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THE PUNCTURED NEIGHBOURHOOD THEOREM IN BANACH ALGEBRAS

By Christoph Schmoeger

Mathematisches Institut I, Universität Karlsruhe

ROYAL IRISH ACADEMY

19 Dawson Street
Dublin 2, Ireland

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Mathematisches Institut I, Universität Karlsruhe

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ABSTRACT

Let A denote a complex Banach algebra with identity e and K an inessential ideal of A. An element $x \in A$ is called *K-Atkinson element* if x is left or right invertible modulo K. This paper contains necessary and sufficient conditions in order that the *nullity* $\mathrm{nul}(x - \lambda e)$[the *defect* $\mathrm{def}(x - \lambda e)$] of a K-Atkinson element x is constant for all complex numbers λ in a neighbourhood of 0.

Introduction

Let X denote a complex Banach space and $\mathscr{L}(X)$ the Banach algebra of all bounded linear operators on X. An operator $T \in \mathscr{L}(X)$ is *semi-Fredholm* if $T(X)$ is closed and if either $\mathrm{nul}(T) = \dim N(T)$ or $\mathrm{def}(T) = \mathrm{codim}\, T(X)$ is finite ($N(T)$ denotes the kernel of T, $T(X)$ denotes the range of T). In [4, theorems 3, 5 and 6] Kato proved the following.

Theorem A. *If $T \in L(X)$ is semi-Fredholm and* $\mathrm{nul}(T) < \infty$ [*resp.* $\mathrm{def}(T) < \infty$], *there exists $\delta > 0$ such that*

(a) *$T - \lambda I$ is semi-Fredholm for $|\lambda| < \delta$;*
(b) $\mathrm{nul}(T - \lambda I)$ *is a constant* $\leqslant \mathrm{nul}(T)$ [*resp.* $\mathrm{def.}(T - \lambda I)$ *is a constant* $\leqslant \mathrm{def}(T)$] *for* $0 < |\lambda| < \delta$;
(c) $\mathrm{nul}(T - \lambda I) = \mathrm{nul}(T)$ [*resp.* $\mathrm{def}(T - \lambda I) = \mathrm{def}(T)$] *for all $|\lambda| < \delta$ if and only if $N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X)$.*

This important result we call the *punctured neighbourhood theorem.*

$T \in \mathscr{L}(X)$ is called *relatively regular* if $TST = T$ for some $S \in \mathscr{L}(X)$. From [5, théorème 2.6] we easily deduce the following.

Theorem B. *If $T \in \mathscr{L}(X)$ is relatively regular, the following conditions are equivalent:*

(a) $N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X)$;
(b) *there exists a neighbourhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $F : U \to \mathscr{L}(X)$ such that*

$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \text{ for all } \lambda \in U.$$

Combining Theorems A and B, we obtain the following.

Theorem C. *If* $T \in \mathscr{L}(X)$ *is a relatively regular semi-Fredholm operator and* $\text{nul}(T) < \infty$ [*resp.* $\text{def}(T) < \infty$], *then the following conditions are equivalent*:

(a) $\text{nul}(T - \lambda I) = \text{nul}(T)$ [*resp.* $\text{def}(T - \lambda I) = \text{def}(T)$] *for all complex numbers* λ *in a neighbourhood of* 0;

(b) $N(T) \subseteq \cap_{n=1}^{\infty} T^n(X)$;

(c) *there exists a neighbourhood* $U \subseteq \mathbb{C}$ *of* 0 *and a holomorphic function* $F\colon U \to \mathscr{L}(X)$ *such that*

$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \text{ for all } \lambda \in U.$$

Relatively regular semi-Fredholm operators are also called *Atkinson operators*. It is well known (see [3, p. 28]) that T in $\mathscr{L}(X)$ is an Atkinson operator if and only if T is left or right invertible modulo $\mathscr{K}(X)$, where $\mathscr{K}(X)$ denotes the closed ideal of compact operators on X.

Let A be a unital complex Banach algebra and K an inessential ideal of A. An element $x \in A$ is defined to be a *K-Atkinson element* of A if it is left or right invertible modulo K.

The aim of this paper is to extend Theorem C to K-Atkinson elements of A. In section 1 we investigate relatively regular elements in Banach algebras and generalise Theorem B. Section 2 deals with Atkinson theory in semisimple Banach algebras. The main result of this section is a generalisation of Theorem C for Atkinson elements in primitive Banach algebras. In section 3 we investigate Atkinson elements in general Banach algebras and present the main results of this paper.

1. Relatively regular elements in Banach algebras

In this paper we always assume that A is a complex Banach algebra with identity $e \neq 0$.

1.1 Definition. (a) Let $x \in A$. We say that x is *relatively regular* if there exists $y \in A$ such that $xyx = x$.

(b) For each subset M of $A (M \neq \emptyset)$ the *left annihilator* and the *right annihilator* are the sets

$$L(M) = \{y \in A : yM = 0\} \text{ and } R(M) = \{y \in A : My = 0\}, \text{ respectively.}$$

If $M = \{z\}$ we simply write $L(z)$ and $R(z)$.

