PARTIAL ISOMETRIES ON BANACH SPACES

CHRISTOPH SCHMOEGER

1. Introduction and terminology

Throughout this paper, X shall denote a complex Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators on X. For an operator $T \in \mathcal{L}(X)$ we write $N(T)$ for its kernel and $T(X)$ for its range. The spectrum, the resolvent set and the spectral radius of $T \in \mathcal{L}(X)$ are denoted by $\sigma(T)$, $\rho(T)$ and $r(T)$, respectively. The *reduced minimum modulus* of T is defined by

$$\gamma(T) = \inf\{\|Tx\| : \text{dist}(x, N(T)) = 1\} \quad (\gamma(T) = \infty \text{ if } T = 0).$$

It is well known that $\gamma(T) > 0$ if and only if $T(X)$ is closed.
We will say that $T \in \mathcal{L}(X)$ is *relatively regular* if there exists an operator $S \in \mathcal{L}(X)$ for which

$$TST = T.$$

In this case S is called a *pseudo inverse* of T. If $T \in \mathcal{L}(X)$ is relatively regular and $S \in \mathcal{L}(X)$ such that

$$TST = T \text{ and } STS = S,$$

then S is called a *generalized inverse* of T. Observe that if S is a pseudo inverse of T, then $S_0 = STS$ is a generalized inverse of T. We recall that in general a pseudo inverse is not unique, and that T is relatively regular if and only if $N(T)$ and $T(X)$ are closed and complemented subspaces of X (see for instance [4]).

If $T \in \mathcal{L}(X)$ has a generalized inverse S, then

$$TS,\ ST,\ I - TS \text{ and } I - ST$$

are projections and

$$(TS)(X) = T(X),\ (ST)(X) = S(X),$$
$$(I - ST)(X) = N(T) \text{ and } (I - TS)(X) = N(S).$$

In the following proposition a useful relation between the reduced minimum modulus and generalized inverses is established. A proof can be found in [10].

2000 *Mathematics Subject Classification.* 47 A 53.
Key words and phrases. generalized inverse, partial isometry.

1.1. Proposition. *Let $T \in \mathcal{L}(X)$, $T \neq 0$, and S be a generalized inverse of T. Then*

$$\frac{1}{\|S\|} \leq \gamma(T) \leq \frac{\|TS\|\,\|ST\|}{\|S\|}.$$

A bounded linear operator T on a complex Hilbert space is said to be a *partial isometry* provided that $\|Tx\| = \|x\|$ for every $x \in N(T)^{\perp}$, that is, T^* is a generalized inverse of T (i.e. $TT^*T = T$). In this case $\|T\| \leq 1$ (see Chapter 13 of [6] for details).

M. Mbekhta has given in [10] the following characterization of partial isometries:

1.2. Theorem. *If T is a bounded linear operator on a complex Hilbert space with $\|T\| \leq 1$, then the following are equivalent:*

(1) *T is a partial isometry,*
(2) *T has a generalized inverse S with $\|S\| \leq 1$.*

Since assertion (2) of the above theorem does not depend on the structure of a Hilbert space, Theorem 1.2 suggests a definition (due to M. Mbekhta) of a partial isometry in the algebra of operators on *Banach* spaces:

1.3. Definition. A bounded linear operator T on a *Banach* space is called a *partial isometry* if T is a contraction and admits a generalized inverse which is a contraction.

Remarks.

(1) Partial isometries are investigated in [10].
(2) In Definition 1.3, the contractive generalized inverse is in general not unique (see [10, page 776].
(3) One of the disadvantages of Definition 1.3 is that, in general, an arbitrary isometry $T \in \mathcal{L}(X)$ (i.e. $\|Tx\| = \|x\|$ for all $x \in X$) does not need to be a partial isometry (indeed an isometry may not have generalized inverse), but we have the following result ([10, Corollary 4.3]):

    An isometry $T \in \mathcal{L}(X)$ is a partial isometry, in the sense of Definition 1.3, if and only if there exists a projection onto $T(X)$ of norm 1.

    There are certain Banach spaces (other than Hilbert spaces) in which all isometries are "partial", including $L^p(\mu)\,(1 \leq p \leq \infty)$, as shown in [1] and [3].
(4) If $T \in \mathcal{L}(X)$ is a partial isometry and S is a contractive generalized inverse of T, then

$$X = S(X) \oplus N(T)$$

and

$$\|Tx\| = \|x\| \text{ for every } x \in S(X).$$

Indeed, we have $X = (ST)(X) \oplus (I - ST)(X) = S(X) \oplus N(T)$. Furthermore, suppose $x = Sy \in S(X)$. Then

$$\|x\| = \|Sy\| = \|STSy\| \leq \|S\| \, \|TSy\| = \|Tx\| \leq \|T\| \, \|x\| \leq \|x\|,$$

thus $\|Tx\| = \|x\|$.

