

# ON PSEUDO-INVERSES OF FREDHOLM OPERATORS

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ABSTRACT. Suppose that  $A$  is a Fredholm operator on a Banach space. We prove that  $A$  has index = 0 (resp.  $\geq 0$ , resp.  $\leq 0$ ) if and only if  $A$  has pseudo-inverse which is invertible (resp. Fredholm and left invertible, resp. Fredholm and right invertible). Furthermore we determine the interior points of some classes of linear operators.

## 1. Terminology

$X$  always denotes a complex Banach space, and the algebra of all bounded linear operators on  $X$  is denoted by  $\mathcal{L}(X)$ .

If  $A \in \mathcal{L}(X)$  we denote by  $N(A)$  the kernel of  $A$  and by  $\alpha(A)$  the dimension of  $N(A)$ .  $A(X)$  denotes the range of  $A$ , and we define  $\beta(A) = \text{codim } A(X)$ .

An operator  $A \in \mathcal{L}(X)$  is called *relatively regular* if there is  $S \in \mathcal{L}(X)$  such that  $ASA = A$ . In this case  $S$  is called a *pseudo-inverse* of  $A$ , and, if  $B = SAS$ , then

$$ABA = A \quad \text{and} \quad BAB = B.$$

$A \in \mathcal{L}(X)$  is called a *Fredholm operator* if  $\alpha(A)$  and  $\beta(A)$  are both finite. In this case we define the *index* of  $A$  by  $\text{ind}(A) = \alpha(A) - \beta(A)$ .

Observe that a Fredholm operator is relatively regular [2, Satz 74.4].

Let  $A \in \mathcal{L}(X)$ . The sequence  $N(A), N(A^2), N(A^3), \dots$  is increasing, while the sequence  $A(X), A^2(X), A^3(X), \dots$  is decreasing. Define  $p(A)$ , the *ascent* of  $A$ , to be the smallest integer  $p \geq 0$  such that  $N(A^p) = N(A^{p+1})$  or  $\infty$  if no such  $p$  exists.

Define  $q(A)$ , the *descent* of  $A$ , to be the smallest integer  $q \geq 0$  with  $A^q(X) = A^{q+1}(X)$  or  $\infty$  if no such  $q$  exists. It is shown in [2, Satz 72.3], that if  $p(A) < \infty$  and  $q(A) < \infty$ , then  $p(A) = q(A)$ .

We define various classes of operators:

$$\Phi(X) = \{A \in \mathcal{L}(X) : A \text{ is Fredholm}\};$$

$$\Phi_\alpha(X) = \{A \in \Phi(X) : \alpha(A) = 0\};$$

$$\Phi_\beta(X) = \{A \in \Phi(X) : \beta(A) = 0\};$$

$$\mathcal{L}(X)^{-1} = \{A \in \mathcal{L}(X) : \alpha(A) = \beta(A) = 0\};$$

$$\mathcal{F}(X) = \{A \in \mathcal{L}(X) : \dim A(X) < \infty\}.$$

Since Fredholm operators are relatively regular,  $\Phi_\alpha(X)$  is the set of all left invertible Fredholm operators and  $\Phi_\beta(X)$  is the class of all right invertible Fredholm operators.

The main results of this paper read as follows:

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**1.1. Theorem.** *If  $A \in \Phi(X)$ , then*

- (a)  $\text{ind}(A) = 0 \Leftrightarrow$  *there is  $S \in \mathcal{L}(X)^{-1}$  such that  $ASA = A$ ;*
- (b)  $\text{ind}(A) \geq 0 \Leftrightarrow$  *there is  $S \in \Phi_\alpha(X)$  such that  $ASA = A$ ;*
- (c)  $\text{ind}(A) \leq 0 \Leftrightarrow$  *there is  $S \in \Phi_\beta(X)$  such that  $ASA = A$ .*

**1.2. Theorem.** *If  $A \in \Phi(X)$ , then  $p(A) = q(A) < \infty$  if and only if there are  $p \in \mathbb{N}_0$  and  $S \in \mathcal{L}(X)^{-1}$  such that  $A^p S A^p = A^p$  and  $A^p S = S A^p$ .*

Proofs of the above result will be given in the next section.

