l(x) = q_r(x)$ and $q_l(x) = p_r(x)$.*

(2) *If $\alpha = \delta_l(x)$ then*

$$p_l(x) < \infty \;\Leftrightarrow\; R(x) \cap x^\alpha \mathcal{A} = \{0\} \; .$$

In this case $p_l(x) = \delta_l(x)$.

(3) *If $\beta = \delta_r(x)$ then*

$$q_l(x) < \infty \;\Leftrightarrow\; L(x^\beta) + \mathcal{A}x = \mathcal{A} \; .$$

In this case $q_l(x) = \delta_r(x)$.

Proof. (1) Proposition 4.1 (2) gives $p_l(x) \leq q_r(x)$. Without loss of generality we assume that $n = p_l(x) < \infty$. Since $x^n, x^{n+1} \in \Phi_g(\mathcal{A})$, x^n and x^{n+1} are relatively regular. Thus $\mathcal{A}x^n = \mathcal{A}p$, $\mathcal{A}x^{n+1} = \mathcal{A}q$ for some $p = p^2,\; q = q^2 \in \mathcal{A}$. Then it follows that

$$(e-p)\mathcal{A} = R(x^n) = R(x^{n+1}) = (e-q)\mathcal{A} \; ,$$

thus $e - q = (e-p)(e-q) = e - q - p + pq$, hence $p = pq$. Then $\mathcal{A}x^n = \mathcal{A}p = \mathcal{A}pq \subseteq \mathcal{A}q = \mathcal{A}x^{n+1} \subseteq \mathcal{A}x^n$. Hence $q_r(x) \leq n = p_l(x)$.

The proof for $q_l(x) = p_r(x)$ is similar.

(2) “$\Rightarrow$”: Put $p = p_l(x)$. Proposition 7.1 (1) gives $R(x) \cap x^p\mathcal{A} = \{0\}$, thus $p \in \Delta_l(x)$ and $\alpha \leq p$.

“$\Leftarrow$” follows from Proposition 7.1 (1).

(3) Similar. ∎

For the rest of this section we always assume that $\mathcal{A}$ is a Banach algebra.

For our further investigation the following concepts will be useful.

For $x \in \mathcal{A}$ we define

$$a_l(x) = \{\mu \in \mathbb{C}\colon \text{there is a neighbourhood } U \text{ of } \mu \text{ and a holomorphic function } f : U \to \mathcal{A} \text{ such that } (\lambda e - x)f(\lambda) = 0 \text{ on } U \text{ and } f(\mu) \neq 0\} .$$

$$a_r(x) = \{\mu \in \mathbb{C}\colon \text{there is a neighbourhood } U \text{ of } \mu \text{ and a holomorphic function } f : U \to \mathcal{A} \text{ such that } f(\lambda)(\lambda e - x) = 0 \text{ on } U \text{ and } f(\mu) \neq 0\} .$$

It is clear that $a_l(x)$ and $a_r(x)$ are open subsets of $\mathbb{C}$.

Proposition 7.4. *Let $x \in \mathcal{A}$.*

(1) *If $\lambda_0 \in a_l(x)$ then $p_l(\lambda_0 e - x) = q_l(\lambda_0 e - x) = \infty$.*

(2) *If $\lambda_0 \in a_r(x)$ then $p_r(\lambda_0 e - x) = q_r(\lambda_0 e - x) = \infty$.*

Proof. [13, Theorem 3.5]. □

Lemma 7.5. *Let $u \in \Phi(\mathcal{A})$.*

(1) $p_l(u) = 0 \;\Leftrightarrow\; \mathrm{nul}(u) = 0$.

(2) $q_l(u) = 0 \;\Leftrightarrow\; \mathrm{def}(u) = 0$.

Proof. (1) $p_l(u) = 0 \;\Leftrightarrow\; R(u) = \{0\} \;\Leftrightarrow\; \mathrm{nul}(u) = 0$.