1.4. Proposition. *If $T \in \mathcal{L}(X)$ is a non-zero partial isometry and S is a contractive generalized inverse of T, then*

$$\|T\| = \|S\| = \|TS\| = \|ST\| = \gamma(T) = 1$$

Proof. $\|T\| = \|TST\| \leq \|T\|\|S\| \, \|T\| \leq \|T\| \, \|S\|$ implies $\|S\| \geq 1$, and so $\|S\| = 1$. Since $(TS)^2 = TS$ and $TS \neq 0$, $1 \leq \|TS\| \leq \|T\| \, \|S\| \leq 1$, thus $\|TS\| = 1$. The same arguments give $\|T\| = \|ST\| = 1$. Finally we obtain $\gamma(T) = 1$, by Proposition 1.1. □

The next result is shown in [10, Proposition 4.2]:

1.5. Proposition. *For $T \in \mathcal{L}(X)$ the following conditions are equivalent:*

(1) *T is a partial isometry;*
(2) *there are two projections P and Q such that $P(X) = T(X)$, $N(Q) = N(T)$, $\|P\| = \|Q\| = 1$ and*

$$\|TQx\| = \|Qx\| \text{ for every } x \in X.$$

Examples.

(1) If $P \in \mathcal{L}(X)$ is a projection and $P \neq 0$, then P is a partial isometry if and only if $\|P\| = 1$.
(2) Let T be the bounded operator on the Banach space $l^1(\mathbb{N})$ defined by

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Let the operator S on $l^1(\mathbb{N})$ be given by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots),$$

then it is easy to see $TST = T$ and $STS = S$. Since $\|T\| = \|S\| = 1$, T is a partial isometry.

2. Spectral properties of partial isometries

In this section we always assume that $T \in \mathcal{L}(X)$ is a non-zero partial isometry and that S is a contractive generalized inverse of T. Recall that then $\|T\| = \|S\| = 1$.

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\overline{\mathbb{D}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. By $\mathcal{L}(X)^{-1}$ we denote the group of all invertible operators in $\mathcal{L}(X)$.

2.1. Proposition. *If $T \in \mathcal{L}(X)^{-1}$ then $S = T^{-1}$ and*

$$\sigma(T) \subseteq \partial \mathbb{D}.$$

Proof. Since $T \in \mathcal{L}(X)^{-1}$ and $TST = T$, it follows that $ST = I$ and $TS = I$. Hence $0 \in \rho(T)$. Now let $\lambda \in \mathbb{C}$ and $0 < |\lambda| < 1$. Then $|\lambda|^{-1} > \|S\| \geq r(S)$, thus $\lambda^{-1} \in \rho(S)$. Therefore we get from

$$(1/\lambda\, I - S)\,(-\lambda T) = \lambda I - T,$$

that $\lambda I - T \in \mathcal{L}(X)^{-1}$, hence $\lambda \in \rho(T)$. This shows that $\mathbb{D} \subseteq \rho(T)$. Since $\lambda \in \rho(T)$ if $|\lambda| > 1 = \|T\|$, we derive that $\sigma(T) \subseteq \partial\,\mathbb{D}$. □

An operator $U \in \mathcal{L}(X)$ is called *decomposably regular* if U is relatively regular and admits a pseudo inverse $V \in \mathcal{L}(X)^{-1}$.

A proof of the next result can be found in [7, Chapter 3.8].

2.2. Proposition. *Suppose that $U \in \mathcal{L}(X)$ is relatively regular. Then the following assertions are equivalent:*

(1) *U is decomposably regular;*
(2) *$N(U)$ and $X/U(X)$ are isomorphic;*
(3) *there are P, $V \in \mathcal{L}(X)$ such that $P^2 = P$, $V \in \mathcal{L}(X)^{-1}$ and $U = VP$;*
(4) *there are Q, $W \in \mathcal{L}(X)$ such that $Q^2 = Q$, $W \in \mathcal{L}(X)^{-1}$ and $U = QW$.*

Examples.

(1) Each projection $P \in \mathcal{L}(X)$ is decomposably regular, since $P = PIP$.
(2) Proposition 2.2 (2) shows that if $\dim X < \infty$, then each operator on X is decomposably regular.
(3) For $U \in \mathcal{L}(X)$ let $\alpha(U) = \dim N(U)$ and $\beta(U) = \operatorname{codim} U(X)$. U is called a *Fredholm operator* if $\alpha(U) < \infty$ and $\beta(U) < \infty$. In this case

$$\operatorname{ind}(U) = \alpha(U) - \beta(U)$$

is called the *index* of U. It follows from [8, §74] that a Fredholm operator U is relatively regular and Proposition 2.2 (2) shows that

$$U \text{ is decomposably regular } \iff \operatorname{ind}(U) = 0.$$

(4) In [14, Theorem 2.1] we have shown that an operator U is an interior point of the set of all decomposably regular operators if and only if U is a Fredholm operator with $\operatorname{ind}(U) = 0$.

