Perturbation properties of some classes of operators

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RIASSUNTO: Sia $X$ uno spazio di Banach complesso e $\mathcal{L}(X)$ l’algebra di di Banach di tutti gli operatori lineari limitati in $X$. Considerate le seguenti famiglie di operatori:

$$D(X) = \{ T \in \mathcal{L}(X) : T(X) \text{ is closed and } N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X) \},$$

$$S(X) = \{ T \in D(X) : T \text{ is relatively regular} \}.$$ 

si determinano i punti interni di $D(X)$ e $S(X)$, si dimostrano inoltre alcuni teoremi di perturbazione.

ABSTRACT: Let $X$ be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on $X$. We consider the following classes of operators:

$$D(X) = \{ T \in \mathcal{L}(X) : T(X) \text{ is closed and } N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X) \},$$

$$S(X) = \{ T \in D(X) : T \text{ is relatively regular} \}.$$ 

We determine the interior points of $D(X)$ and $S(X)$ and prove some perturbation theorems.

1 – Introduction and terminology

Throughout this paper $X$ denotes a Banach space over the complex
field \( \mathbb{C} \) and \( \mathcal{L}(X) \) the Banach algebra of all bounded linear operators on \( X \). If \( T \in \mathcal{L}(X) \) we denote by \( N(T) \) the kernel of \( T \) and by \( \alpha(T) \) the dimension of \( N(T) \). The range of \( T \) is denoted by \( T(X) \) and we define \( \beta(T) = \text{codim } T(X) \).

\( T \in \mathcal{L}(X) \) is called relatively regular if \( TST = T \) for some \( S \in \mathcal{L}(X) \). \( \mathcal{R}(X) \) will denote the set of all relatively regular operators.

We shall make use of the following results [1, p. 10]:

1. \( T \in \mathcal{R}(X) \) if and only if \( N(T) \) and \( T(X) \) are closed complemented subspaces of \( X \).
2. If \( TST = T \) for some \( S \in \mathcal{L}(X) \), then \( TS \) is a projection onto \( T(X) \) and \( I - ST \) is a projection onto \( N(T) \).

An operator \( T \) is called an Atkinson operator if \( T \in \mathcal{R}(X) \) and at least one of \( \alpha(T), \beta(T) \) is finite. The set of Atkinson operators will be denoted by \( \mathcal{A}(X) \).

We write \( \mathcal{C}(X) \) for the set of operators having closed range. The class of semi-Fredholm operators is defined by

\[
\mathcal{SF}(X) = \{ T \in \mathcal{C}(X) : \alpha(T) < \infty \text{ or } \beta(T) < \infty \}
\]

We have \( \mathcal{A}(X) \subseteq \mathcal{SF}(X) \). The index of \( T \in \mathcal{SF}(X) \) is given by \( \text{ind}(T) = \alpha(T) - \beta(T) \).

The following result is well known (for proofs see [1] and [3]).

**Theorem 1.** Let \( T \in \mathcal{A}(X) \) (resp. \( T \in \mathcal{SF}(X) \)). Then there exists \( \delta > 0 \) such that

(a) \( T - B \in \mathcal{A}(X) \) (resp. \( T - B \in \mathcal{SF}(X) \)), \( \alpha(T - B) \leq \alpha(T), \beta(T - B) \leq \beta(T) \) and \( \text{ind}(T - B) = \text{ind}(T) \) for all \( B \in \mathcal{L}(X) \) with \( \|B\| < \delta \);

(b) \( \alpha(T - \lambda I) \) is a constant \( \leq \alpha(T), \beta(T - \lambda I) \) is a constant \( \leq \beta(T) \) for \( 0 < |\lambda| < \delta \).

The above theorem shows that \( \mathcal{A}(X) \) and \( \mathcal{SF}(X) \) are open subsets of \( \mathcal{L}(X) \). Furthermore, the continuity of the index shows that the *jump* of \( T \in \mathcal{SF}(X) \)

\[
j(T) = \begin{cases} 
\alpha(T) - \alpha(T - \lambda I) & \text{if } \alpha(T) < \infty \\
\beta(T) - \beta(T - \lambda I) & \text{if } \beta(T) < \infty 
\end{cases}
\]

is unambiguously defined.
Proposition 1. If \( T \in SF(X) \) then \( j(T) = 0 \iff N(T) \subseteq \bigcap_{n \geq 1} T^n(X) \).

Proof. [14, Proposition 2.2].