Since A has an identity, we have $L(zA) = L(z)$ and $R(Az) = R(z)$.

The first proposition of this section is easily deduced from [5, lemme 2.3].

1.2 Proposition. *Let* $x \in A$. *Then*

(a) $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A \Leftrightarrow \bigcup_{n=1}^{\infty} R(x^n) \subseteq xA;$

(b) $L(x) \subseteq \bigcap_{n=1}^{\infty} Ax^n \Leftrightarrow \bigcup_{n=1}^{\infty} L(x^n) \subseteq Ax.$

The following proposition is a generalisation of [5, proposition 2.4, corollaire 2.5].

1.3 Proposition. *Suppose that* $x, y \in A$ *and* $xyx = x$.

(a) *If* $\bigcup_{n=1}^{\infty} R(x^n) \subseteq xA$, *then*

$$yR(x^n) \subseteq R(x^{n+1}) \text{ and } y^n R(x) \subseteq R(x^{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

(b) *If* $\bigcup_{n=1}^{\infty} L(x^n) \subseteq Ax$, *then*

$$L(x^n)y \subseteq L(x^{n+1}) \text{ and } L(x)y^n \subseteq L(x^{n+1}) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

PROOF. (a) Take $z \in R(x^n)$, then $z \in xA$, hence $z = xa$ for some $a \in A$. It follows that $x^{n+1}yz = x^{n+1}yxa = x^n(xyx)a = x^n(xa) = x^n z = 0$, therefore $yz \in R(x^{n+1})$. This proves that

$$yR(x^n) \subseteq R(x^{n+1}) \qquad (n \in \mathbb{N} \cup \{0\}). \tag{1.1}$$

We now prove by induction that for $n \geq 1$,

$$y^n R(x) \subseteq R(x^{n+1}). \tag{1.2}$$

By (1.1), (1.2) holds for $n = 1$. Now suppose that (1.2) holds for some integer $n \geqslant 1$. We derive $y^{n+1}R(x) = y(y^n R(x)) \subseteq yR(x^{n+1}) \subseteq R(x^{n+2})$ by (1.1).

(b) Similar. ■

Now we are in a position to present the generalisation of Theorem B.

1.4 Theorem. *Suppose that* $x \in A$ *is relatively regular. The following conditions are equivalent*:

(a) $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A;$

(b) $L(x) \subseteq \bigcap_{n=1}^{\infty} Ax^n;$

(c) *there is a neighbourhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $f: U \to A$ such that*

$$(x - \lambda e)f(\lambda)(x - \lambda e) = x - \lambda e$$

for all $\lambda \in U$.

Proof. (a) $\Rightarrow$ (c): Take $y \in A$ such that $xyx = x$ and put $p = e - yx$, $q = xy$. Then we have

$$p^2 = p,\ q^2 = q,\ pA = R(x) \text{ and } qA = xA. \tag{1.3}$$

Define U and f by $U = \{\lambda \in \mathbb{C} : |\lambda| < \|y\|^{-1}\}$ and

$$f(\lambda) = (e - \lambda y)^{-1}y = y(e - \lambda y)^{-1} = \sum_{n=0}^{\infty} \lambda^n y^{n+1} \quad (\lambda \in U).$$

Since $p \in R(x)$, we derive $f(\lambda)(x - \lambda e)p = -\lambda f(\lambda)p$, hence

$$\begin{aligned}(x - \lambda e)f(\lambda)(x - \lambda e)p &= (\lambda e - x) \sum_{n=0}^{\infty} \lambda^{n+1} y^{n+1} p \\ &= \sum_{n=0}^{\infty} \lambda^{n+2} y^{n+1} p - \sum_{n=0}^{\infty} \lambda^{n+1} \underbrace{(xy)}_{=q} y^n p \\ &= \sum_{n=0}^{\infty} \lambda^{n+2} y^{n+1} p - q \sum_{n=0}^{\infty} \lambda^{n+1} y^n p.\end{aligned}$$

Since $y^n p \in y^n pA = y^n R(x)$, we obtain $y^n p \in R(x^{n+1})$(Proposition 1.3(a)). It follows, by Proposition 1.2(a), that $y^n p \in xA = qA$ for all $n \in \mathbb{N} \cup \{0\}$. Using that qA is a closed right ideal, we conclude that

$$\sum_{n=0}^{\infty} \lambda^{n+1} y^n p \in qA \qquad (\lambda \in U).$$

This gives

$$q \sum_{n=0}^{\infty} \lambda^{n+1} y^n p = \sum_{n=0}^{\infty} \lambda^{n+1} y^n p \qquad (\lambda \in U),$$

since $q^2 = q$. Therefore

$$(x - \lambda e)f(\lambda)(x - \lambda e)p = \sum_{n=0}^{\infty} \lambda^{n+2} y^{n+1} p - \sum_{n=0}^{\infty} \lambda^{n+1} y^n p = -\lambda p \tag{1.4}$$

for all $\lambda \in U$.

