ON A QUESTION OF MBEKHTA

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ABSTRACT. The present paper deals with a question of M. Mbekhta concerning partial isometries on Banach spaces

1. Introduction

Throughout this paper, X shall denote a Banach space and $\mathcal{L}(X)$ the algebra of all bounded linear operators on X. X^* denotes the dual space of X. For an operator $T \in \mathcal{L}(X)$ we write T^* for its adjoint, $N(T)$ for its kernel and $T(X)$ for its range.

We will say that $T \in \mathcal{L}(X)$ has a *generalized inverse* if there is an operator $S \in \mathcal{L}(X)$ for which

$$TST = T \quad \text{and} \quad STS = S\,. \tag{1.1}$$

The operator S is called a *generalized inverse* of T. We recall that in general a generalized inverse is not unique and that T has a generalized inverse if and only if $N(T)$ and $T(X)$ are closed and complemented subspaces of X (see for instance, [3]). Observe that if (1.1) holds then TS, ST, $I - TS$ and $I - ST$ are projections, $T(X) = TS(X)$, $S(X) = ST(X)$, $N(T) = (I - ST)(X)$ and $N(S) = (I - TS)(X)$, hence

$$X = T(X) \oplus N(S) \quad \text{and} \quad X = S(X) \oplus N(T)\,. \tag{1.2}$$

A bounded linear operator T on a *Hilbert* space is said to be a *partial isometry* provided that $\|Tx\| = \|x\|$ for every $x \in N(T)^\perp$, that is

$$TT^*T = T\,.$$

In this case T is a contraction (see Chapter 13 in [5] for details).

In [7] M. Mebekhta has given the following characterization of partial isometries on Hilbert spaces:

1.1. Theorem. *If T is a contraction on a Hilbert space, then the following are equivalent:*

(1) *T is a partial isometry;*
(2) *T has a contractive generalized inverse.*

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Since assertion (2) of Theorem 1.1 does not depend on the structure of a Hilbert space, Theorem 1.1 suggests the following definition of a partial isometry on a *Banach* space. This definition is due to M. Mbekhta [7].

1.2. Definition. An operator $T \in \mathcal{L}(X)$ is called a *partial isometry* if T is a contraction and admits a generalized inverse which is a contraction.

Remarks.

(1) As mentioned by Mbekhta in [7], one of the disadvantages of Definition 1.2 is that, in general, an isometry on X (i.e. $\|Tx\| = \|x\|$ for all $x \in X$) does not need to be a partial isometry. Indeed an isometry may not have a generalized inverse.
(2) In Definition 1.2, the contractive generalized inverse is not unique, as is shown by an example in [7, page 776].

The following proposition collects some properties of partial isometries on Banach spaces. Proofs can be found in [7].

1.3. Proposition. *If $T \in \mathcal{L}(X)$ is a non-zero partial isometry and S is a contractive generalized inverse of T then:*

(1) $\|T\| = \|S\| = \|TS\| = \|ST\| = 1$*;*
(2) $S(X) \subseteq \{x \in X : \|Tx\| = \|x\|\}$.

If T is a partial isometry on a *Hilbert* space H and S is a contractive generalized inverse of T, then $S = T^*$ (see [7, Corollary 3.3]). Hence T has a unique contractive generalized inverse. Furthermore, by (1.2),

$$T^*(H) = S(H) = \{x \in H : \|Tx\| = \|x\|\}\,. \tag{1.3}$$

In view of Proposition 1.3 (2) and (1.3) the following question, due to M Mbekthta [7], arises:

1.4. Question. If $T \in \mathcal{L}(X)$ is a partial isometry on a *Banach* space X and S is a contractive generalized inverse of T, does

$$S(X) = \{x \in X : \|Tx\| = \|x\|\}\,? \tag{1.4}$$

The following example, provide in [7], shows that in general (1.4) does not hold.

1.5. Example. Let $X = \mathbb{C}^2$ be equipped with the norm $\|(x,y)\| = |x| + |y|$, and consider the operator

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X)\,.$$

Take

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then it is easy to see that $T^2 = T$, $\|T\| = \|S\| = 1$ and that $TST = T$ and $STS = S$. Thus T is a partial isometry and T and S are contractive generalized inverses of T. For $(0,1) \in X$ we have $T(0,1) = (-1,0)$, thus $\|T(0,1)\| = \|(0,1)\| = 1$, but $(0,1) \notin S(X)$.