We now list various classes of bounded linear operators which will be discussed:

\[ SF_0(X) = \{ T \in SF(X) : \alpha(T) = 0 \text{ or } \beta(T) = 0 \} ; \]
\[ B(X) = \{ T \in L(X) : N(T) \subseteq T(X) \} ; \]
\[ M(X) = \{ T \in L(X) : T \text{ is left or right invertible in } L(X) \} ; \]
\[ D(X) = \{ T \in C(X) : N(T) \subseteq \bigcap_{n \geq 1} T^n(X) \} ; \]
\[ S(X) = \{ T \in R(X) : N(T) \subseteq \bigcap_{n \geq 1} T^n(X) \} . \]

It is well known that \( M(X) \) is open. \( SF_0(X) \) is open by Theorem 1. An operator in \( S(X) \) is called an operator of Saphar type. Such operators have an important property:

\[ T \in S(X) \text{ if and only if there is a neighbourhood } U \subset U \text{ of } 0 \text{ and a holomorphic function } F : U \to L(X) \text{ such that} \]
\[ (T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \text{ for all } \lambda \in U. \]

For a proof see [7, Théorème 2.6] or [12, Theorem 1.4].

2 – Interior points of \( D(X) \) and \( S(X) \)

If \( \mathcal{H} \) is a subset of \( L(X) \) we write \( \text{int}(\mathcal{H}) \) for the set of interior points of \( \mathcal{H} \).

Proposition 2. If \( T \in \text{int}(B(X)) \) then \( N(T) = \{0\} \text{ or } T(X) = X. \)
PROOF. There exists $\delta > 0$ such that

$$S \in \mathcal{B}(X) \text{ whenever } ||T - S|| < \delta.$$  

Suppose that $N(T) \neq \{0\}$ and $T(X) \neq X$. Then there are $x_0, y_0 \in X$ with $x_0 \neq 0$, $Tx_0 = 0$, $y_0 \notin T(X)$ and $||Ty_0|| = \delta/2$. Since $y_0 \notin T(X)$ and $N(T) \subseteq T(X)$, we have $y_0 \notin N(T)$. An application of the Hahn-Banach extension theorem shows the existence of a continuous linear functional $f$ such that

$$\alpha = f(x_0) \neq 0, \quad f(y_0) = 0 \quad \text{and} \quad ||f|| = 1$$

(see [6, Satz 36.3]). Define $S \in \mathcal{L}(X)$ by

$$Sx = Tx + f(x)Ty_0 \quad (x \in X).$$

It follows that $||Tx - Sx|| = ||f(x)||||Ty_0|| \leq ||x||\delta/2$, thus $||T - S|| < \delta$, hence $S \in \mathcal{B}(X)$. Since $S(X) \subseteq T(X)$, we conclude that

$$N(S) \subseteq T(X).$$

Now put $z = y_0 - x_0/\alpha$. It results that

$$Sz = Ty_0 + f \left(y_0 - \frac{1}{\alpha}x_0\right)Ty_0 = Ty_0 - Ty_0 = 0.$$  

This gives $z \in T(X)$, hence $y_0 = z + x_0/\alpha \in T(X) + N(T) = T(X)$ which contradicts $y_0 \notin T(X)$.

It is shown in [10] that neither $D(X)$ nor $S(X)$ are open subsets of $\mathcal{L}(X)$. But the following perturbation results are valid:

\begin{itemize}
  \item Suppose $T \in S(X)$ (resp. $T \in D(X)$), $B \in \mathcal{L}(X)$ and
  \item $B \left(\bigcap_{n=1}^{\infty} T^n(X)\right) \subseteq \bigcap_{n=1}^{\infty} T^n(X)$. If $||B||$ is sufficiently small
  \item then $T - B \in S(X)$ (resp. $T - B \in D(X)$).
\end{itemize}

(For proofs see [1, p. 150] (resp. [10, Corollaire 3.6]).)

Therefore a natural question arises: What are the interior points of $S(X)$ and $D(X)$? The following result gives an answer.
Theorem 2.

(a) \(\text{int}(\mathcal{D}(X)) = \text{int}(\mathcal{B}(X) \cap \mathcal{C}(X)) = \mathcal{SF}_0(X)\).

(b) \(\text{int}(\mathcal{S}(X)) = \text{int}(\mathcal{B}(X) \cap \mathcal{R}(X)) = \mathcal{M}(X)\).

Proof. (a) By Theorem 1 and Proposition 1, \(\mathcal{SF}_0(X) \subseteq \mathcal{D}(X)\). Since \(\mathcal{SF}_0(X)\) is open and \(\mathcal{SF}_0(X) \subseteq \mathcal{D}(X) \subseteq \mathcal{B}(X) \cap \mathcal{C}(X)\), we have

\[
\mathcal{SF}_0(X) \subseteq \text{int}(\mathcal{D}(X)) \subseteq \text{int}(\mathcal{B}(X) \cap \mathcal{C}(X)).
\]

If \(T \in \text{int}(\mathcal{B}(X) \cap \mathcal{C}(X))\) then \(T \in \text{int}(\mathcal{B}(X))\), thus \(a(T) = 0\) or \(b(T) = 0\), by Proposition 2. Since \(T(X)\) is closed, we derive \(T \in \mathcal{SF}_0(X)\).