## 2. Proofs

**2.1. Proposition.** *Suppose that  $A \in \mathcal{L}(X)$  is relatively regular and  $B$  is a pseudo-inverse of  $A$  with  $ABA = A$  and  $BAB = B$ .*

- (a)  *$AB, BA, I - AB$  and  $I - BA$  are projections with*

$$\begin{aligned} (AB)(X) &= A(X), \quad (BA)(X) = B(X) \\ (I - AB)(X) &= N(B) \quad \text{and} \quad (I - BA)(X) = N(A). \end{aligned}$$

- (b) *If  $A \in \Phi(X)$ , then  $B \in \Phi(X)$ ,  $\alpha(B) = \beta(A)$ ,  $\beta(B) = \alpha(A)$  and  $\text{ind}(B) = -\text{ind}(A)$ .*

*Proof.* Easy verification. □

**2.2. Proposition.** *Let  $A$  and  $B$  as in Proposition 2.1 and suppose that  $A \in \Phi(X)$ . Then there are  $R \in \Phi(X)$  and  $F \in \mathcal{F}(X)$  such that*

$$BF = 0 \quad \text{and} \quad A = R + F.$$

*Furthermore we have:*

- (a) *if  $\text{ind}(A) = 0$ , then  $R \in \mathcal{L}(X)^{-1}$ ;*
- (b) *if  $\text{ind}(A) \geq 0$ , then  $R \in \Phi_\beta(X)$ ;*
- (c) *if  $\text{ind}(A) \leq 0$ , then  $R \in \Phi_\alpha(X)$ .*

*Proof.* By Proposition 2.1,  $(AB)(X) = A(X)$  and  $(I - AB)(X) = N(B)$ . Hence

$$X = A(X) \oplus N(B).$$

Since  $\alpha(A) < \infty$ , there is  $P \in \mathcal{L}(X)$  such that  $P^2 = P$  and  $P(X) = N(A)$ . Let  $n = \alpha(A)$ ,  $m = \beta(A)$  and  $p = \min\{n, m\}$ . Let  $\{x_1, \dots, x_n\}$  be a basis of  $N(A)$ . Then there are  $x_1^*, \dots, x_n^* \in X^*$  linearly independent with

$$Px = \sum_{j=1}^n x_j^*(x) x_j \quad (x \in X).$$

If  $\{y_1, \dots, y_m\}$  is a basis of  $N(B)$ , define  $F \in \mathcal{F}(X)$  by

$$Fx = \sum_{j=1}^p x_j^*(x) y_j \quad (x \in X).$$

Then  $F(X) \subseteq N(B)$ , thus  $BF = 0$ . Let  $R = A - F$ . It is shown in the proof of Satz 77.2 in [2] that (a), (b) and (c) hold. □

*Proof of Theorem 1.1.* Let  $B, F$  and  $R$  as in Proposition 2.2. Then  $BA = BR + BF = BR$ , hence  $A = ABA = ABR$

(a)  $\Rightarrow$ : Since  $\text{ind}(A) = 0$ , we have  $R \in \mathcal{L}(X)^{-1}$ . Thus  $AR^{-1} = AB$ , hence  $A = ABA = AR^{-1}A$ .

$\Leftarrow$ : By the Index-theorem([2, Satz 71.3]),

$$\text{ind}(A) = 2 \text{ind}(A) + \text{ind}(S) = 2 \text{ind}(A),$$

hence  $\text{ind}(A) = 0$ .

(b)  $\Rightarrow$ : Since  $\text{ind}(A) \geq 0$ , there is  $S \in \mathcal{L}(X)$  such that  $RS = I$ . From  $R \in \Phi_\beta(X)$  we get, by Proposition 2.1 (b),  $S \in \Phi_\alpha(X)$ . From  $A = ABR$  it results that  $AS = AB$ , hence  $A = ABA = ASA$ .

$\Leftarrow$ : Use the Index-theorem to see that  $\text{ind}(A) \geq 0$ .

(c)  $\Rightarrow$ : Since  $\text{ind}(A) \leq 0$ , we have  $\text{ind}(B) \geq 0$ , by Proposition 2.1(b). Apply Proposition 2.2 to  $B$ . Hence there are  $F_0 \in \mathcal{F}(X)$ ,  $R_0 \in \Phi_\beta(X)$  such that

$$AF_0 = 0 \quad \text{and} \quad B = R_0 + F_0.$$

Let  $S = R_0$ . From  $AB = AR_0 + AF_0 = AS$ , we derive  $A = ABA = ASA$ .