1.6. Proposition. *If $T \in \mathcal{L}(X)$ is a partial isometry, then the following assertions are equivalent:*

(1) *There is a contractive generalized inverse S of T such that* (1.4) *holds.*
(2) (1.4) *holds for every contractive generalized inverse of T.*

Proof. We only have to show that (1) implies (2). Hence assume that S and S_0 are contractive generalized inverses of T and that (1.4) holds for S. It follows from Proposition 1.3 (2) that $S_0(X) \subseteq S(X)$, therefore $S_0T(X) \subseteq ST(X)$. This gives $STS_0T = S_0T$, thus $ST = S_0T$, hence $S(X) \subseteq S_0(X)$, and so $S_0(X) = S(X)$. $\square$

In this paper we show that in the case of a *strictly convex* Banach space, Question 1.4 has an affirmative answer. Furthermore we show that a partial isometry on a strictly convex Banach space with a strictly convex dual space has a unique contractive generalized inverse, and we give some corollaries of these results.

2. Results

We say that the Banach space X is *strictly convex* if the assumptions

$$x, y \in X,\ \|x\| = \|y\| = 1 \quad \text{and} \quad x \neq y$$

imply that $\|x + y\| < 2$.

We say that the norm of X is *Gâteaux-differentiable* if, for all $x \in X \setminus \{0\}$ and for all $h \in X$, the limit

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

exists when $t \to 0\,(t \in \mathbb{R})$. The Banach space X is called *smooth* if its norm is Gâteaux-differentiable. The duality between strict convexity and smoothness reads as follows (see [1]):

If X^* is smooth, then X is strictly convex; if X^* is strictly convex, then X is smooth. Hence, if X is reflexive, then X is smooth (strictly convex) if and only if X^* is strictly convex (smooth).

Examples.

(1) If $X = l^p$ or $X = L^p\,(1 < p < \infty)$, then X and X^* are strictly convex (see [6, §121]).

(2) Let $X = \mathbb{R}^2$ equipped with the norm

$$\|(x, y)\| = (x^2 + y^2/4)^{1/2} + \frac{|y|}{2}\,,$$

then X is strictly convex, but X^* is not strictly convex (see [6, Aufgabe 121.2]).

(3) Each Hilbert space is strictly convex ([6, §121]).

The main results of this paper read as follows:

2.1. Theorem. *If X is a strictly convex Banach space and $T \in \mathcal{L}(X)$ is a partial isometry with contractive generalized inverse S, then*

$$S(X) = \{x \in X : \|Tx\| = \|x\|\}\,.$$

and

$$S_0T = ST$$

for each contractive generalized inverse S_0 of T.

2.2. Theorem. *If X and X^* are both strictly convex and if $T \in \mathcal{L}(X)$ is a partial isometry, then T has a unique contractive generalized inverse.*

Remark. As an immediate consequence of Theorem 2.2 we obtain [7, Corollary 3.3]: *a partial isometry on a Hilbert space has a unique contractive generalized inverse.*

Proof of Theorem 2.1. We have, by Proposition 1.3 (2) that $S(X) \subseteq \{x \in X : \|Tx\| = \|x\|\}$. Now let $x \in X$ and $\|Tx\| = \|x\|$. We can assume that $1 = \|x\| = \|Tx\|$. By (1.2) there are $u \in S(X)$ and $v \in N(T)$ such that $x = u + v$. In view of Proposition 1.3 (2) we have $\|Tu\| = \|u\|$, thus

$$1 = \|x\| = \|Tx\| = \|Tu\| = \|u\| .$$

We have to show that $v = 0$. Assume to the contrary that $v \neq 0$. Then $u \neq x$. Since X is strictly convex, it follows that $\|x + u\| < 2$. But

$$\begin{aligned} 1 &= \|Tu\| = \|T(u + \frac{1}{2}v)\| \leq \|T\| \, \|u + \frac{1}{2}v\| \\ &= \|u + \frac{1}{2}v\| = \frac{1}{2}\|2u + v\| = \frac{1}{2}\|x + u\| < 1 , \end{aligned}$$

a contradiction. Hence we have $v = 0$, and so $x = u \in S(X)$.
Now suppose that S_0 is also a contractive generalized inverse of T. Then $S_0(X) = \{x \in X : \|Tx\| = \|x\|\}$, thus $S(X) = S_0(X)$. It follows that $ST(X) = S_0T(X)$. Since $N(ST) = N(T) = N(S_0T)$, we get $ST = S_0T$. □