By (1.3), we have $e - p = yx$. Thus

$$\begin{aligned}(x-\lambda e)f(\lambda)(x-\lambda e)(e-p) &= (x-\lambda e)(e-\lambda y)^{-1}(yx-\lambda y)yx \\ &= (x-\lambda e)(e-\lambda y)^{-1}(\underbrace{(yx)^2}_{=yx}-\lambda yyx) \\ &= (x-\lambda e)(e-\lambda y)^{-1}(e-\lambda y)yx \\ &= (x-\lambda e)yx = xyx - \lambda yx \\ &= x - \lambda(e-p) = x - \lambda e + \lambda p.\end{aligned}$$

Together with (1.4) this shows

$$\begin{aligned}(x-\lambda e)f(\lambda)(x-\lambda e) &= (x-\lambda e)f(\lambda)(x-\lambda e)(p+(e-p)) \\ &= -\lambda p + x - \lambda e + \lambda p \\ &= x - \lambda e\end{aligned}$$

for all $\lambda \in U$.

(c) $\Rightarrow$ (a): Put $p(\lambda) = (x-\lambda e)f(\lambda)$ $(\lambda \in U)$. Hence $p(\lambda)^2 = p(\lambda)$ and $p(\lambda)A = (x-\lambda e)A$. Take $a \in R(x^n)$. Then

$$0 = x^n a = (x - \lambda e + \lambda e)^n a = \sum_{k=1}^{n} \binom{n}{k} (x-\lambda e)^k \lambda^{n-k} a + \lambda^n a.$$

Thus

$$a \in (x-\lambda e)A = p(\lambda)A \text{ for all } \lambda \in U \backslash \{0\},$$

therefore $a = p(\lambda)a \quad (\lambda \in U\backslash\{0\})$.

The continuity of p shows that $a = p(0)a \in p(0)A = xA$. This proves $\bigcup_{n=1}^{\infty} R(x^n) \subseteq xA$. From Proposition 1.2(a) we obtain $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A$.

Similar arguments show that (b) and (c) are equivalent (use Proposition 1.2(b) and Proposition 1.3(b)). ■

In section 3 we need the following corollary which is implicitly contained in the preceding proof.

1.5 Corollary. *Let x, $y \in A$ and $xyx = y$. Suppose that $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A$ or $L(x) \subseteq \bigcap_{n=1}^{\infty} Ax^n$. If the function f is defined by $f(\lambda) = (e-\lambda y)^{-1} y$ for $|\lambda| < \|y\|^{-1}$, then*

$$(x-\lambda e)f(\lambda)(x-\lambda e) = x - \lambda e \qquad (|\lambda| < \|y\|^{-1}).$$

2. The punctured neighbourhood theorem in primitive Banach algebras

For the convenience of the reader we shall summarise some definitions, notations and results of Atkinson and Fredholm theory in semisimple Banach algebras (see [1], [6] and [8] for details). In the second part of this section we state and prove the generalisation of Theorem C in primitive Banach algebras.

Given a left ideal L of A the *quotient* is the ideal $L{:}A = \{a \in A : aA \subseteq L\}$. The quotient of a maximal left ideal is called a *primitive* ideal. We denote the set of all primitive ideals by $\Pi(A)$.

Observe that each $P \in \Pi(A)$ is closed.

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denote the left regular representation of A on the Banach space Ae_0, that is $\hat{x}(y) = xy$, $y \in Ae_0$ (recall that $\mathscr{L}(Ae_0)$ denotes the set of all bounded linear operators on Ae_0). Note that

$$\hat{x}(Ae_0) = xAe_0$$

and

$$\{y \in Ae_0 : \hat{x}(y) = 0\} = R(x) \cap Ae_0 = R(x)e_0.$$

It follows from [1, F.2.1] that dim xAe_0, dim $R(x)e_0$ and dim(Ae_0/xAe_0) are independent of the choice of $e_0 \in \mathrm{Min}(A)$.

2.4 Definition. For $x \in A$ we define the *rank* of x by $\mathrm{rank}(x) = \dim xAe_0$. The *nullity* of x is defined to be $\mathrm{nul}(x) = \dim R(x)e_0$. The defect of x is defined by $\mathrm{def}(x) = \dim\,(Ae_0/xAe_0)$.

2.5 Proposition. (a) $x = 0 \Leftrightarrow \mathrm{rank}(x) = 0$;
(b) $\mathrm{soc}(A) = \{x \in A : \mathrm{rank}(x) < \infty\}$;
(c) $x \in \Phi_l(A,\ I(A))$ [*resp.* $x \in \Phi_r(A,\ I(A))$] $\Leftrightarrow x$ *relatively regular and* $\mathrm{nul}(x) < \infty$ [*resp.* $\mathrm{def}(x) < \infty$].

