GENERALIZED PROJECTIONS IN BANACH ALGEBRAS

CHRISTOPH SCHMOEGER

Abstract. In this note we investigate generalized projections in Banach algebras. Our results generalize results obtained for bounded linear operators on Hilbert spaces.

1. Introduction and terminology

A bounded linear operator on a Hilbert space is said to be a *generalized projection* if

$$A^2 = A^* .$$

For properties and characterizations of generalized projection operators see [3], [4], and [5] (and the references there). The aim of this note are generalizations of some of the results obtained in [3] and [4] to elements of complex Banach algebras.

Throughout this paper $\mathscr{A}$ will denote a complex unital Banach algebra with unit $\mathbf{1}$.

If $a \in \mathscr{A}$, then we denote the spectrum and the spectral radius of a by $\sigma(a)$ and $r(a)$, respectively.

An element $h \in \mathscr{A}$ is said to be *hermitian* if $\| \exp{(ith)}\| = 1$ for all $t \in \mathbb{R}$. $\mathscr{H}(\mathscr{A})$ denotes the set of hermitian elements of $\mathscr{A}$. It is well-known that a bounded linear operator H on a complex Hilbert space is hermitian if and only if $H = H^*$.

We collect the basic properties of $\mathscr{H}(\mathscr{A})$ (for proofs see [2]):

1.1. Proposition.
(1) *$\mathscr{H}(\mathscr{A})$ is closed real subspace of $\mathscr{A}$ and $\mathscr{H}(\mathscr{A}) \cap i\mathscr{H}(\mathscr{A}) = \{0\}$.*
(2) *If $h, k \in \mathscr{H}(\mathscr{A})$, then*

$$i(hk - kh) \in \mathscr{H}(\mathscr{A}),\ \sigma(h) \subseteq \mathbb{R} \ \text{ and } \ r(h) = \|h\| .$$

The following example (due to M. J. Crabb, see [2, page 57]) shows that, in contrast to the operator situation, if $h \in \mathscr{H}(\mathscr{A})$, then it does not follow that $h^2 \in \mathscr{H}(\mathscr{A})$.

1.2. Example. *Let $\mathscr{A} = \mathbb{C}^3$ with pointwise multiplication and let $p : \mathscr{A} \to [0, \infty)$ be defined by*

$$p(\alpha, \beta, \gamma) = \sup\{|\lambda^{-1}\alpha + \beta + \lambda\gamma| : \lambda \in \mathbb{C},\ |\lambda| = 1\} .$$

Define the norm $\| \cdot \|$ on $\mathscr{A}$ by

$$\|a\| = \sup\{p(xa) : x \in \mathscr{A},\ p(x) = 1\} .$$

Then $(\mathscr{A}, \| \cdot \|)$ is a complex (commutative) Banach algebra with unit $\mathbf{1}$= $(1, 1, 1)$. Let $h_0 = (-1, 0, 1)$, then

$$h_0 \in \mathscr{H}(\mathscr{A}),\ h_0^2 \notin \mathscr{H}(\mathscr{A}) .$$

Date: 5th November 2007.
1991 *Mathematics Subject Classification.* 46H99.
Key words and phrases. generalized projection, normal element.

Let $\mathscr{J}(\mathscr{A}) = \{h + ik : h,\, k \in \mathscr{H}(\mathscr{A})\}$. Since $\mathscr{H}(\mathscr{A}) \cap i\mathscr{H}(\mathscr{A}) = \{0\}$, each element of $\mathscr{J}(\mathscr{A})$ has a *unique* representation of the form $h + ik$ with $h,\, k \in \mathscr{H}(\mathscr{A})$. Therefore we may define a linear involution * on $\mathscr{J}(\mathscr{A})$ by

$$(h + ik)^* = h - ik\,.$$

We say that $a \in \mathscr{J}(\mathscr{A})$ is *normal* if $aa^* = a^*a$. If $a = h + ik \in \mathscr{J}(\mathscr{A})\,(h,\, k \in \mathscr{H}(\mathscr{A}))$, then it is easy to see that a is normal if and only if $hk = kh$.