Proof of Theorem 2.2. Let S and S_0 be contractive generalized inverses of T. Theorem 2.1 shows that $ST = S_0T$, thus

$$T^*S^* = T^*S_0^* . \tag{2.1}$$

Since X^* is strictly convex and T^* is a partial isometry with contractive generalized inverses S^* and S_0^*, we obtain as above that

$$S^*T^* = S_0^*T^* . \tag{2.2}$$

From (2.1) and (2.2) we now obtain that

$$S^* = (S^*T^*)S^* = (S_0^*T^*)S^* = S_0^*(T^*S^*) = S_0^*T^*S_0^* = S_0^* ,$$

therefore $S = S_0$. □

2.3. Corollary. *If X^* is strictly convex and if $T \in \mathcal{L}(X)$ is a partial isometry with contractive generalized inverses S and S_0, then*

$$TS = TS_0 \quad \textit{and} \quad N(S) = N(S_0) .$$

Proof. As in the proof of Theorem 2.2 we obtain $S^*T^* = S_0^*T^*$, thus $(TS)^* = (TS_0)^*$. Hence $TS = TS_0$ and $N(S) = N(S_0)$. □

2.4. Corollary. *If X is strictly convex, $P \in \mathcal{L}(X)$, $P^2 = P$ and $\|P\| = 1$, then we have:*

(1) *$P(X) = \{x \in X : \|Px\| = \|x\|\}$;*
(2) *if $S \in \mathcal{L}(X)$, $PSP = P$, $SPS = S$ and $\|S\| = 1$, then*

$$S^2 = S, \; SP = P \quad \textit{and} \quad PS = S .$$

Proof. Since $\|P\| = 1$, P is a partial isometry on X and P is a contractive generalized inverse of itself. Thus, (1) follows from Theorem 2.1.
For the proof of (2) observe that S is a contractive generalized inverse of P, therefore; by Theorem 2.1, $SP = P^2 = P$. From this we get

$$S^2 = SPS(SP)S = SPSPS = SPS = S .$$

Therefore S is a partial isometry with contractive generalized inverses S and P. Theorem 2.1 shows now that $S^2 = PS$, hence $S = PS$. □

2.5. Corollary. *Suppose that X and X^* are strictly convex and that $Y \neq \{0\}$ is a closed and complemented subspace of X. Then there is at most one projection $P \in \mathcal{L}(X)$ such that $\|P\| = 1$ and $P(X) = Y$.*

Proof. Let P and Q be projections with $\|P\| = \|Q\| = 1$ and $P(X) = Q(X) = Y$. Then $P = QP$ and $Q = PQ$, thus $P = P^2 = P(QP)$ and $Q = Q^2 = Q(PQ)$. This shows that P is a partial isometry with contractive generalized inverses P and Q. By Theorem 2.2 it results that $P = Q$. □

2.6. Corollary. *Let $T \in \mathcal{L}(X)$ be a partial isometry.*

(1) *If X is strictly convex and T right invertible, then there is exactly one right inverse of T with norm* 1.
(2) *If X^* is strictly convex and T is left invertible, then there is exactly one left inverse of T with norm* 1.

Proof. (1) Let S and S_0 be right inverses of T such that $\|S\| = \|S_0\| = 1$. Then $TS = TS_0 = I$. It follows that S and S_0 are contractive generalized inverses of T. Using Theorem 2.1 we obtain $ST = S_0T$. Hence $S_0 = S_0TS_0 = STS_0 = S$.