2.3. Theorem.

(1) *If $\mathbb{D} \cap \rho(S) \neq \emptyset$ or $\mathbb{D} \cap \rho(T) \neq \emptyset$, then T and S are both decomposably regular.*
(2) *Suppose that T is not decomposably regular, then*

$$\sigma(T) = \sigma(S) = \overline{\mathbb{D}}.$$

Proof. (1) Assume that $\mathbb{D} \cap \rho(S) \neq \emptyset$. Take $\lambda_0 \in \mathbb{D} \cap \rho(S)$. Then $\|\lambda_0 T\| = |\lambda_0|\|T\| = |\lambda_0| < 1$, thus $\lambda_0 T - I \in \mathcal{L}(X)^{-1}$. Since $\lambda_0 I - S \in \mathcal{L}(X)^{-1}$, the operator

$$R = (\lambda_0 T - I)^{-1}\,(\lambda_0 I - S) \in \mathcal{L}(X)^{-1}.$$

From

$$(\lambda_0 T - I)ST = \lambda_0 TST - ST = \lambda_0 T - ST = (\lambda_0 I - S)T$$

we see that

$$ST = (\lambda_0 T - I)^{-1} (\lambda_0 I - S)T = RT,$$

hence $T = T(ST) = TRT$. Therefore T is decomposably regular. On the other hand

$$S = (ST)S = RTS = R(TS),$$

thus, by Proposition 2.2 (3), S is decomposably regular.
If $\mathbb{D} \cap \rho(T) \neq \emptyset$, the same arguments show that T and S are decomposably regular.

(2) By (1) we must have $\mathbb{D} \subseteq \sigma(T)$ and $\mathbb{D} \subseteq \sigma(S)$. Since the spectrum of an operator is always closed, we derive $\overline{\mathbb{D}} \subseteq \sigma(T)$ and $\overline{\mathbb{D}} \subseteq \sigma(S)$. From $\|T\| = \|S\| = 1$, we see that $\sigma(T), \sigma(S) \subseteq \overline{\mathbb{D}}$. □

2.4. Corollary.

1. *If $r(T) < 1$ or $r(S) < 1$ then both T and S are decomposably regular.*
2. *If T is a Fredholm operator and* $\operatorname{ind}(T) \neq 0$*, then*

$$\sigma(T) = \sigma(S) = \overline{\mathbb{D}}\,.$$

Remark. Since each projection with norm 1 is a partial isometry and decomposably regular we see that in general the implication in Corollary 2.4 (1) cannot be reversed.

2.5. Corollary. *Suppose that T is not decomposably regular. Then*

$$\{r(R) : R \textit{ is a pseudo inverse of } T\} = [1, \infty).$$

Proof. Let $M = \{r(R) : R \text{ is a pseudo inverse of } T\}$ and $\alpha = \inf M$. Assume that $\alpha < 1$. Hence there is $R \in \mathcal{L}(X)$ such that $TRT = T$ and $r(R) < 1$. Take a complex number λ_0 with $r(R) < |\lambda_0| < 1$. Then $\lambda_0 \in \rho(R)$ and $\lambda_0^{-1} \in \rho(T)$, since $r(T) = 1$, by Theorem 2.3 (2). Therefore

$$V = (\lambda_0 T - I)\,(\lambda_0 I - R) \in \mathcal{L}(X)^{-1}.$$

As in the proof of Theorem 2.3 (1) we conclude that $TVT = T$, thus T is decomposably regular, a contradiction. Therefore $\alpha \geq 1$. Theorem 2.3 (1) shows that $r(S) = 1$, hence $1 = \min M$, thus $M \subseteq [1, \infty)$. Now take $\beta \in [1, \infty)$. Since $T \notin \mathcal{L}(X)^{-1}$, $TS \neq I$ or $ST \neq I$. Then it follows from [12, Corollary 4] that there is a pseudo inverse B of T with $r(B) = \beta$. Hence $\beta \in M$, and so $M = [1, \infty)$. □

2.6. Proposition. *Suppose that $T \notin \mathcal{L}(X)^{-1}$. Then*

$$\{\|R\| : R \textit{ is a pseudo inverse of } T\} = [1, \infty).$$

Proof. Let $M = \{\|R\| : R \text{ is a pseudo inverse of } T\}$. If $R \in \mathcal{L}(X)$ and $TRT = T$, then $1 = \|T\| = \|TRT\| \leq \|T\|^2 \|R\| = \|R\|$, thus $M \subseteq [1, \infty)$. Theorem 4 in [12] shows that $[\|S\|, \infty) \subseteq M$. Since $\|S\| = 1$, we get $M = [1, \infty)$. □

Now we introduce a further class of relatively regular operators: an operator $U \in \mathcal{L}(X)$ is called *holomorphically regular* if there is a neighbourhood $\Omega \subseteq \mathbb{C}$ of 0 and a holomorphic function $F : \Omega \to \mathcal{L}(X)$ such that

$$(U - \lambda I)\, F(\lambda)\, (U - \lambda I) = U - \lambda I \text{ for all } \lambda \in \Omega.$$

2.7. Proposition. *For $U \in \mathcal{L}(X)$ the following assertions are equivalent:*

(1) *U is holomorphically regular;*
(2) *U is relatively regular and $N(U) \subseteq \bigcap_{n=1}^{\infty} U^n(X)$.*

Proof. cf. [13, Theorem 1.4]. □

Examples.