PROOF. (a) and (b) [1, F.2.4]. (c) [8, theorem 3.4]. ■

We now state the first punctured neighbourhood theorem.

2.6 Theorem. *Let A be a primitive Banach algebra, K an inessential ideal of A, and let $x \in \Phi_l(A, K)$. Then there exists $\delta > 0$ such that*
(a) $x - \lambda e \in \Phi_l(A, K)$ *for* $|\lambda| < \delta$;
(b) $\mathrm{nul}(x - \lambda e) = \text{const.} \leq \mathrm{nul}(x)$ *for* $0 < |\lambda| < \delta$;
(c) *the following conditions are equivalent*:
(i) $\mathrm{nul}(x - \lambda e) = \mathrm{nul}(x)$ *for all* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \delta$;
(ii) $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A$:
(iii) $L(x) \subseteq \bigcap_{n=1}^{\infty} Ax^n$;
(iv) *there is a neighbourhood* $U \subseteq \mathbb{C}$ *of* 0 *and a holomorphic function* $f\colon U \to A$ *such that*

$$(x - \lambda e)f(\lambda)(x - \lambda e) = x - \lambda e \text{ for all } \lambda \in U.$$

PROOF. (a) and (b) are known [6, lemma 4.3; 8, theorem 3.8, 5.4], but we shall give proofs, because the proof of (c) depends on some technical informations obtained in the proof of (a) and (b).

Since $x \in \Phi_l(A, K)$, there is $p = p^2 \in \mathrm{soc}(A) \cap K$ such that

$$Ax = A(e - p) \tag{2.1}$$

(Proposition 2.3(b)). Therefore $yx = e - p$ for some $y \in A$. Now fix $e_0 \in \mathrm{Min}(A)$. We have $R(x) = R(Ax) = pA$, thus

$$\text{nul}(x) = \dim R(x)e_0 = \dim pAe_0 = \text{rank}(p). \tag{2.2}$$

Put $\delta_0 = \|y\|^{-1}$and define the holomorphic function g by

$$g(\lambda) = \sum_{n=0}^{\infty} \lambda^n y^n = (e - \lambda y)^{-1} \ (|\lambda| < \delta_0).$$

Since $y(x - \lambda e) = yx - \lambda y = e - \lambda y - p$, we obtain

$$g(\lambda)y(x - \lambda e) = e - g(\lambda)p \text{ and } g(\lambda)p \in K. \tag{2.3}$$

This shows (a). Next we show $R(x - \lambda e) \subseteq g(\lambda)pA$ $(|\lambda| < \delta_0)$. Let $a \in R(x - \lambda e)$, then $0 = g(\lambda)y(x - \lambda e)a = a - g(\lambda)pa$ and thus $a = g(\lambda)pa \in g(\lambda)pA$. Consequently,

$$R(x - \lambda e) \subseteq g(\lambda)pA \qquad (|\lambda| < \delta_0) \tag{2.4}$$

and

$$a = g(\lambda)pa \ (|\lambda| < \delta_0, a \in R \qquad (x - \lambda e)). \tag{2.5}$$

From (2.4) we derive for $|\lambda| < \delta_0$:

$$\text{nul}(x - \lambda e) \leqslant \text{rank}(g(\lambda)p) = \text{rank}(p) = \text{nul}(x). \tag{2.6}$$

Again by (2.4), for each λ with $|\lambda| < \delta_0$, there is a subspace R_λ of $g(\lambda)pAe_0$ such that

$$R(x - \lambda e)e_0 \oplus R_\lambda = g(\lambda)pAe_0. \tag{2.7}$$

It follows that $(x - \lambda e)g(\lambda)pAe_0 = (x - \lambda e)R_\lambda$ and that

$$\begin{aligned} \dim (x - \lambda e)g(\lambda)pAe_0 &= \text{rank}((x - \lambda e)g(\lambda)p) \\ &= \dim (x - \lambda e)R_\lambda = \dim R_\lambda \\ &\leqslant \dim R_\lambda + \dim R(x - \lambda e)e_0 \\ &= \dim R_\lambda + \text{nul}(x - \lambda e) \\ &= \dim g(\lambda)pAe_0 = \text{rank}(g(\lambda)p) \\ &= \text{nul}(x) \qquad (|\lambda| < \delta_0). \end{aligned} \tag{2.8}$$

Next define the holomorphic function h by

$$h(\lambda) = (x - \lambda e)g(\lambda)p \qquad (|\lambda| < \delta_0).$$

Equation (2.8) shows that $m = \max_{|\lambda|<\delta_0} \text{rank}(h(\lambda))$ exists.