If $\mathscr{S}$ is a commutative subset of $\mathscr{A}$, then the *centralizer* of $\mathscr{S}$ is given by

$$\Gamma(\mathscr{S}) = \{x \in \mathscr{A} : xs = sx \quad \text{for every } s \in \mathscr{S}\}\,.$$

We have (see [7, 11.21])

$$\mathscr{S} \subseteq \Gamma(\Gamma(\mathscr{S}))$$

and $\Gamma(\Gamma(\mathscr{S}))$ *is a commutative Banach algebra* (with unit $\mathbf{1}$).

Notation. For a normal element $a \in \mathscr{J}(\mathscr{A})$, $a = h + ik$ with $h,\, k \in \mathscr{H}(\mathscr{A})$, let

$$\mathscr{B}(a) = \Gamma(\Gamma(\{h, k\}))$$

and let Δ_a denote the set of all nontrivial complex homomorphisms of $\mathscr{B}(a)$.

1.3. Proposition. *Suppose that* $a = h + ik \in \mathscr{J}(\mathscr{A})\,(h, k \in \mathscr{H}(\mathscr{A}))$ *is normal. Then*
(1) $\sigma(x) = \{\varphi(x) : \varphi \in \Delta_a\}$ *for all* $x \in \mathscr{B}(a)$*;*
(2) $\varphi(h),\, \varphi(k) \in \mathbb{R}$ *for all* $\varphi \in \Delta_a$*;*
(3) $\varphi(a^*) = \overline{\varphi(a)}$ *for all* $\varphi \in \Delta_a$*, and* $\sigma(a^*) = \{\overline{\lambda} : \lambda \in \sigma(a)\}$*;*
(4) *if* $x,\, y \in \mathscr{B}(a)$*, then*

$$\sigma(x + y) \subseteq \sigma(x) + \sigma(y),\; \sigma(xy) \subseteq \sigma(x)\sigma(y)\,.$$

Proof. (1) [7, Theorem 11.9, Theorem 11.22].
(2) Follows from (1) and Proposition 1.1 (2).
(3) $\varphi(a^*) = \varphi(h - ik) = \varphi(h) - i\varphi(k) = \overline{\varphi(h) + i\varphi(k)} = \overline{\varphi(a)}$.
(4) [7, Theorem 11.23]. □

2. Generalized projections in Banach algebras

An element $a \in \mathscr{J}(\mathscr{A})$ is called a *generalized projection* if $a^2 = a^*$. We say that $a \in \mathscr{J}(\mathscr{A})$ is a *partial isometry* if $a = aa^*a$.

2.1. Theorem. *Suppose that* $a = h + ik \in \mathscr{J}(\mathscr{A})$ *is a generalized projection. Then we have:*
(1) *a is normal;*
(2) $\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}$*;*
(3) *if, in addition,* $hk \in \mathscr{H}(\mathscr{A})$*, then*

$$(a^*)^2 = a \quad \text{and} \quad a^4 = a\,.$$

Remark. If $\mathscr{A}$ is the Banach algebra of all bounded linear operators on a complex Hilbert space, then the condition $hk \in \mathscr{H}(\mathscr{A})$ in part (3) of the above theorem can be dropped, since a is normal.

Proof of Theorem 2.1. (1) Since $a^2 = a^*$,

$$aa^* = aa^2 = a^2 a = a^* a\,.$$

(2) Let $\lambda \in \sigma(a)$. Then $\lambda = \varphi(a)$ for some $\varphi \in \Delta_a$, hence, by Proposition 1.3,

$$\overline{\lambda} = \varphi(a^*) = \varphi(a^2) = \varphi(a)^2 = \lambda^2\,.$$

If $\lambda \neq 0$, then $|\lambda| = 1$ and $\lambda^3 = \lambda\lambda^2 = \lambda\overline{\lambda} = 1$.
(3) From $a^2 = a^*$ we see that

$$h^2 - h - k^2 = i(-k - 2hk)\,.$$

Proposition 1.3 (4) and Proposition 1.1 (2) show that

$$\sigma(h^2 - h - k^2),\ \sigma(-k - 2hk) \subseteq \mathbb{R}\,,$$

hence

$$\sigma(-k - 2hk) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\}\,,$$

thus $r(-k - 2hk) = 0$. Since $hk \in \mathscr{H}(\mathscr{A})$, $-k - 2hk \in \mathscr{H}(\mathscr{A})$. Now use Proposition 1.1 (2) to get $k = -2hk$, hence $h^2 - k^2 = h$. Therefore

$$(a^*)^2 = h^2 - 2ihk - k^2 = h - 2ihk = h - (-ik) = h + ik = a$$

and

$$a^4 = (a^*)^2 = a\,.$$

□

2.2. Theorem. *Let $a = h + ik \in \mathscr{J}(\mathscr{A})$ and suppose that*

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}\,.$$

We have:
(1) *if a is normal, then $r(a^2 - a^*) = 0$;*
(2) *if $hk, h^2, k^2 \in \mathscr{H}(\mathscr{A})$, then a is a generalized projection.*