$\Leftarrow$ : We have  $\text{ind}(A) \leq 0$ , by the Index-theorem. □

*Proof of Theorem 1.2.*

(a)  $\Rightarrow$  (b): Let  $p = p(A) = q(A) < \infty$ . Satz 72.4 in [2] gives

$$X = N(A^p) \oplus A^p(X).$$

From [2, Satz 101.2] we see that 0 is a pole if the resolvent  $(\lambda I - A)^{-1}$ . Let  $P$  be the associated spectral projection. Hence

$$P(X) = N(A^p) \quad \text{and} \quad N(P) = A^p(X).$$

Then  $PA = AP$  by [2, Satz 99.1]. Let  $F = AP + P$  and  $R = A(I - P) - P$ . Then  $A = R + F$ . The proof of Satz 77.4 in [2] shows that  $R$  is invertible in  $\mathcal{L}(X)$ . Furthermore we have  $RF = FR$ ,  $AF = FA$  and  $AR = RA$ . Since  $F(X) \subseteq P(X) = N(A^p)$ , we get  $A^p F = 0$ , thus  $A^{p+1} = A^p R$ .

*Case 1:*  $p = 0$ . With  $S = I$  we are done.

*Case 2:*  $p = 1$ . We have  $A^2 = AR$ . Let  $S = R^{-1}$ . Then  $AS = SA$  and  $A = A^2 S = ASA$ .

*Case 3:*  $p > 1$ . Let  $A_0 = A^p$ . Satz 71.2 in [2] shows that  $A_0$  is a Fredholm operator. From

$$N(A_0^2) = N(A^{2p}) = N(A^p) = N(A_0),$$

and

$$A_0^2(X) = A^{2p}(X) = A^p(X) = A_0(X),$$

we conclude that  $p(A_0) = q(A_0) \leq 1$ . Case 1 and Case 2 show that there is an invertible operator  $S$  in  $\mathcal{L}(X)$  with  $A_0 S = S A_0$  and  $A_0 = A_0 S A_0$ .

(b)  $\Rightarrow$  (a): Assume that  $p \in \mathbb{N}_0$ ,  $S \in \mathcal{L}(X)$  is invertible,  $A^p S = S A^p$  and  $A^p = A^p S A^p$ . Then  $A^{2p} S = A^p$ . It follows that  $A^p(X) = A^{2p}(S(X)) = A^{2p}(X)$ , thus  $q(A) < \infty$ . Furthermore  $N(A^{2p}) = N(A^p)$ , hence  $p(A) < \infty$ . □

### 3. Interior points of some classes of operators

For a subset  $\mathcal{M}$  of  $\mathcal{L}(X)$  let  $\text{cl}(\mathcal{M})$  and  $\text{int}(\mathcal{M})$  denote the closure and the interior of  $\mathcal{M}$ , respectively.

#### Notation.

$$\begin{aligned}\Phi_+(X) &= \{A \in \Phi(X) : \text{ind}(A) \geq 0\}; \\ \Phi_-(X) &= \{A \in \Phi(X) : \text{ind}(A) \leq 0\}; \\ \Phi_0(X) &= \{A \in \Phi(X) : \text{ind}(A) = 0\}; \\ \mathcal{R}(X) &= \{A \in \mathcal{L}(X) : A \text{ is relatively regular}\}; \\ \mathcal{A}(X) &= \{A \in \mathcal{R}(X) : \alpha(A) < \infty \text{ or } \beta(A) < \infty\}; \\ \mathcal{R}_\alpha(X) &= \{A \in \mathcal{R}(X) : ABA = A \text{ for some } B \in \Phi_\alpha(X)\}; \\ \mathcal{R}_\beta(X) &= \{A \in \mathcal{R}(X) : ABA = A \text{ for some } B \in \Phi_\beta(X)\}; \\ \mathcal{R}_0(X) &= \{A \in \mathcal{R}(X) : ABA = A \text{ for some } B \in \mathcal{L}(X)^{-1}\}.\end{aligned}$$

Operators of the class  $\mathcal{A}(X)$  are called *Atkinson operators* or *relatively regular semi-Fredholm operators*.

#### 3.1. Proposition.

- (a)  $\Phi_+(X)$ ,  $\Phi_-(X)$ ,  $\Phi_0(X)$  and  $\mathcal{A}(X)$  are open subsets of  $\mathcal{L}(X)$ .
- (b)  $\mathcal{R}_\alpha(X) \cup \mathcal{R}_\beta(X) \subseteq \text{cl}(\Phi(X))$ .