By [8, lemma 5.2], there is a positive $\delta \leqslant \delta_0$ such that $\text{rank}(h(\lambda)) = \dim (x - \lambda e)g(\lambda)pAe_0 = \dim R_\lambda = m$ for $0 < |\lambda| < \delta$. Use (2.8) to derive

$$\text{nul}(x) = m + \text{nul}(x - \lambda e) \text{ whenever } 0 < |\lambda| < \delta. \tag{2.9}$$

Hence (b) follows.

Proof of (*c*). A simple computation (observe that $xp = 0$) yields

$$h(\lambda) = \sum_{n=1}^{\infty} \lambda^n (xy^n p - y^{n-1}p) \qquad (|\lambda| < \delta). \tag{2.10}$$

By Proposition 2.5 and (2.9), this gives

$$\text{nul}(x - \lambda e) = \text{nul}(x) \text{ for all } \lambda \text{ with } |\lambda| < \delta \Leftrightarrow m = 0$$

$$\Leftrightarrow \text{rank}(h(\lambda)) = 0 \ (|\lambda| < \delta) \Leftrightarrow h(\lambda) = 0 \text{ for all } \lambda \text{ with } |\lambda| < \delta \Leftrightarrow$$

$$xy^n p = y^{n-1} p \text{ for all } n \in \mathbb{N}. \tag{2.11}$$

Therefore condition (i) and (2.11) are equivalent.

(i) implies (ii): We prove by induction that for $n \geqslant 1$

$$p = x^n y^n p; \tag{2.12}$$

(2.11) shows that (2.12) holds for $n = 1$. Now suppose that (2.12) holds for some integer $n \geqslant 1$. This gives $x^{n+1}y^{n+1}p = x^n(xy^{n+1}p) = x^n y^n p = p$ by (2.11). Thus (2.12) is proved, and we have $p \in \bigcap_{n=1}^{\infty} x^n A$. Since $pA = R(x)$, it follows that $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A$.

(ii) implies (i): For each $n \in \mathbb{N}$ there is $x_n \in A$ such that

$$p = x^n x_n. \tag{2.13}$$

Since $yx = e - p$, it is easy to see (by induction) that

$$y^{n+1}x^{n+1} = e - p - \sum_{k=1}^{n} y^k p x^k \qquad (n \in \mathbb{N}). \tag{2.14}$$

We now prove by induction that (2.11) holds. By (2.13), $p = xx_1$, thus $xyp = xyxx_1 = x(e-p)x_1 = xx_1 - xpx_1 = xx_1 = p$. Now suppose that (2.11) holds for 1, 2, . . . , n. Use (2.13) and (2.14) to derive

$$\begin{aligned} xy^{n+1}p &= xy^{n+1}x^{n+1}x_{n+1} = x\left(e - p - \sum_{k=1}^{n} y^k p x^k\right)x_{n+1} \\ &= \left(x - \sum_{k=1}^{n} \underbrace{xy^k p}_{=y^{k-1}p} \, x^k\right)x_{n+1} \\ &= \left(e - \sum_{k=1}^{n} y^{k-1} p x^{k-1}\right)xx_{n+1} \\ &= \left(e - p - \sum_{k=1}^{n-1} y^k p x^k\right)xx_{n+1} \\ &= y^n x^n x x_{n+1} = y^n x^{n+1} = y^n p. \end{aligned}$$

This proves (2.11). The proof of (i) $\Leftrightarrow$ (ii) is now complete. Since x is relatively regular (Proposition 2.3(c)), the equivalence of (ii), (iii) and (iv) follows from Theorem 1.4. ■

The second punctured neighbourhood theorem reads as follows.

2.7 Theorem. *Let A be a primitive Banach algebra, K an inessential ideal of A, and let $x \in \Phi_r(A, K)$. Then there exists $\delta > 0$ such that*

(a) $x - \lambda e \in \Phi_r(A, K)$ *for* $|\lambda| < \delta$;

(b) $\text{def}(x - \lambda e) = \text{const.} \leqslant \text{def}(x)$ *for* $0 < |\lambda| < \delta$;

(c) *the following conditions are equivalent*:

(i) $\operatorname{def}(x - \lambda e) = \operatorname{def}(x)$ *for all* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \delta$;
(ii) $R(x) \subseteq \bigcap_{n=1}^{\infty} x^n A$;
(iii) $L(x) \subseteq \bigcap_{n=1}^{\infty} Ax^n$;
(iv) *there is a neighbourhood* $U \subseteq \mathbb{C}$ *of* 0 *and a holomorphic function* $f\colon U \to A$ *such that*

$$(x - \lambda e)f(\lambda)(x - \lambda e) = x - \lambda e \text{ for all } \lambda \in U.$$

We omit the proof, because this theorem can be proved in the same way as Theorem 2.6, if one uses the following proposition.