Before we give a proof of the above theorem we have the following corollary, which generalizes Theorem 2 in [3] and Theorem 1 in [4] (see Remark (2) below).

2.3. Corollary. *For $a = h + ik \in \mathscr{J}(\mathscr{A})$ with $h, k, hk, h^2, k^2 \in \mathscr{H}(\mathscr{A})$ the following assertions are equivalent:*
(1) *a is a normal partial isometry and $a^4 = a$;*
(2) *a is normal and $a^4 = a$;*
(3) *$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}$;*
(4) *$a^2 = a^*$.*

Proof. (1) $\Rightarrow$ (2): Clear.
(2) $\Rightarrow$ (3): Use the spectral mapping theorem ([7, Theorem 10.28]).
(3) $\Rightarrow$ (4): Theorem 2.2 (2).
(4) $\Rightarrow$ (1): By Theorem 2.1 (1), a is normal, and part (3) of Theorem 2.1 gives $aa^*a = aa^2a = a^4 = a$. □

Remarks. (1) The implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in Corollary 2.3 are valid without the assumptions that $hk, h^2, k^2 \in \mathscr{H}(\mathscr{A})$.
(2) If $\mathscr{A}$ is the Banach algebra of all bounded linear operators on a complex Hilbert space, then the condition $hk \in \mathscr{H}(\mathscr{A})$ is equivalent to the normality of a, thus the assumptions $h^2, k^2 \in \mathscr{H}(\mathscr{A})$ in Theorem 2.2 (2) and Corollary 2.3 can be dropped in the operator situation.

Proof of Theorem 2.2. Let $b = a^2 - a^*$.
(1) Since a is normal, $hk = kh$, thus $b \in \mathscr{B}(a)$. Take $\lambda \in \sigma(b)$. Then $\lambda = \varphi(b) = \varphi(a)^2 - \overline{\varphi(a)}$ for some $\varphi \in \Delta_a$.
Case 1: $\varphi(a) = 0$. Then we have $\lambda = 0$.
Case 2: $\varphi(a) \neq 0$. Since $\varphi(a) \in \sigma(a)$, $\varphi(a)^3 = 1$.
It follows that

$$\lambda\varphi(a) = \varphi(a)^3 - \overline{\varphi(a)}\varphi(a) = 1 - 1 = 0\,,$$

and so $\lambda = 0$. Hence $\sigma(b) = \{0\}$.
(2) Since h, k, hk, h^2 and k^2 are all hermitian, it follows from [1, Theorem 2.14] (see also [6]) that $hk = kh$. Thus a is normal. From

$$b = h^2 + 2ihk - k^2 - (h - ik)$$

we get

$$i(2hk + k) = b - (h^2 - h - k^2)\,.$$

By Proposition 1.1 and Proposition 1.3,

$$\sigma(2hk + k) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\}\,.$$

Since $2hk + k \in \mathscr{H}(\mathscr{A})$, $2hk + k = 0$, thus

$$b = h^2 - h - k^2 \in \mathscr{H}(\mathscr{A})\,.$$

Since, by (1), $r(b) = 0$, we conclude that $b = 0$, hence $a^2 = a^*$. □

2.4. Example. *Let $\mathscr{A}$ and h_0 as in Example 1.2. Observe that*

$$\sigma((\alpha, \beta, \gamma)) = \{\alpha, \beta, \gamma\} \qquad ((\alpha, \beta, \gamma) \in \mathscr{A})\,.$$

Hence

$$\sigma(a) = \{0\} \Leftrightarrow a = 0 \qquad (a \in \mathscr{A})\,. \tag{2.1}$$