(2) Let S and S_0 be left inverses of T with $\|S\| = \|S_0\| = 1$. Then S^* and S_0^* are right inverses of T^* with $\|S^*\| = \|S_0^*\| = 1$. By (1), $S^* = S_0^*$, therefore $S = S_0$. □

Definitions.

(1) An operator $U \in \mathcal{L}(X)$ is called *hermitian* if $\|\exp(itU)\| = 1$ for every $t \in \mathbb{R}$.
(2) Let $T \in \mathcal{L}(X)$. We will say that $T^+ \in \mathcal{L}(X)$ is the *Moore-Penrose inverse* of T if T^+ is a generalized inverse of T and the projections TT^+ and T^+T are hermitian.
(3) $T \in \mathcal{L}(X)$ is called an *MP-partial isometry* if T is a contraction and admits a contractive Moore-Penrose inverse (see [7]).

Remarks.

(1) A bounded linear operator has at most one Moore-Penrose inverse (see [8]).
(2) It is well-known that a bounded linear operator U on a *Hilbert* space is hermitian if and only if $U = U^*$ (see [2]).
(3) If $T \in \mathcal{L}(X)$ is an MP-partial isometry, then T is a partial isometry in the sense of Definition 1.2.

2.7. Corollary. *Let $T \in \mathcal{L}(X)$ be an MP-partial isometry and S a contractive generalized inverse of T.*

(1) *If X is strictly convex, then $ST = T^+T$.*
(2) *If X^* is strictly convex, then $TS = TT^+$.*
(3) *If X and X^* are strictly convex, then $S = T^+$.*

Proof. (1) follows from Theorem 2.1 and (2) follows from Corollary 2.3.

(3) is obtained from Theorem 2.2. □

Question. (see [7, p. 780]) Let $T \in \mathcal{L}(X)$ be an MP-partial isometry. Does

$$T^+(X) = \{x \in X : \|Tx\| = \|x\|\}\,?$$

The following example gives a negative answer to this question.

Example. Let $X = \mathbb{C}^2$ be equipped with the norm $\|(x, y)\| = \max\{|x|, |y|\}$ and consider the operator

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X)\,.$$

Then $T^2 = T$ and $\|T\| = 1$, therefore T ia a contractive generalized inverse for itself. It is easy to see that

$$\exp(itT) = \begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix},$$

thus T is hermitian. Therefore T is an MP-partial isometry and $T^+ = T$. Take $(x, y) = (1, 1)$, then $T(1, 1) = (1, 0)$ and $\|T(1, 1)\| = 1 = \|(1, 1)\|$, but $(1, 1) \notin T^+(X)$.

If $T \in \mathcal{L}(X) \setminus \{0\}$ has a generalized inverse S, then $S \neq 0$ and $\|T\| = \|TST\| \leq \|T\|^2\|S\|$, thus $\|T\|\,\|S\| \geq 1$.

We say that $T \in \mathcal{L}(X)$ is a *generalized partial isometry* if $T = 0$ or if T has a generalized inverse S such that $\|T\|\,\|S\| = 1$. Clearly, a partial isometry is a generalized partial isometry.
There are no restrictions on the norm for generalized partial isometries, every λI is a generalized partial isometry, where $\lambda \in \mathbb{C}$.

2.8. Corollary. *Suppose that $T \in \mathcal{L}(X) \setminus \{0\}$ is a generalized partial isometry.*

(1) *If X is strictly convex and S and S_0 are generalized inverses of T such that $\|T\|\,\|S\| = \|T\|\,\|S_0\| = 1$, then*

$$S(X) = \{x \in X : \|Tx\| = \|T\|\,\|x\|\}$$

and

$$ST = S_0T\,.$$

(2) *If X and X^* are both strictly convex, then there is exactly one generalized inverse S of T with $\|T\|\,\|S\| = 1$.*

Proof. Let $\alpha = \|T\|^{-1}$, $T_1 = \alpha T$, $S_1 = \frac{1}{\alpha}S$ and $S_2 = \frac{1}{\alpha}S_0$. Then $T_1S_iT_1 = T_1$, $S_iT_1S_i = S_i$, $\|T_1\| = 1$ and $\|S_i\| = 1\,(i = 1, 2)$. Hence T_1 is a partial isometry with contractive generalized inverses S_1 and S_2.