(2) Use Proposition 7.3 (1) to get

$$q_l(u) = 0 \;\Leftrightarrow\; p_r(u) = 0 \;\Leftrightarrow\; L(u) = \{0\} \;\Leftrightarrow\; \mathrm{def}(u) = 0 .$$ □

In Theorem 3.11 (4) we have seen that if $x \in \Phi_g(\mathcal{A})$, then there is $\delta > 0$ such that $\lambda e - x \in \Phi(\mathcal{A})$ for $0 < |\lambda| < \delta$. This and Theorem 3.6 (3) (ii) show

Proposition 7.6. *If $x \in \Phi_g(\mathcal{A})$, then there is $\epsilon > 0$ and there are $n, m \in \mathbb{N}_0$ such that*

$$\mathrm{nul}(\lambda e - x) = n \textit{ and } \mathrm{def}(\lambda e - x) = m \quad \textit{for} \quad 0 < |\lambda| < \epsilon .$$

Theorem 7.7. *Let* $x \in \Phi_g(\mathcal{A})$*.*

(1) *If* $p_l(x)$ $(= q_r(x)) < \infty$*, then there is* $\epsilon > 0$ *such that*

$$p_l(\lambda e - x) = \mathrm{nul}(\lambda e - x) = 0 \text{ for } 0 < |\lambda| < \epsilon \,.$$

(2) *If* $q_l(x)$ $(= p_r(x)) < \infty$*, then there is* $\epsilon > 0$ *such that*

$$q_l(\lambda e - x) = \mathrm{def}(\lambda e - x) = 0 \text{ for } 0 < |\lambda| < \epsilon \,.$$

(3) *The following assertions are equivalent:*
- (i) $p_l(x)$ $(= q_r(x)) = \infty$.
- (ii) $0 \in a_l(x)$.
- (iii) *There is* $\epsilon > 0$ *with* $\mathrm{nul}(\lambda e - x) > 0$ *for* $|\lambda| < \epsilon$.

(4) *The following assertions are equivalent:*
- (i) $q_l(x) = (p_r(x)) = \infty$.
- (ii) $0 \in a_r(x)$.
- (iii) *There is* $\epsilon > 0$ *with* $\mathrm{def}(\lambda e - x) > 0$ *for* $|\lambda| < \epsilon$.

Proof. We only prove (1) and (3).

(1) Define the bounded linear operator $T : \mathcal{A} \to \mathcal{A}$ by $Ta = xa$ ($a \in \mathcal{A}$). Then $p(T) = p_l(x)$ and $T^n(\mathcal{A}) = x^n\mathcal{A}$ ($n \in \mathbb{N}$). Since $x^n \in \Phi_g(\mathcal{A})$, x^n is relatively regular, thus $T^n(\mathcal{A})$ is closed. Lemma 2.5 in [9] shows that there is $\epsilon > 0$ with $p(\lambda I - T) = 0$ for $0 < |\lambda| < \epsilon$. Use Lemma 7.5 to conclude that (1) holds.

(3) (i) $\Rightarrow$ (ii): Write $\mathcal{M} = \bigcap_{n=1}^{\infty} x^n\mathcal{A}$. As above, each $x^n\mathcal{A}$ is closed, thus $\mathcal{M}$ is a closed subspace of $\mathcal{A}$. We have

$$x\mathcal{M} = \mathcal{M} \,. \tag{7.8}$$

In fact, since the inclusion "$\subseteq$" is clear, we only have to show that $\mathcal{M} \subseteq x\mathcal{M}$. Since $\Delta_l(x) \neq \emptyset$ (Proposition 7.2), there is $\alpha \in \mathbb{N}_0$ with

$$R(x) \cap x^\alpha\mathcal{A} = R(x) \cap x^{\alpha+k}\mathcal{A} \quad \text{for} \quad k \geq 0 \,.$$

Take $y \in \mathcal{M}$. Then there is a sequence $(u_k)_{k=1}^{\infty}$ in $\mathcal{A}$ such that $y = x^{\alpha+k}u_k$ for $k \geq 1$. Put $z_k = x^{\alpha}u_1 - x^{\alpha+k-1}u_k$. Then $xz_k = 0$, thus $z_k \in R(x) \cap x^{\alpha}\mathcal{A} = R(x) \cap x^{\alpha+k-1}\mathcal{A}$ for all $k \geq 1$. It follows that

$$x^{\alpha}u_1 = z_k + x^{\alpha+k-1}u_k \in x^{\alpha+k-1}\mathcal{A}\ (k \geq 1)\ .$$

Hence $x^{\alpha}u_1 \in \mathcal{M}$ and therefore $y = x^{\alpha+1}u_1 \in x\mathcal{M}$. The proof of (7.8) is now complete.