(b) Since \(\mathcal{M}(X)\) is open and \(\mathcal{M}(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{B}(X) \cap \mathcal{R}(X)\), we have

\[
\mathcal{M}(X) \subseteq \text{int}(\mathcal{S}(X)) \subseteq \text{int}(\mathcal{B}(X) \cap \mathcal{R}(X)).
\]

Let \(T \in \text{int}(\mathcal{B}(X) \cap \mathcal{R}(X))\). There is \(S \in \mathcal{L}(X)\) with \(TST = T\). Proposition 2 shows that \((I - ST)(X) = N(T) = \{0\}\) or \(TS(X) = T(X) = X\), thus \(ST = I\) or \(TS = I\), therefore \(T \in \mathcal{M}(X)\).

Remark. If \(X\) is a Hilbert space, then \(\mathcal{C}(X) = \mathcal{R}(X)\) [1, p. 12], hence \(\mathcal{D}(X) = \mathcal{S}(X)\). In this special case it was shown in [8, Théorème 6.5] that \(\text{int}(\mathcal{D}(X)) = \mathcal{M}(X)\).

Corollary 1. If \(X\) is a Hilbert space then \(\text{int}(\mathcal{S}(X))\) is dense in \(\mathcal{L}(X)\).

Proof. \(\mathcal{M}(X)\) is dense in \(\mathcal{L}(X)\) [4, Problem 140]. Now use Theorem 2.

Corollary 2.

(a) \(\text{int}(\{T \in \mathcal{SF}(X) : j(T) = 0\}) = \mathcal{SF}_0(X)\).

(b) \(\text{int}(\{T \in \mathcal{A}(X) : j(T) = 0\}) = \mathcal{M}(X)\).

Proof. (a) follows from \(\mathcal{SF}_0(X) \subseteq \{T \in \mathcal{SF}(X) : j(T) = 0\} \subseteq \mathcal{D}(X)\) (Proposition 1) and from Theorem 2.

(b) follows from \(\mathcal{M}(X) \subseteq \{T \in \mathcal{A}(X) : j(T) = 0\} \subseteq \mathcal{S}(X)\) and from Theorem 2.
3 – The reduced minimum modulus of operators in $\mathcal{D}(X)$

By definition, the reduced minimum modulus $\gamma(T)$ of $T \in \mathcal{L}(X) \setminus \{0\}$ is given by

$$\gamma(T) = \inf \left\{ \frac{\|Tx\|}{d(x, N(T))} : x \in X, Tx \neq 0 \right\}.$$ 

($d(x, N(T))$ denotes the distance of $x$ to $N(T)$.) Observe that $\gamma(T) > 0$ if and only if $T \in \mathcal{C}(X)$ [3, Theorem IV. 1.6].

**Proposition 3.** Let $T \in \mathcal{L}(X)$.

(a) If $T \in \mathcal{D}(X)$ then $T^n \in \mathcal{D}(X)$ for all $n \in \mathbb{N}$.

(b) If $T \in \mathcal{D}(X)$ then $\gamma(T^{n+m}) \geq \gamma(T^n)\gamma(T^m)$ for all $n, m \in \mathbb{N}$.

(c) If $T \in \mathcal{R}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$ then $\|S\|^{-1} \leq \gamma(T)$.

(d) If $T \in \mathcal{S}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$ then $T^nS^nT^n = T^n$

for each $n \in \mathbb{N}$.

**Proof.** (a) [11, Satz 6]. (b) [2, Lemma 1]. (c) [2, Lemma 4]. (d) [13, Proposition 2].

We denote by $\sigma(T)$ the spectrum of $T \in \mathcal{L}(X)$ and by $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ ($= \lim_{n \to \infty} \|T^n\|^{1/n}$) the spectral radius of $T$. $\partial \sigma(T)$ denotes the boundary of $\sigma(T)$.

**Proposition 4.** Let $T \in \mathcal{L}(X)$.

(a) If $\mu \in \partial \sigma(T)$ then $T - \mu I \notin \mathcal{D}(X)$.

(b) If $T \in \mathcal{D}(X)$ then

$$\sup_{n \geq 1} \gamma(T^n)^{1/n} \leq \min \{|\mu| : \mu \in \partial \sigma(T)\},$$

the sequence $(\gamma(T^n)^{1/n})_{n \geq 1}$ converges and

$$\lim_{n \to \infty} \gamma(T^n)^{1/n} = \sup_{n \geq 1} \gamma(T^n)^{1/n}.$$

(c) If $T \in \mathcal{S}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$ then

$$r(S)^{-1} \leq \lim_{n \to \infty} \gamma(T^n)^{1/n}.$$
PROOF. (a) follows from [11, Satz 2].