2.8 Proposition. *Let* K *be an inessential ideal of the primitive Banach algebra* A. *If* $x \in \Phi_r(A, K)$ *and* $e_0 \in \operatorname{Min}(A)$, *then*

$$\operatorname{def}(x) = \dim(e_0 L(x)).$$

PROOF. By Proposition 2.3(b), there is an idempotent $p \in \operatorname{soc}(A) \cap K$ such that $xA = (e - p)A$. Hence $L(x) = Ap$ and $Ae_0 = xAe_0 \oplus pAe_0$. Thus $\operatorname{def}(x) = \operatorname{rank}(p)$. According to [7, lemma 2.3], there are $e_1, \ldots, e_n \in \operatorname{Min}(A)$ such that $p = e_1 + \cdots + e_n$ and $e_i e_j = 0$ $(i \neq j)$. It follows that

$$pA = e_1 A \oplus \cdots \oplus e_n A \text{ and } Ap = Ae_1 \oplus \ldots \oplus Ae_n.$$

This gives

$$pAe_0 = e_1 Ae_0 \oplus \cdots \oplus e_n Ae_0 \quad \text{and} \quad e_0 A_p = e_0 Ae_1 \oplus \cdots \oplus e_0 Ae_n.$$

Since $\dim e_j Ae_0 = \dim e_0 Ae_j = 1$ [1, F.2.1], we have

$$\begin{aligned}\operatorname{def}(x) &= \operatorname{rank}(p) = \dim(pAe_0) = n = \dim(e_0 Ap)\\ &= \dim(e_0 L(x)). \blacksquare\end{aligned}$$

3. The general form of the punctured neighbourhood theorem

In this section we assume that A is an arbitrary Banach algebra. Thus $\operatorname{soc}(A)$ might not exist, but the quotient algebra $A' = A/\operatorname{rad}(A)$ is semisimple [2, proposition 24.21], hence A' has a socle.

We write x' for the coset $x + \operatorname{rad}(A) \in A'$ $(x \in A)$, and if $S \subseteq A$ write $S' = \{x' : x \in S\}$.

In order to extend Atkinson and Fredholm theory we need the following important fact: if $P \in \Pi(A)$, then the quotient algebra A/P is primitive [2, proposition 26.9].

3.1 Definition. The *presocle* of A is defined by $\operatorname{psoc}(A) = \{x \in A : x' \in \operatorname{soc}(A')\}$. The *ideal of inessential elements* is defined to be

$$I(A) = \bigcap \{P : P \in \Pi(A) \text{ and } \operatorname{psoc}(A) \subseteq P\}.$$

An ideal K of A is *inessential* if $K \subseteq I(A)$.

Observe that $\operatorname{psoc}(A)$ is an ideal of A and that $\operatorname{soc}(A) = \operatorname{psoc}(A)$ if A is semisimple.

If K is an inessential ideal of A, the sets

$$\Phi_l(A, K),\ \Phi_r(A, K),\ \mathscr{A}(A, K) \text{ and } \Phi(A, K)$$

are defined as in Definition 2.2. If $K = I(A)$ we write $\Phi_l(A)$, $\Phi_r(A)$, $\Phi(A)$, $\mathscr{A}(A)$ instead of $\Phi_l(A, I(A))$, $\Phi_r(A, I(A))$, $\Phi(A, I(A))$, $\mathscr{A}(A, I(A))$.

In the following proposition we collect some properties of Atkinson and Fredholm elements. The reader is referred to [1], [6] and [8] for further information and details.

3.2 Proposition. (a) $\Phi_l(A) = \Phi_l(A, \mathrm{psoc}(A))$, $\Phi_r(A) = \Phi_r(A, \mathrm{psoc}(A))$.

(b) *If* $x \in \Phi_l(A)$ [*resp.* $x \in \Phi_r(A)$] *there exist* $P_1, \dots, P_n \in \Pi(A)$ *such that*

$$x + P \in \Phi_l(A/P) [\textit{resp. } x + P \in \Phi_r(A/P)] \textit{ for all } P \in \Pi(A)$$

and

$$\mathrm{nul}(x + P) = 0\ [\textit{resp. } \mathrm{def}(x + P) = 0 \textit{ for all } P \in \Pi(A) \backslash \{P_1, \dots, P_n\}.$$

PROOF. [6, prop. 2.19, theorem 2.22]. ■

In view of part (b) of the preceding proposition the concepts of nullity and defect can be extended as follows.