(7.8) and the open mapping theorem show that there is a constant $\gamma > 0$ such that

$$\text{for each } y \in \mathcal{M} \quad \text{there is} \quad z \in \mathcal{M} \quad \text{with} \quad xz = y \text{ and } \|z\| \leq \gamma\|y\|\ .$$

Since $R(x) \cap \mathcal{M} = R(x) \cap x^{\alpha}\mathcal{A}$ and $p_l(x) = \infty$, we get some $a_0 \in R(x) \cap \mathcal{M}$ with $a_0 \neq 0$ (Proposition 7.1 (1)).

Now use (7.9) to construct a sequence $(a_n)_{n=1}^{\infty}$ such that

$$xa_{n+1} = a_n \quad \text{and} \quad \|a_n\| \leq \gamma^n\|a_0\| \quad \text{for} \quad n \in \mathbb{N}\ .$$

Put $U = \{\lambda \in \mathbb{C} :\ |\lambda| < 1/\gamma\}$ and $f(\lambda) = \sum\limits_{n=0}^{\infty} a_n\lambda^n$. Then f is holomorphic on U and a simple computation gives

$$(\lambda e - x)f(\lambda) = -xa_0 = 0 \quad \text{for each} \quad \lambda \in U\ .$$

From $f(0) = a_0 \neq 0$ we derive $0 \in a_l(x)$.

(ii) $\Rightarrow$ (iii): Since $a_l(x)$ is open, there is $\epsilon > 0$ such that $\lambda \in a_l(x)$ for $|\lambda| < \epsilon$. Take $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| < \epsilon$. Then there is a neighbourhoud V of λ_0 and a holomorphic $f : V \to \mathcal{A}$ with $f(\lambda_0) \neq 0$ and $(\lambda_0 e - x)f(\lambda_0) = 0$. This shows that $R(\lambda_0 e - x) \neq \{0\}$, thus $\mathrm{nul}(\lambda_0 e - x) > 0$.

(iii) $\Rightarrow$ (i) Assume to the contrary that $p_l(x) < \infty$. (1) shows that there is a positive $\delta \leq \epsilon$ such that $\mathrm{nul}(\lambda e - x) = 0$ for $0 < |\lambda| < \delta$, a contradiction. □

For $x \in \mathcal{A}$ we define

$$\Phi_g(x) = \{\lambda \in \mathbb{C} : \ \lambda e - x \in \Phi_g(\mathcal{A})\} \ .$$

It is clear that $\mathbb{C} \backslash \sigma(x) \subseteq \Phi_g(x)$. From Theorem 3.11 (4) we see that $\Phi_g(x)$ is open.

THEOREM 7.10. *Let $x \in \mathcal{A}$ and let C be a connected component of $\Phi_g(x)$.*

(1) *Either $p_l(\lambda e - x) < \infty$ for all $\lambda \in C$ or $p_l(\lambda e - x) = \infty$ for all $\lambda \in C$.*

(2) *Either $q_l(\lambda e - x) < \infty$ for all $\lambda \in C$ or $q_l(\lambda e - x) = \infty$ for all $\lambda \in C$.*

PROOF. We only show (1). The proof for (2) is similar since $q_l(\lambda e - x) = p_r(\lambda e - x)$.

Put $M = \{\lambda \in C : \ p_l(\lambda e - x) < \infty\}$. Theorem 7.7 (1) shows that M is open.

Take $\lambda_0 \in C \backslash M$, hence $p_l(\lambda_0 e - x) = \infty$. Theorem 7.7 (3) gives $\lambda_0 \in a_l(x)$. Since $a_l(x)$ is open, there is $\epsilon > 0$ such that $\lambda \in a_l(x)$ if $|\lambda - \lambda_0| < \epsilon$. Then it follows from Proposition 7.4 (1) that $p_l(\lambda e - x) = \infty$ if $|\lambda - \lambda_0| < \epsilon$. Hence $C \backslash M$ is open. Since C is connected, $M = \emptyset$ or $C = M$. □

THEOREM 7.11. *For $x \in \Phi_g(\mathcal{A})$ the following assertions are equivalent:*

(1) 0 *is a boundary point of* $\sigma(x)$.

(2) 0 *is an isolated point of* $\sigma(x)$.

(3) 0 *is a pole of* $(\lambda e - x)^{-1}$.

PROOF. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear.