(b) Fix \( \mu \in \partial \sigma(T) \) such that \( |\mu| = \min \{|\lambda| : \lambda \in \partial \sigma(T)\} \) and suppose that \( |\mu| < \gamma(T^m)^{1/m} \) for some \( m \in \mathbb{N} \). Thus \( |\mu^m| < \gamma(T^m) \). Since \( T^m \in \mathcal{D}(X) \) (Proposition 3(a)), Théorème 2.10 in [9] gives \( T^m - \mu^m I \in \mathcal{D}(X) \).

[11, Satz 6] implies now that \( T - \mu I \in \mathcal{D}(X) \), but this contradicts (a). Hence \( \gamma(T^m)^{1/m} \leq |\mu| \) for each \( m \in \mathbb{N} \).

(b) follows from [2, remarks in connection with Lemma 1].

(c) By Proposition 3(d), \( T^n S^n T^n = T^n \) for all \( n \in \mathbb{N} \). Part (c) of Proposition 3 implies that \( ||S^n||^{-1} \leq \gamma(T^n) \) for each \( n \in \mathbb{N} \), hence

\[
 r(S)^{-1} = \lim_{n \to \infty} \frac{1}{||S^n||^{1/n}} \leq \lim_{n \to \infty} \gamma(T^n)^{1/n}.
\]

The following theorem is another perturbation result for operators in \( \mathcal{D}(X) \) which generalizes Théorème 2.10 in [9].

**Theorem 3.** If \( T \in \mathcal{D}(X) \), \( B \in \mathcal{L}(X) \), \( TB = BT \) and \( r(B) < \lim_{n \to \infty} \gamma(T^n)^{1/n} \), then \( T - B \in \mathcal{D}(X) \).

**Proof.** Since \( r(B) = \inf_{k \geq 1} ||B^k||^{1/k} < \sup_{n \geq 1} \gamma(T^n)^{1/n} \), there exists \( k \in \mathbb{N} \) such that \( ||B^{k+1}|| < \gamma(T^{k+1}) \). By Proposition 3(a), \( T^{k+1} \in \mathcal{D}(X) \), thus \( T^{k+1} - B^{k+1} \in \mathcal{D}(X) \), by [10, Corollaire 3.6], since \( B \left( \bigcap_{n=1}^{\infty} T^n(X) \right) \subseteq \bigcap_{n=1}^{\infty} T^n(X) \). \( TB = BT \) implies

\[
 T^{k+1} - B^{k+1} = (T - B)(T^k + T^{k-1}B + \cdots + TB^{k-1} + B^k).
\]

Therefore [11, Satz 5] shows that \( T - B \in \mathcal{D}(X) \). □

The next result is proved in [10, Théorème 3.7]. It is now an immediate consequence of the last theorem.

**Theorem 4.** Let \( T, Q \in \mathcal{L}(X) \). If \( Q \) is quasi-nilpotent and commutes with \( T \), then

\[
 T \in \mathcal{D}(X) \text{ if and only if } T - Q \in \mathcal{D}(X).
\]
We close this paper with a perturbation result concerning operators in $S(X)$. For the proof we need the following proposition.

**Proposition 5.** If $A, B \in \mathcal{L}(X)$ commute and $AB \in S(X)$, then $A, B \in S(X)$.

**Proof.** [5, Theorem 10].

**Theorem 5.** Let $T, Q \in \mathcal{L}(X)$. If $Q$ is quasi-nilpotent and commutes with $T$, then

$$T \in S(X) \text{ if and only if } T - Q \in S(X).$$

**Proof.** It suffices to prove the implication $T \in S(X) \implies T - Q \in S(X)$. Put $S \in \mathcal{L}(X)$ such that $TST = T$. By Proposition 3(a),(d), $T^n \in S(X)$ and $T^n S^n T^n = T^n$ for each $n \in \mathbb{N}$. Put $S_n := S^n T^n S^n$ ($n \in \mathbb{N}$). It follows that $T^n S_n T^n = T^n$, $S_n T^n S_n = S_n$ and $||S_n||^{1/n} \leq ||S||^2 ||T||$. There exists $k \in \mathbb{N}$ such that $||Q^{k+1}||^{1/(k+1)} < (||S||^2 ||T||)^{-1}$, thus $||Q^{k+1}|| < ||S_{k+1}||^{-1}$. By [1, Theorem 9 in Section 5.2], $T^{k+1} - Q^{k+1} \in S(X)$. $TQ = QT$ implies

$$T^{k+1} - Q^{k+1} = (T - Q)(T^k + T^{k-1}Q + \cdots + TQ^{k-1} + Q^k),$$

hence $T - Q \in S(X)$, by Proposition 5.

**REFERENCES**


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