(1) If $U \in \mathcal{L}(X)$ is right or left invertible in $\mathcal{L}(X)$, then U is holomorphically regular. Indeed, suppose that V is a right inverse of U, thus $UV = I$. It follows that $U^nV^n = I$ for all $n \in \mathbb{N}$. Hence $1 \leq \|U^n\|^{1/n}\|V^n\|^{1/n}$ for all $n \in \mathbb{N}$, and so $1 \leq r(U)r(V)$, thus $r(U) \neq 0 \neq r(V)$. Let $\Omega = \{\lambda \in \mathbb{C} : |\lambda| < r(V)^{-1}\}$ and $F(\lambda) = V(I - \lambda V)^{-1}\,(\lambda \in \Omega)$. Then it it easy to see that
$$(U - \lambda I)\,F(\lambda)\,(U - \lambda I) = U - \lambda I$$
for every $\lambda \in \Omega$.
Similar arguments show that U is holomorphically regular if U is left invertible.
(2) Let $U \in \mathcal{L}(X)$ be a Fredholm operator, then it is well-known that there is $\rho > 0$ such that $U - \lambda I$ is a Fredholm operator for $|\lambda| < \rho$ and that there are non-negative integers α_0 and β_0 such that
$$\alpha_0 = \alpha(U - \lambda I) \leq \alpha(U),\ \beta_0 = \beta(U - \lambda I) \leq \beta(U) \text{ for } 0 < |\lambda| < \rho.$$
It is shown in [15] that U is holomorphically regular if and only if
$$\alpha(U - \lambda I) = \alpha(U) \text{ and } \beta(U - \lambda I) = \beta(U) \text{ for } |\lambda| < \rho.$$

We say that $U \in \mathcal{L}(X)$ is *holomorphically decomposably regular* if there is a neighbourhood $\Omega \subseteq \mathbb{C}$ of 0 and a holomorphic function $F : \Omega \to \mathcal{L}(X)$ such that $F(\lambda) \in \mathcal{L}(X)^{-1}$ for all $\lambda \in \Omega$ and
$$(U - \lambda I)\,F(\lambda)\,(U - \lambda I) = U - \lambda I \text{ for all } \lambda \in \Omega.$$

2.8. Theorem. *If T is holomorphically regular and if $T \notin \mathcal{L}(X)^{-1}$, then*

(1) *$\sigma(T) = \overline{\mathbb{D}}$ and $r(S) = 1$;*
(2) *if $F(\lambda) = (I - \lambda S)^{-1}S$ for $\lambda \in \mathbb{D}$, then*
$$(T - \lambda I)\,F(\lambda)\,(T - \lambda I) = T - \lambda I$$
and
$$F(\lambda)\,(T - \lambda I)\,F(\lambda) = F(\lambda)$$
for every $\lambda \in \mathbb{D}$;
(3) *if $\mathbb{D} \cap \rho(S) \neq \emptyset$, then S is decomposably regular and T is holomorphically decomposably regular;*
(4) *for each $n \in \mathbb{N}$, T^n is a non-zero partial isometry and a contractive generalized inverse of T^n is given by $S^nT^nS^n$.*

Proof. (1) Let $\Omega = \{\lambda \in \mathbb{C} : |\lambda| r(S) < 1\}$ and $F(\lambda) = (I - \lambda S)^{-1}S$. We have shown in [13, Corollary 1.5] that

$$(*) \qquad (T - \lambda I)\, F(\lambda)\, (T - \lambda I) \quad \text{for} \quad \lambda \in \Omega.$$

Now take $\lambda_0 \in \Omega$ and assume that $\lambda_0 \in \rho(T)$. By $(*)$, $F(\lambda_0) = (T - \lambda_0 I)^{-1}$, thus

$$S(I - \lambda_0 S)^{-1} = (I - \lambda_0 S)^{-1} S = (T - \lambda_0 I)^{-1},$$

therefore $S(T - \lambda_0 I) = (T - \lambda_0 I)S = I - \lambda_0 S$, and so $TS = ST = I$, a contradiction, since $T \notin \mathcal{L}(X)^{-1}$. Hence we have shown that $\Omega \subseteq \sigma(T)$. Since $\sigma(T)$ is bounded, $r(S) > 0$, $r(T) > 0$ and

$$\overline{\Omega} = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{r(S)}\} \subseteq \sigma(T) \subseteq \overline{\mathbb{D}}.$$

¿From this it follows that $r(S) \geq 1$, consequently $r(S) = 1$ and $\sigma(T) = \overline{\mathbb{D}}$.