For each $x \in \mathscr{A}(A)$ we define the *nullity* and *defect functions* $\Pi(A) \to \mathbb{Z} \cup \{\infty\}$ by

$$\nu(x)(P) = \mathrm{nul}(x + P) \text{ and } \delta(x)(P) = \mathrm{def}(x + P).$$

If $x \in \mathscr{A}(A)$ we define

$$\mathrm{nul}(x) = \begin{cases} \sum_{P \in \Pi(A)} \nu(x)(P) & \text{for } x \in \Phi_l(A), \\ \infty & \text{for } x \notin \Phi_l(A), \end{cases}$$

$$\mathrm{def}(x) = \begin{cases} \sum_{P \in \Pi(A)} \delta(x)(P) & \text{for } x \in \Phi_r(A), \\ \infty & \text{for } x \notin \Phi_r(A), \end{cases}$$

Note that if A is primitive and $x \in \Phi_l(A)$ [resp. $x \in \Phi_r(A)$], $\nu(x)(P) = 0$ [resp. $\delta(x)(P) = 0$] for all $P \in \Pi(A)$, $P \neq \{0\}$ and $\mathrm{nul}(x) = \nu(x)(\{0\})$ [resp. $\mathrm{def}(x) = \delta(x)(\{0\})$](see [1, p. 38] and [8, remark 4.5] for details). So our two definitions for nullity [defect] coincide.

Now we present the general form of the punctured neighbourhood theorem.

3.3 Theorem. *Let K be an inessential ideal of A. If $x \in \Phi_l(A, K)$ [resp. $x \in \Phi_r(A, K)$] then there is a positive δ such that*

(a) $x - \lambda e \in \Phi_l(A, K)$ [*resp.* $x - \lambda e \in \Phi_r(A, K)$] *for* $|\lambda| < \delta$;

(b) $\mathrm{nul}(x - \lambda e) = \text{const.} \leq \mathrm{nul}(x)$ [*resp.* $\mathrm{def}(x - \lambda e) = \text{const.} \leq \mathrm{def}(x)$] *for* $0 < |\lambda| < \delta$;

(c) *the following conditions are equivalent*:

(i) $\text{nul}(x - \lambda e) = \text{nul}(x)$[*resp.* $\text{def}(x - \lambda e) = \text{def}(x)$] *for all* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \delta$;

(ii) *for each* $P \in \Pi(A)$, $\nu(x - \lambda e)(P) = \nu(x)(P)$ [*resp.* $\delta(x - \lambda e)(P) = \delta(x)(P)$] *for all* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \delta$;

(iii) $R(x') \subseteq \cap_{n=1}^{\infty}(x')^n A'$;

(iv) $L(x') \subseteq \cap_{n=1}^{\infty} A'(x')^n$;

(v) *there is a neighbourhood* $U \subseteq \mathbb{C}$ *of* 0 *and a holomorphic function* $\bar{f}: U \to A/\text{rad}(A)$ *such that*

$$(x' - \lambda e')\bar{f}(\lambda)(x' - \lambda e') = x' - \lambda e' \text{ for all } \lambda \in U.$$

PROOF. (a) and (b) are known [6, theorem 4.4; 8, theorem 4.8, 5.5].

(c) Let $x \in \Phi_l(A, K)$ (the proof for the case $x \in \Phi_r(A, K)$ is similar). Since $\Phi_l(A, K) \subseteq \Phi_l(A, I(A)) = \Phi_l(A, \text{psoc}(A))$ (Proposition 3.2(a)), there is $a \in A$ such that $ax - e \in \text{psoc}(A)$, hence $a'x' - e' \in \text{soc}(A')$. It follows that $x' \in \Phi_l(A', \text{soc}(A'))$. Proposition 2.3(c) shows that x' is relatively regular (recall that A' is semisimple). Hence there exists $y \in A$ such that

$$xyx - x \in \text{rad}(A) = \cap\{P : P \in \Pi(A)\}; \tag{3.1}$$

thus

$$(x + P)(y + P)(x + P) = x + P \text{ for all } P \in \Pi(A). \tag{3.2}$$

By [6, theorem 2.22] and [8, theorem 5.5], there exist $\delta < 0$, $P_1, \ldots, P_n \in \Pi(A)$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$ with

$$\text{nul}(x - \lambda e + P_j) = \alpha_j \leq \text{nul}(x + P_j) \quad (j = 1, \ldots, n, 0 < |\lambda| < \delta), \tag{3.3}$$

$$\text{nul}(x - \lambda e + P) = \text{nul}(x + P) = 0 \quad (P \in \Pi(A)\backslash\{P_1, \ldots, P_n\}, |\lambda| < \delta) \tag{3.4}$$

$$\text{and } \text{nul}(x - \lambda e) = \sum_{j=i}^{n} \alpha_j \leq \text{nul}(x) \quad (0 < |\lambda| < \delta). \tag{3.5}$$

By (3.3), (3.4) and (3.5), the equivalence of (i) and (ii) is clear.

(i) implies (v): Put $U := \{\lambda \in \mathbb{C} : |\lambda| < \|y\|^{-1}\}$. Since $\|y + P\| \leq \|y\|$ $(P \in \Pi(A))$, it follows that the functions

$$f(\lambda) := (e - \lambda y)^{-1} y$$

and

$$f_P(\lambda) := f(\lambda) + P = ((e - \lambda y)^{-1} + P)(y + P) = (e - \lambda y + P)^{-1}(y + P)$$

$(P \in \Pi(A))$ are holomorphic in U.