(1) Since $S(X) = S_1(X)$, we derive from Theorem 2.1 that $S(X) = \{x \in X : \|T_1x\| = \|x\|\} = \{x \in X : \|Tx\| = \|T\|\,\|x\|\}$. Furthermore we obtain $S_1T_1 = S_2T_1$, thus $ST = S_0T$.

(2) In view of Theorem 2.2 we get $S_1 = S_2$, hence $S = S_0$. □

For an operator $T \in \mathcal{L}(X) \setminus \{0\}$ the *reduced minimum modulus* is defined by

$$\gamma(T) = \inf\{\|Tx\| : x \in X,\, \mathrm{dist}(x, N(T)) = 1\}\,.$$

It is a classical fact that $\gamma(T) > 0$ if and only if $T(X)$ is closed, and that $\gamma(T) = \gamma(T^*)$ (see [4] or [6]).

A proof of the following proposition can be found in [7].

2.9. Proposition. *Let $T \in \mathcal{L}(X) \setminus \{0\}$ and $S \in \mathcal{L}(X)$ be a generalized inverse of T. Then*

$$\frac{1}{\|S\|} \leq \gamma(T) \leq \frac{\|TS\|\,\|ST\|}{\|S\|}\,.$$

If T is as in Proposition 2.9, then

$$\gamma(T) \geq \sup \left\{ \frac{1}{\|S\|} : S \in \mathcal{L}(X),\, TST = T,\, STS = S \right\}.$$

2.10. Corollary. *If $T \in \mathcal{L}(X) \setminus \{0\}$ is a generalized partial isometry then*

$$\gamma(T) = \|T\|.$$

Proof. Let S be a generalized inverse of T such that $\|T\|\,\|S\| = 1$. Then $\|TS\| \leq \|T\|\,\|S\| = 1$ and $\|ST\| \leq 1$, hence, by Proposition 2.9,

$$\|T\| = \frac{1}{\|S\|} \leq \gamma(T) \leq \frac{1}{\|S\|} = \|T\|.$$

□

We say that $T \in \mathcal{L}(X)$ is a *semi-Fredholm operator* if $T(X)$ is closed and $\dim N(T) < \infty$ or $\operatorname{codim} T(X) < \infty$.

The following result is well-known in the case of partial isometries on Hilbert spaces ([5, Problem 101]).

2.11. Theorem. *Let X be an arbitrary Banach space.*

(1) *If $T \in \mathcal{L}(X)\setminus\{0\}$ is a generalized partial isometry, $U \in \mathcal{L}(X)$ and $\dim N(T) < \dim N(U)$, then*

$$\|T - U\| \geq \|T\|.$$

(2) *If $T_1, T_2 \in \mathcal{L}(X)$ are generalized partial isometries and $\|T_1 - T_2\| < \min\{\|T_1\|, \|T_2\|\}$, then*

$$\dim N(T_1) = \dim N(T_2) \quad \textit{and} \quad \operatorname{codim} T_1(X) = \operatorname{codim} T_2(X).$$

Proof. (1) Since T is semi-Fredholm and $\|T\| = \gamma(T)$, we have $\|T\| \leq \|T - U\|$ by [4, Theorem V.1.6]. (2) follows immediately from (1) by duality. □

2.12. Corollary. *If the generalized partial isometry $T \in \mathcal{L}(X)$ is semi-Fredholm and $\dim N(T) \neq \operatorname{codim} T(X)$, then*

$$\|T - S\| \geq \min\{\|T\|,\, \|T\|^{-1}\}$$

for each generalized inverse S of T with $\|T\|\,\|S\| = 1$.

Proof. Assume to the contrary that $\|T-S\| < \min\{\|T\|,\, \|T\|^{-1}\} = \min\{\|T\|,\, \|S\|\}$. It follows from Theorem 2.11 that

$$\dim N(S) = \dim N(T) \quad \text{and} \quad \operatorname{codim} S(X) = \operatorname{codim} T(X).$$

But (1.2) shows that $\dim N(S) = \operatorname{codim} T(X)$, thus $\dim N(T) = \operatorname{codim} T(X)$, a contradiction. □

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