(2) The proof of (1) shows that $\Omega = \mathbb{D}$ and that

$$(T - \lambda I)\, F(\lambda)\, (T - \lambda I) = T - \lambda I \text{ for } \lambda \in \mathbb{D}.$$

Now take $\lambda \in \mathbb{D}$. Then

$$\begin{aligned} F(\lambda)\,(T - \lambda I)\, F(\lambda) &= (I - \lambda S)^{-1}\,(ST - \lambda S)\, F(\lambda) \\ &= (I - \lambda S)^{-1}\,(I - \lambda S - (I - ST))\, F(\lambda) \\ &= (I - (I - \lambda S)^{-1}\,(I - ST))\, S(I - \lambda S)^{-1} \\ &= F(\lambda) - (I - \lambda S)^{-1}\, \underbrace{(I - ST)S}_{=0}\, (I - \lambda S)^{-1} \\ &= F(\lambda). \end{aligned}$$

(3) Theorem 2.3 (1) shows that T and S are decomposably regular. Take $R \in \mathcal{L}(X)$ with $TRT = T$ and $R \in \mathcal{L}(X)^{-1}$. As in the proof of (1), $r(R) > 0$. Let $\Omega_0 = \{\lambda \in \mathbb{C} : |\lambda| < r(R)^{-1}\}$ and $G(\lambda) = (I - \lambda R)^{-1}R$ for $\lambda \in \Omega_0$. Then $G(\lambda) \in \mathcal{L}(X)^{-1}$ for $\lambda \in \Omega_0$ and as above

$$(T - \lambda I)\, G(\lambda)\, (T - \lambda I) = T - \lambda I \quad (\lambda \in \Omega_0).$$

(4) By Proposition 9 in [12], $T^n S^n T^n = T^n$ for all $n \in \mathbb{N}$. Let $S_n = S^n T^n S^n$ $(n \in \mathbb{N})$. Then

$$T^n S_n T^n = T^n \text{ and } S_n T^n S_n = S_n \quad (n \in \mathbb{N}).$$

Since $r(T) = 1$, $T^n \neq 0$. Furthermore we have $\|T^n\| \leq \|T\|^n = 1$ and $\|S_n\| \leq \|S\|^n\, \|T\|^n\, \|S\|^n = 1$. □

For $U \in \mathcal{L}(X)$ let $\sigma_p(U)$ denote the set of eigenvalues of U.

2.9. Corollary. *Suppose that T is right or left invertible but not invertible.*

(1) $\sigma(T) = \sigma(S) = \overline{\mathbb{D}}$;
(2) *if T is right invertible, then* $\mathbb{D} \subseteq \sigma_p(T)$ *and* $\mathbb{D} \cap \sigma_p(S) = \emptyset$;
(3) *if T is left invertible, then* $\mathbb{D} \subseteq \sigma_p(S)$ *and* $\mathbb{D} \cap \sigma_p(T) = \emptyset$.

Proof. (1) If T is right (left) invertible, then S is left (right) invertible, hence T and S are holomorphically regular. Theorem 2.8 (1) gives the result.

(2) Let $R \in \mathcal{L}(X)$ with $TR = I$. From $TST = T$ we derive $I = TR = TSTR = TS$. Let $\lambda \in \mathbb{D}$. Then $(T-\lambda I)\, S(I-\lambda S)^{-1} = (I-\lambda S)\,(I-\lambda S)^{-1} = I$ and $N(T-\lambda I) = (I - S(I-\lambda S)^{-1}\,(T-\lambda I))(X)$. Since $\lambda \in \sigma(T)$, $I - S(I-\lambda S)^{-1}\,(T-\lambda I) \neq 0$, therefore $N(T-\lambda I) \neq \{0\}$. If $Sx = \lambda x$ for some $x \in X$, then $x = TSx = \lambda Tx$, hence $Tx = \lambda^{-1}x$. Since $|\lambda^{-1}| > 1 = r(T)$, we derive $x = 0$, thus $\lambda \notin \sigma_p(S)$.

(3) Similar. □

2.10. Corollary. *If T is holomorphically regular and $T \notin \mathcal{L}(X)^{-1}$ then*

$$\{r(R) : R \text{ is a pseudo inverse of } T\} = [1, \infty).$$

Proof. Let $M = \{r(R) : R \in \mathcal{L}(X) \text{ and } TRT = T\}$ and $\alpha = \inf M$. If $R \in \mathcal{L}(X)$ and $TRT = T$, then, by [12, Proposition 9],

$$T^nR^nT^n = T^n \quad (n \in \mathbb{N}),$$

thus $\|T^n\|^{1/n} \leq \|T^n\|^{2/n}\|R^n\|^{1/n}$ for $n \in \mathbb{N}$. This gives, since $r(T) = 1$ (Theorem 2.8 (1)),

$$1 = r(T) \leq r(T)^2\, r(R) = r(R),$$

thus $\alpha \geq 1$. By Theorem 2.8 (1), $r(S) = 1$, hence $\alpha = 1 = \min M$, and so $M \subseteq [1, \infty)$. Now proceed as in the proof of Corollary 2.5 to derive that $[1, \infty) \subseteq M$. □

3. Partial isometries with an index

Recall that for an operator $U \in \mathcal{L}(X)$, the dimension of $N(U)$ is denoted by $\alpha(U)$ and the codimension of $U(X)$ is denoted by $\beta(U)$. If $\alpha(U)$ and $\beta(U)$ are not both infinite, we say that U *has an index*. The *index* $\text{ind}(U)$ is then defined by

$$\text{ind}(U) = \alpha(U) - \beta(U),$$

with the understanding, that for any real number r,

$$\infty - r = \infty \quad \text{and} \quad r - \infty = -\infty$$

(we agree to let $-(-\infty) = \infty$).