(1) $\Rightarrow$ (3): By C we denote the connected component of $\Phi_g(x)$ for which $0 \in C$. There is some $\epsilon > 0$ such that for $U = \{\lambda \in \mathbb{C} : \ |\lambda| < \epsilon\}$ we have $U \subseteq C$ and $U \cap (\mathbb{C} \backslash \sigma(x)) \neq \emptyset$. For $\lambda \in U \cap (\mathbb{C} \backslash \sigma(x))$ we have $p_l(\lambda e - x) = q_l(\lambda e - x) = 0$. Theorem 7.10 shows now that $p_l(x), \ q_l(x) < \infty$. Observe that $p_l(x), \ q_l(x) > 0$. Proposition 4.1 (1) shows then that $0 < p_l(x) = q_l(x) < \infty$. From [18, Theorem 15.6] (see also [11]) we conclude that 0 is a pole of $(\lambda e - x)^{-1}$. □

Examples for $\Phi_g(x)$.

(1) If $x \in \mathrm{soc}(\mathcal{A})$ then $\Phi_g(x) = \mathbb{C}$.

(2) If $x \in \overline{\mathrm{soc}(\mathcal{A})} \backslash \mathrm{soc}(\mathcal{A})$ then $\Phi_g(x) = \mathbb{C} \backslash \{0\}$ (see Theorem 6.6).

(3) If x is a Riesz element then $\mathbb{C} \backslash \{0\} \subseteq \Phi_g(x)$.

(4) Let $x \in \mathcal{A}$ with $x^2 = x$. Then $\sigma(x) \subseteq \{0, 1\}$, thus $\mathbb{C} \backslash \{0, 1\} \subseteq \Phi_g(x)$. We also have $(e - x)^2 = e - x$. Therefore, by Example 3.9 (5), $0, 1 \in \Phi_g(x)$. Thus $\Phi_g(x) = \mathbb{C}$.

We close this paper with

Theorem 7.12. *Suppose that* $\dim \mathcal{A} = \infty$ *and let* $x \in \mathcal{A}$. *The following assertions are equivalent:*

(1) $\Phi_g(x) = \mathbb{C}$.

(2) *There are* $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ *with* $\lambda_i \neq \lambda_j$ *for* $i \neq j$ *and*

$$\prod_{j=1}^{m} (x - \lambda_j e) \in \mathrm{soc}(\mathcal{A}) .$$

Proof. (1) $\Rightarrow$ (2): Take $\mu \in \sigma(x)$. Since $\mu \in \Phi_g(x)$, it follows from Theorem 3.11 (4) that there is an open neighbourhood U_μ of μ with

$$x - \lambda e \in \Phi(\mathcal{A}) \ for \ \lambda \in U_\mu \backslash \{\mu\} . \tag{7.13}$$

Since $\sigma(x) \subseteq \bigcup_{\mu \in \sigma(x)} U_\mu$ and $\sigma(x)$ is compact, there are $\lambda_1, \ldots, \lambda_n \in \sigma(x)$ such that

$$\sigma(x) \subseteq \bigcup_{j=1}^{n} U_{\lambda_j} .$$

This and (7.13) show that $\sigma_\Phi(x) = \sigma(\widetilde{x}) \subseteq \{\lambda_1, \ldots, \lambda_n\}$. Since $\dim \mathcal{A} = \infty$, $\sigma(\widetilde{x}) \neq \emptyset$, thus $\sigma(\widetilde{x}) = \{\lambda_1, \ldots, \lambda_m\}$ with $m \leq n$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Define the polynomial p by $p(\lambda) = \prod_{j=1}^{m} (\lambda - \lambda_j)$. Then

$$\sigma_\Phi(p(x)) = \sigma(\widetilde{p(x)}) = \sigma(p(\widetilde{x})) = p(\sigma(\widetilde{x})) = \{0\} .$$

It follows that $p(x)$ is a Riesz element. Since $x - \lambda_j e \in \Phi_g(\mathcal{A})$ for $j = 1, \dots, m$, we have $p(x) \in \Phi_g(\mathcal{A})$ (see Theorem 5.2 (1)). Now use Theorem 6.7 to get $p(x) \in \mathrm{soc}(\mathcal{A})$.

(2) $\Rightarrow$ (1): Let p denote the polynomial $p(\lambda) = \prod_{j=1}^{m} (\lambda - \lambda_j)$. Since $p(x) \in \mathrm{soc}(\mathcal{A})$, $\widehat{0} = \widehat{p(x)} = p(\widehat{x})$. Proposition 4.6 yields $\widehat{x} - \lambda\widehat{e} \in \widehat{\mathcal{A}}^g$ for each $\lambda \in \mathbb{C}$, thus $\Phi_g(x) = \mathbb{C}$. ∎

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