Fix $P \in \Pi(A)$. By (ii) $\text{nul}(x - \lambda e + P) = \text{nul}(x + P)$ $(|\lambda| < \delta)$, thus

$$R(x + P) \subseteq \bigcap_{n=1}^{\infty} (x^n + P)A/P$$

(Theorem 2.6(c)). From (3.2) and Corollary 1.5 we derive

$$(x - \lambda e + P)(f(\lambda) + P)(x - \lambda e + P) = x - \lambda e + P \text{ for all } \lambda \in U. \tag{3.6}$$

Since $P \in \Pi(A)$ was arbitrary, (3.6) gives

$$(x-\lambda e)f(\lambda)(x-\lambda e)-(x-\lambda e)\in P$$

for all $P\in\Pi(A)$ and all $\lambda\in U$. Consequently

$$(x-\lambda e)f(\lambda)(x-\lambda e)-(x-\lambda e)\in \mathrm{rad}(A)\ \text{ for all } \lambda\in U.$$

Now define the function $\bar{f}\colon U\to A/\mathrm{rad}(A)$ by $\bar{f}(\lambda)=f(\lambda)+\mathrm{rad}(A)$, and (v) follows.

(v) implies (i): [9, lemma 2.1] shows that there is a neighbourhood $V\subseteq U$ of 0 and a holomorphic function $g\colon V\to A$ such that

$$g(\lambda)+\mathrm{rad}(A)=\bar{f}(\lambda)\qquad(\lambda\in V).$$

This gives $(x-\lambda e)g(\lambda)(x-\lambda e)-(x-\lambda e)\in\mathrm{rad}(A)=\cap\{P:P\in\Pi(A)\}$ for all $\lambda\in V$. Hence

$$(x-\lambda e+P)(g(\lambda)+P)(x-\lambda e+P)=x-\lambda e+P \tag{3.7}$$

for all $\lambda\in V$ and all $P\in\Pi(A)$.

From (3.7), Theorem 2.6, (3.3) and (3.4), we obtain

$$\mathrm{nul}(x-\lambda e+P)=\mathrm{nul}(x+P)\ \text{ for all } P\in\Pi(A) \text{ and } |\lambda|<\delta.$$

Thus $\mathrm{nul}(x-\lambda e)=\mathrm{nul}(x)$ for $|\lambda|<\delta$. This shows that (v) implies (i).

Since x' is relatively regular, the equivalence of (iii), (iv) and (v) follows from Theorem 1.4. The proof is now complete. ■

An immediate consequence of the last theorem is the following corollary.

3.4 Corollary. *Let A be semisimple and K an inessential ideal of A. If $x\in\Phi_l(A,K)$ [resp. $x\in\Phi_r(A,K)$] then the following conditions are equivalent:*

(a) $\mathrm{nul}(x-\lambda e)=\mathrm{nul}(x)$ [*resp.* $\mathrm{def}(x-\lambda e)=\mathrm{def}(x)$] *in a neighbourhood of* 0;

(b) $R(x)\subseteq\cap_{n=1}^{\infty}x^nA$;

(c) $L(x)\subseteq\cap_{n=1}^{\infty}Ax^n$;

(d) *there is a neighbourhood $U\subseteq\mathbb{C}$ of 0 and a holomorphic function $f\colon U\to A$ such that*

$$(x-\lambda e)f(\lambda)(x-\lambda e)=x-\lambda e\ \textit{ for all } \lambda\in U.$$

REFERENCES

[1] BARNES, B. A., MURPHY, G. J., SMYTH, M. R. F. and WEST, T. T. 1982 *Riesz and Fredholm theory in Banach algebras*. Boston–London–Melbourne. Pitman.

[2] BONSALL, F. F. and DUNCAN, J. 1973 *Complete normed algebras*. Berlin–Heidelberg–New York. Springer.

[3] CARADUS, S. R. 1974 *Operator theory of the pseudo-inverse*. Queen's Papers in Pure and Applied Math. No. 38.

[4] KATO, T. 1958 Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. Anal. Math.* **6**, 261–322.

[5] MBEKHTA, M. 1987 Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux. *Glasgow Math. J.* **29**, 159–75.

[6] ROWELL, J. W. 1984 Unilateral Fredholm theory and unilateral spectra. *Proc. R. Ir. Acad.* **84A**, 69–85.

[7] Ó SEARCÓID, M. 1987 Nilpotent decomposition of left and right Fredholm elements of a prime ring. *Proc. R. Ir. Acad.* **87A**, 187–209.

[8] Schmoeger, Ch. 1991 Atkinson theory and holomorphic functions in Banach algebras. *Proc. R. Ir. Acad.* **91A**, 113–27.

[9] Vasilescu, F.-H. 1971 Residual properties for closed operators on Fréchet spaces. *Ill. J. Math.* **15**, 377–86.