We say that $U \in \mathcal{L}(X)$ is a *semi-Fredholm operator*, if $U(X)$ is closed and U has an index.

Observe that if $T \in \mathcal{L}(X)$ is a partial isometry with an index, then T is semi-Fredholm.

We write $\mathcal{SF}(X)$ for the set of all semi-Fredholm operators on X (see [5] or [8] for properties of this class of operators).

3.1. Proposition. *Let $T \in \mathcal{L}(X)$ be a non-zero partial isometry and $U \in \mathcal{L}(X)$.*

(1) *If $\alpha(T) < \alpha(U)$ then $\|T - U\| \geq 1$.*

(2) *If U has closed range and $\beta(T) < \beta(U)$ then $\|T - U\| \geq 1$.*

Proof. (1) By Lemma V.1.1 in [5] there is $x \in N(U)$ such that $1 = \|x\| = \operatorname{dist}(x, N(T))$, hence, by Proposition 1.4,

$$1 = \gamma(T) \leq \|Tx\| = \|Tx - Ux\| \leq \|T - U\| \, \|x\| = \|T - U\|.$$

(2) We denote by X^* the dual space of X and by R^* the adjoint of $R \in \mathcal{L}(X)$. By [5, Theorem IV.2.3], $\beta(T) = \alpha(T^*)$ and $\beta(U) = \alpha(U^*)$, therefore $\alpha(T^*) < \alpha(U^*)$. Since T^* is a non-zero partial isometry, it follows from (1) that $1 \leq \|T^* - U^*\| = \|T - U\|$. ☐

3.2. Corollary. *If T_1 and T_2 are partial isometries on X and if $\|T_1 - T_2\| < 1$, then*

$$\alpha(T_1) = \alpha(T_2) \quad \text{and} \quad \beta(T_1) = \beta(T_2).$$

Proof. If $T_1 = 0$, then $\|T_2\| < 1$, hence $T_2 = 0$ (since T_2 is partial isometry) and we are done. So we can assume that $T_1 \neq 0$. ¿From Proposition 3.1 we derive (let $T = T_1$ and $U = T_2$) that $\alpha(T_1) \geq \alpha(T_2)$ and $\beta(T_1) \geq \beta(T_2)$. By symmetry we also get $\alpha(T_2) \geq \alpha(T_1)$ and $\beta(T_2) \geq \beta(T_1)$. ☐

Remark. Corollary 3.2 generalizes [6, Problem 101].

In the following proposition we collect some properties of semi-Fredholm operators.

3.3. Proposition. *Let $U \in \mathcal{L}(X)$.*

(1) *If U is relatively regular and V is a generalized inverse of U, then*

$$\alpha(V) = \beta(U) \quad \text{and} \quad \beta(V) = \alpha(U).$$

Furthermore,

$$U \in \mathcal{SF}(X) \Longleftrightarrow V \in \mathcal{SF}(X)$$

and in this case

$$\operatorname{ind}(U) = -\operatorname{ind}(V).$$

(2) *If $U \in \mathcal{SF}(X)$ then*

$$U - \lambda I \in \mathcal{SF}(X) \quad \text{and} \quad \operatorname{ind}(U - \lambda I) = \operatorname{ind}(U)$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < \gamma(U)$ and there are integers α_0 and β_0 such that

$$\alpha_0 = \alpha(U - \lambda I) \leq \alpha(U) \quad \text{and} \quad \beta_0 = \beta(U - \lambda I) \leq \beta(U)$$

for $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \gamma(U)$.

(3) *If U is a relatively regular semi-Fredholm operator, then U is holomorphically regular if and only if*

$$\alpha(U - \lambda I) = \alpha(U) \quad \text{and} \quad \beta(U - \lambda I) = \beta(U)$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < \gamma(U)$.

Proof. (1) Since

$$X = (UV)(X) \oplus (I - UV)(X) = U(X) \oplus N(V)$$

and

$$X = (VU)(X) \oplus (I - VU)(X) = V(X) \oplus N(U),$$

the result follows.

(2) is shown in [5, Theorem V.1.6] and a proof of (3) is given in [15]. □

3.4. Corollary. *Suppose that T is a holormorphically regular partial isometry with an index and that S is a contractive generalized inverse of T. Then:*

(1) $T - \lambda I \in \mathcal{SF}(X)$ *and*
$$\alpha(T - \lambda I) = \alpha(S) \quad \text{and} \quad \beta(T - \lambda I) = \beta(S)$$
for each $\lambda \in \mathbb{D}$.

(2) $\sigma(T) = \sigma(S) = \overline{\mathbb{D}}$ *if* $T \notin \mathcal{L}(X)^{-1}$.

Proof. (1) Since $T \in \mathcal{SF}(X)$, $T \neq 0$. Thus $\gamma(T) = 1$, by Proposition 1.4. The assertions follow now from Proposition 3.3.