The following properties are shown in [2]:

$$\mathscr{A} = \{\alpha\mathbf{1} + \beta h_0 + \gamma h_0^2 : \alpha, \beta, \gamma \in \mathbb{C}\}\,,$$

$$\mathscr{H}(\mathscr{A}) = \{\alpha\mathbf{1} + \beta h_0 : \alpha, \beta \in \mathbb{R}\}\,. \tag{2.2}$$

Hence we have

$$\mathscr{J}(\mathscr{A}) = \{\xi\mathbf{1} + \eta h_0 : \xi, \eta \in \mathbb{C}\}\,,$$

and each element of $\mathscr{J}(\mathscr{A})$ is normal. Furthermore, if

$$a = \xi\mathbf{1} + \eta h_0 \in \mathscr{J}(\mathscr{A}),\ \textit{then}\ a^* = \overline{\xi}\mathbf{1} + \overline{\eta}h_0\,.$$

For $a \in \mathscr{J}(\mathscr{A})$ the following assertions are equivalent:

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = 1\}\,; \tag{2.3}$$

$$a^2 = a^*\,; \tag{2.4}$$

$$a \in \{(\lambda, \lambda, \lambda) : \lambda = 0 \ \textit{or}\ \lambda^3 = 1\}\,; \tag{2.5}$$

$$a^4 = a\,. \tag{2.6}$$

Proof. (2.3) $\Rightarrow$ (2.4): Since a is normal, we have $\sigma(a^2 - a^*) = \{0\}$ (Theorem 2.2), thus, by (2.1), $a^2 = a^*$.
(2.4) $\Rightarrow$ (2.5): Let $a = \xi\mathbf{1} + \eta h_0$ ($\xi, \eta \in \mathbb{C}$). Since $a^2 = a^*$,

$$((\xi-\eta)^2, \xi^2, (\xi+\eta)^2) = (\overline{\xi} - \overline{\eta}, \overline{\xi}, \overline{\xi} + \overline{\eta}) .$$

Case 1: $\xi \neq 0$. Then $\eta^2 = -\overline{\eta}$ and $\eta^2 = \overline{\eta}$, thus $\xi = \eta = 0$, hence $a = (0, 0, 0)$.
Case 2: $\xi \neq 0$. Since $\xi^2 = \overline{\xi}$,

$$\overline{\xi} - \overline{\eta} = (\xi - \eta)^2 = \xi^2 - 2\xi\eta + \eta^2 = \overline{\xi} - 2\xi\eta + \eta^2$$

and

$$\overline{\xi} + \overline{\eta} = [\xi + \eta)^2 = \xi^2 + 2\xi\eta + \eta^2 = \overline{\xi} + 2\xi\eta + \eta^2 ,$$

hence

$$\eta^2 = 2\xi\eta - \overline{\eta} = -\eta^2 ,$$

therefore $\eta = 0$. From $\xi^2 = \overline{\xi}$ it follows that $|\xi| = 1$ and $\xi^3 = \xi\overline{\xi} = 1$.
(2.5) $\Rightarrow$ (2.6): Clear.
(2.6) $\Rightarrow$ (2.3): Use the spectral mapping theorem ([7, Theorem 10.28]). □

Our next results generalize results obtained in [4] for complex $n \times n$ matrices.

2.5. Theorem. *Let $a, b \in \mathscr{J}(\mathscr{A})$ be generalized projections. Then $a + b$ is a generalized projection if and only if $ab = ba = 0$.*

Proof. We have

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^* + ab + ba + b^* .$$

If $ab = ba = 0$, then $(a + b)^2 = a^* + b^* = (a + b)^*$, hence $a + b$ is a generalized projection.
Conversely, assume that $(a + b)^2 = (a + b)^*$. Then $ab + ba = 0$. Applying a on the left and on the right gives

$$a^*b + aba = 0 = aba + ba^* ,$$

hence $a^*b = ba^*$. Since a is normal, a^* is normal. Proposition 2.1 in [8] implies now that $ab = ba$, thus $ab = ba = 0$. □