*Proof.* (a) follows from [2, Satz 82.4] and (b) is shown in [3, Theorem 3]. □

From Proposition 3.1 (a) and Theorem 1.1 we get

$$\begin{aligned}\Phi_0(X) &\subseteq \text{int}(\mathcal{R}_0(X)), \quad \Phi_+(X) \subseteq \text{int}(\mathcal{R}_\alpha(X)), \\ \Phi_-(X) &\subseteq \text{int}(\mathcal{R}_\beta(X)) \quad \text{and} \quad \mathcal{A}(X) \subseteq \text{int}(\mathcal{R}(X)).\end{aligned}$$

We can be more precise:

#### 3.2. Theorem.

- (a)  $\text{int}(\mathcal{R}(X)) = \mathcal{A}(X)$ ;
- (b)  $\text{int}(\mathcal{R}_0(X)) = \Phi_0(X)$ ;
- (c)  $\text{int}(\mathcal{R}_\alpha(X)) = \Phi_+(X)$ ;
- (d)  $\text{int}(\mathcal{R}_\beta(X)) = \Phi_-(X)$ .

*Proof.* We only have to show the inclusion “ $\subseteq$ ”.

(a) Let  $A \in \text{int}(\mathcal{R}(X))$ . Suppose that  $A \notin \mathcal{A}(X)$ . Then  $\alpha(A) = \beta(A) = \infty$ . From [1, Theorem V. 2.6] we know that there is a compact  $K \in \mathcal{L}(X)$  such that the range  $(A + \lambda K)(X)$  is not closed for all  $\lambda \in \mathbb{C} \setminus \{0\}$ . Since  $A$  is an interior point of  $\mathcal{R}(X)$ ,  $A + \lambda K$  has closed range for  $|\lambda|$  sufficiently small, a contradiction.

(b), (c) and (d) Let  $\gamma \in \{0, \alpha, \beta\}$  and  $A \in \text{int}(\mathcal{R}_\gamma(X))$ . Hence  $A \in \text{int}(\mathcal{R}(X))$ , thus  $A \in \mathcal{A}(X)$ , by (a). Proposition 3.1 (b) shows that there is a sequence  $(A_n)$  in  $\Phi(X)$  such that  $\|A_n - A\| \rightarrow 0$  ( $n \rightarrow \infty$ ). Since  $\mathcal{A}(X)$  is open, the stability of the index ([2, Satz 82.4]) shows that

$$\text{ind}(A_n) = \text{ind}(A) \text{ for } n \text{ sufficiently large.}$$

Thus  $\text{ind}(A)$  is finite, and so  $A \in \Phi(X)$ . Since  $A \in \mathcal{R}_\gamma(X)$ , Theorem 1.1 completes the proof. □

### 3.3. Theorem.

$$\Phi_\alpha(X) \cup \Phi_\beta(X) = \text{int}(\{A \in \Phi(X) : N(A) \subseteq A(X)\}).$$

*Proof.* Since  $\Phi_\alpha(X)$  and  $\Phi_\beta(X)$  are open, the inclusion “ $\subseteq$ ” is clear. Now suppose that  $A \in \text{int}(\{A \in \Phi(X) : N(A) \subseteq A(X)\})$ . Then there is  $\epsilon > 0$  such that

(\*) if  $B \in \mathcal{L}(X)$  and  $\|A - B\| < \epsilon$ , then  $B \in \Phi(X)$  and  $N(B) \subseteq B(X)$ .

Assume that  $\alpha(A) > 0$  and  $\beta(A) > 0$ . Then there are  $x_0, y_0 \in X$  with  $x_0 \neq 0$ ,  $x_0 \in N(A)$ ,  $y_0 \notin A(X)$  and  $\|Ay_0\| = \frac{\epsilon}{2}$ . It follows that  $y_0 \notin N(A)$ . The Hahn-Banach theorem shows that there is  $x^* \in X^*$  such that

$$\alpha = x^*(x_0) \neq 0, \quad x^*(y_0) = 0 \quad \text{and} \quad \|x^*\| = 1.$$

Define  $B \in \mathcal{L}(X)$  by

$$Bx = Ax + x^*(x)Ay_0 \quad (x \in X).$$

Then  $\|(A - B)x\| \leq \frac{\epsilon}{2}\|x\|$ , thus  $\|A - B\| < \epsilon$ . By (\*),  $B \in \Phi(X)$  and  $N(B) \subseteq B(X)$ . Since  $B(X) \subseteq A(X)$ ,  $N(B) \subseteq A(X)$ . We have  $y_0 - x_0/\alpha \in N(B)$ , thus  $y_0 \in A(X) + N(A) = A(X)$ , a contradiction.  $\square$

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