(2) If $T \notin \mathcal{L}(X)^{-1}$, then $S \notin \mathcal{L}(X)^{-1}$, hence (1) shows that $\alpha(T - \lambda I) > 0$ for all $\lambda \in \mathbb{D}$ or $\beta(T - \lambda I) > 0$ for all $\lambda \in \mathbb{D}$. Therefore $\mathbb{D} \subseteq \sigma(T)$, and so $\sigma(T) = \overline{\mathbb{D}}$. By symmetry, we also derive $\sigma(S) = \overline{\mathbb{D}}$. □

3.5. Corollary. *Let T_1 and T_2 be partial isometries such that $\|T_1 - T_2\| < 1$.*

(1) $T_1 \in \mathcal{SF}(X) \Longleftrightarrow T_2 \in \mathcal{SF}(X)$.

(2) *If* $T_1 \in \mathcal{SF}(X)$ *and* $\operatorname{ind}(T_1) \neq 0$, *then*
$$T_1 - \lambda I,\, T_2 - \lambda I \in \mathcal{SF}(X)$$
and
$$\operatorname{ind}(T_1 - \lambda I) = \operatorname{ind}(T_2 - \lambda I) \neq 0$$
for all $\lambda \in \mathbb{D}$.
Furthermore
$$\sigma(T_1) = \sigma(T_2) = \overline{\mathbb{D}}.$$

Proof. (1) follows from Corollary 3.2.

(2) Use (1), Corollary 3.2 and Proposition 3.3 (2). □

3.6. Corollary. *Suppose that T is a partial isometry with an index* $\operatorname{ind}(T) \neq 0$. *Then*
$$\|T - S\| \geq 1$$
for each contractive generalized inverse S of T.

Proof. Assume to the contrary that S is a contractive generalized inverse of T such that $\|T - S\| < 1$. Proposition 3.3 (1) shows that $S \in \mathcal{SF}(X)$ and $\operatorname{ind}(S) = -\operatorname{ind}(T)$. But $\operatorname{ind}(S) = \operatorname{ind}(T)$, by Corollary 3.5. Hence $\operatorname{ind}(T) = 0$, a contradiction. □

Remark. The condition $\operatorname{ind}(T) \neq 0$ in Corollary 3.6 can not be dropped without changing the conclusion. Indeed, if $P \in \mathcal{L}(X)$ is a projection with $\|P\| = 1$ and $\alpha(P) < \infty$, then P is a partial isometry. ¿From $X = P(X) \oplus N(P)$ we see that $\alpha(P) = \beta(P) < \infty$, thus $\operatorname{ind}(P) = 0$. But there is a contractive generalized inverse S with $\|P - S\| < 1$: take $S = P$.

3.7. Corollary. *If T is a partial isometry with an index* $\operatorname{ind}(T) \neq 0$ *on a Hilbert space, then* $\|T - T^*\| \geq 1$.

4. Orthogonality and Moore-Penrose inverses

Recall that a bounded linear operator T on a *Hilbert* space H is a partial isometry if and only if $TT^*T = T$. In this case the ranges of T and T^* are closed, hence

$$(*) \qquad N(T)^\perp = T^*(H) \quad \text{and} \quad N(T^*)^\perp = T(H).$$

Furthermore T has a unique contractive generalized inverse $S = T^*$ (see [10, Corollary 3.3]).

Now let x and y be vectors in a *Banach* space X. Following R. C. James [9], we say that x and y are *orthogonal* if

$$\|x\| \leq \|x + \alpha y\| \quad \text{for each} \quad \alpha \in \mathbb{C}.$$

In this case we write $x \perp y$. For $M, N \subseteq X$ we define the relation $M \perp N$ by $x \perp y$ for all $x \in M$ and all $y \in N$.

For our next result recall that if T is a non-zero partial isometry on the Banach space X and if S is a contractive generalized inverse of T, then $\|T\| = \|S\| = \|TS\| = \|ST\| = 1$ and

$$S(X) \oplus N(T) = X = T(X) \oplus N(S).$$

4.1. Theorem. *Let $T \in \mathcal{L}(X)$ be a non-zero partial isometry and S a contractive generalized inverse of T.*

(1) *If $N(T) \neq \{0\}$, then*

$$N(T) \perp S(X) \Longleftrightarrow \|I - ST\| = 1.$$

(2) *If $N(S) \neq \{0\}$, then*

$$N(S) \perp T(X) \Longleftrightarrow \|I - TS\| = 1.$$

Proof. (1) First suppose that $N(T) \perp S(X)$. Let $x \in X$. Then $x = u + v$ with $u \in S(X)$ and $v \in N(T)$. Hence

$$(I - ST)x = (I - ST)u + v = v.$$

Since $v \perp u$, we derive

$$\|(I - ST)x\| = \|v\| \leq \|u + v\| = \|x\|.$$

Therefore $\|I - ST\| \leq 1$. Since $I - ST$ is a non-zero projection, $\|I - ST\| \geq 1$, and so $\|I - ST\| = 1$.