2.6. Theorem. *Let $a, b \in \mathscr{J}(\mathscr{A})$ be generalized projections, $a = h + ik$ ($h, k \in \mathscr{H}(\mathscr{A})$). If in addition $hk \in \mathscr{H}(\mathscr{A})$, then*

$$(b - a)^2 = (b - a)^* \Leftrightarrow ab = ba = a^* .$$

Proof. By Theorem 2.1 (3), $(a^*)^2 = a$ and $a^4 = a$. First assume that $ab = ba = a^*$. Then

$$(b - a)^2 = b^2 - ba - ab + a^2 = b^* - a^* = (b - a)^* ,$$

hence $b - a$ is a generalized projection.
Now assume that $(b - a)^2 = (b - a)^*$. It follows that

$$2a^* = ab + ba . \tag{2.7}$$

Multiplying (2.7) from the left by a^* gives

$$2a = 2(a^*)^2 = a^*ab + a^*ba .$$

Multiplying (2.7) from the right by a^* gives

$$2a = 2(a^*)^2 = aba^* + baa^* .$$

Hence, since $aa^* = a^3 = a^*a$,

$$a^3b + a^*ba = aba^2 + ba^3 . \tag{2.8}$$

Multiplying (2.8) from the right by a yields

$$a^3ba + a^*ba^* = aba^*a + ba ,$$

thus

$$ba = a^3ba + a^*ba^* - aba^3 . \tag{2.9}$$

Similar arguments show that

$$ab = aba^3 + a^*ba^* - a^3ba , \tag{2.10}$$

thus

$$ab + ba = 2a^*ba^* . \tag{2.11}$$

Now use (2.7) to get

$$a^2 = a^*ba^* . \tag{2.12}$$

From (2.12) we see that

$$aba = (a^*)^2b(a^*)^2 = a^*(a^*ba^*)a^* = (a^*)^3 ,$$

hence

$$aba = a^6 = a^4a^2 = a^3 = a^*a = aa^* , \tag{2.13}$$

thus

$$a^*ba = a^2ba = aa^*a - a^4 = a , \tag{2.14}$$

and

$$aba^* = aba^2 = a^*a^2 = a^4 = a . \tag{2.15}$$

Then (2.9)–(2.15) show that $ab = a^* = ba$. □

3. q-generalized projections

If $q \in \mathbb{N}$ and $q > 1$, a *q-generalized projection* is an element $a \in \mathscr{J}(\mathscr{A})$ such that $a^q = a^*$ (q-generalized projection operators on complex Hilbert spaces are considered in [5]).

With obvious modifications of the proofs in Section 2, we see that the following results a true:

3.1. Theorem. *If $a \in \mathscr{J}(\mathscr{A})$ is a q-generalized projection, then a is normal and*

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C}, \lambda^{q+1} = 1\} .$$

3.2. Theorem. *Let $a = h + ik \in \mathscr{J}(\mathscr{A})(h, k \in \mathscr{H}(\mathscr{A}))$ and suppose that*

$$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^{q+1} = 1\} .$$

(1) *If a is normal, then $r(a^q - a^*) = 0$.*
(2) *If $h^nk^m \in \mathscr{H}(\mathscr{A})$ for $n, m \in \{0, 1, \ldots, q\}$ then a is normal and $a^q = a^*$.*

References

[1] E. Berkson: *Hermitian projections and orthogonality in Banach spaces.* Proc. London Math. Soc., **24** (1972), 101–118.

[2] F. F. Bonsall and J. Duncan: *Numerical ranges of operators on normed spaces and elements of normed algebras.* Cambridge University Press (1971).

[3] H. Du and Y. Li: *The spectral characterization of generalized projections.* Linear Algebra Appl., **400** (2005), 313–318.

[4] I. Groß and G. Trenkler: *Generalized and hypergeneralized projectors.* Linear Algebra Appl., **264** (1997), 463–474.

[5] L. Lebtahi and N. Thome: *A note on k-generalized projections.* Linear Algebra Appl., **420** (2007), 572–575.

[6] L. S. Murphy: *A note on hermitian elements of a Banch algebra.* J. London Math. Soc., **6** (1973), 427–428.

[7] W. Rudin: *Functional Analysis.* Mc Graw-Hill (1973).

[8] Ch. Schmoeger: *Commutativity up to a factor in Banach algebras.* Demonstratio Math., **38** (2005), 905–910.

Christoph Schmoeger
Institut für Analysis
Universität Karlsruhe (TH)
Englerstrasse 2
76128 Karlsruhe
Germany
E-mail address: christoph.schmoeger@math.uni-karlsruhe.de
URL: http://www.mathematik.uni-karlsruhe.de/mi1weis/~schmoeger/