Now assume that $\|I - ST\| = 1$. Take $x \in S(X)$ and $y \in N(T)$. Then, for all $\alpha \in \mathbb{C}$,

$$\|y\| = \|(I - ST)(y + \alpha x) \leq \|I - ST\| \, \|y + \alpha x\| = \|y + \alpha x\|.$$

(2) can be proved analogously. □

An operator $U \in \mathcal{L}(X)$ is called *hermitian* if $\| \exp(itU)\| = 1$ for every real numbert t.

Let $T \in \mathcal{L}(X)$ be a relatively regular operator. We will say that an operator $T^+ \in \mathcal{L}(X)$ is the *Moore-Penrose inverse* of T if T^+ is a generalized inverse of T and the projections TT^+ and T^+T are hermitian.

4.2. Proposition.

(1) *If* $U, V \in \mathcal{L}(X)$ *are hermitian and* $\alpha, \beta \in \mathbb{R}$, *then* $\alpha U + \beta V$ *is hermitian.*
(2) *If* $U \in \mathcal{L}(X)$ *is hermitian, then* $\|U\| = r(U)$.
(3) *If* $P \in \mathcal{L}(X)$ *is a hermitian projection then* $\|P\| = 0$ *or* $\|P\| = 1$.
(4) *If* $T \in \mathcal{L}(X)$ *is relatively regaular, then* T *has at most one Moore-Penrose inverse.*

Proof. (1) follows from [2, Lemma 38.2].

(2) is shown in [2, Theorem 11.17].

(3) If $P \neq 0$, then $1 \in \sigma(P) \subseteq \{0, 1\}$, thus $r(P) = 1$, hence $\|P\| = 1$, by (2).

(4) is shown in [11]. □

The following class of partial isometries is introduced in [10]:

Let $T \in \mathcal{L}(X)$ be a partial isometry. T is called an *MP-partial isometry* if T admits a contractive Moore-Penrose inverse.

Remarks.

(1) Every hermitian projection is an MP-partial isometry.
(2) If T is an MP-partial isometry, then T is a partial isometry in the sense of Definition 1.3. Moreover, these two notions are equivalent in the case of a Hilbert space, since $T^+ = T^*$, by Proposition 4.2 (4).

4.3. Corollary. *Let* $T \in \mathcal{L}(X)$ *be a non-zero MP-partial isometry. Then*

$$N(T) \perp T^+(X) \quad \textit{and} \quad N(T^+) \perp T(X)\,.$$

Proof. If $N(T) = \{0\}$ or $N(T^+) = \{0\}$, then there is nothing to prove. So we assume that $N(T) \neq \{0\}$ and $N(T^+) \neq \{0\}$, hence $T(X) \neq X \neq T^+(X)$. Let $P = I - TT^+$ and $Q = I - T^+T$. Thus P and Q are non-zero projections. Since TT^+ and T^+T are hermitian, P and Q are hermitian, by Proposition 4.2 (1). Because of Proposition 4.2 (3) we derive that $\|P\| = \|Q\| = 1$. The result follows now from Theorem 4.1. □

References

[1] T. Ando, *Contractive projections in* L_p *spaces*, Pac. J. Math. 17 (1966), 391-405.
[2] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer (1973).
[3] S. L. Campbell, G. D. Faulkner, *Isometries, projections and Wold decompositions*, Res. Notes in Math., 38, Pitman, Boston (1979), 85-114.
[4] S. R. Caradus, *Generalized inverse and operator theory*, Queen's papers in pure and appl. math. 50, Queen's Univ. Kingston, Ont., 1978.
[5] S. Goldberg, *Unbounded linear operators*, McGraw–Hill (1966).
[6] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand (1967).
[7] R. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker, New York - Basel, 1988.
[8] H. Heuser, *Funktionalanalysis*, 3. ed. Teubner, Stuttgart, 1991.

[9] R. C. James, *Orthogonality and linear functionals in normed linear space*, Trans. Amer. Math. Soc. 61 (1947), 265-292.
[10] M. Mbekhta, *Partial isometries and generalized inverses*, Acta Sci. Math. (Szeged) 70 (2004), 767-781.
[11] V. Rakocevic, *Moore-Penrose inverse in Banach algebras*, Proc. Roy. Irish Acad., Sect. A, 88 (1988), 57-60.
[12] Ch. Schmoeger, *Spectral properties of generalized inverses*, Demonstratio Math. 37 (2004), 137-148.
[13] Ch. Schmoeger, *The punctured neighbourhood theorem in Banach algebras*, Proc. Roy. Irish Acad., Sect. A, 91 (1991), 205-218.
[14] Ch. Schmoeger, *On decomposably regular operators*, Port. Math. 54 (1997), 41-50.
[15] T. T. West, *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. Roy. Irish Acad., Sect. A, 87 (1987), 137-146.

Christoph Schmoeger, Mathematisches Institut I, Universität Karlsruhe (TH), Englerstrasse 2, 76128 Karlsruhe, Germany
E-mail address: `christoph.schmoeger@math.uni-karlsruhe.de`