One-dimensional degenerate operators in L^p–spaces

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Abstract

We comprehensively study a one dimensional elliptic operator degenerating of first order at the boundary in an L^p setting. Here the coefficient of the drift term determines the regularity and the possible boundary conditions of the problem.

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1 Introduction

The study of degenerate elliptic operators started in the fifties and has been the object of many researches in a wide generality, since the seminal works of W. Feller [5], [6] in one dimension and J.J. Kohn and L. Nirenberg [12] in higher dimensions. Of particular interest is the case of degeneracy at the boundary for second-order elliptic operators, and naturally the results heavily depend on the behavior of the coefficients at the boundary, i.e., on the order and direction of degeneracy. The general setting is presented in the classical book [17]. A challenging borderline case occurs if the diffusion coefficients fully degenerate of first order in the normal direction to the boundary. In this case the drift term in normal direction is (roughly speaking) of the same order as the diffusion part and the sign and size of the drift coefficients play a crucial role. Here the model problem is given by the operator

$$L = -y\Delta + b_0 \cdot \nabla_x + b\partial_y$$

on the halfspace $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > 0\}$, where $b_0 \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$ and $\Delta = \Delta_x + \partial_y^2$. It was shown in [8] that $-L$ with domain $D(L) = \{u \in W^{1,p}(\mathbb{R}^n_+) \mid u(\cdot, 0) = 0,\ Lu \in L^p(\mathbb{R}^n_+)\}$ generates an analytic semigroup on $L^p(\mathbb{R}^n_+)$ if $b > -1/p$. (See also [11] and [14] for related results.) We note that in the case of tangential first order degeneracy (where $y\Delta$ is replaced by $y\Delta_x + \partial_y^2$ in the model problem) the corresponding result holds for all $b \in \mathbb{R}$, see [9] and [10]. The techniques of [8] heavily depend on the condition $b > -1/p$ which allows to control gradient terms by the operator L employing a Hardy type inequality which seems to fail for $b \le -1/p$ and without Dirichlet boundary conditions. In fact, it was noted in Example 2.11 of [8] that the generation result of this paper breaks down for $b \le -1/p$, already in the one dimensional case. To our knowledge, for $b \le -1/p$ so far there is no investigation of the

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domain of L and it is not known whether L generates an analytic semigroup on L^p, except for the symmetric case $b = -1$.

In this note we study the one dimensional case, i.e., the operator

$$A = -xD^2 + bD,$$

comprehensively. It turns out that in the case $b \le -1/p$ the behavior of A is rather complex and has several quite unexpected features. We want to use the gained insights in future work on the n–dimensional case. But we are convinced that the precise results and direct computations in the one dimensional case are of independent interest.

To simplify the analysis we work on the interval $(0, 1)$ and impose Dirichlet boundary conditions $u(1) = 0$, throughout. On the interval $(\varepsilon, 1)$, we can equip the operator A with Dirichlet ($u(\varepsilon) = 0$) or Neumann ($u'(\varepsilon) = 0$) boundary conditions at $x = \varepsilon$, letting finally $\varepsilon \to 0$. In Proposition 2.4 we first recall that the Dirichlet approximation yields a generator $-A_p$ with domain D_p on $L^p(0, 1)$ for all $p \in (1, \infty)$ and $b \in \mathbb{R}$, cf. [2]. The generated semigroup is analytic by Theorem 2.13, which essentially follows from known results for spaces of continuous functions in [3] and [14]. It is clear that A_p is a restriction of the operator A defined on the maximal domain

$$D_{p,\max} = \big\{u \in L^p(0, 1) \cap W^{2,p}_{\mathrm{loc}}((0, 1]) \,\big|\, Au \in L^p(0, 1), u(1) = 0\big\}.$$

For a more detailed investigation we further use the domains

$$\begin{aligned} D_{p,\mathrm{en}} &= \left\{u \in D_{p,\max} \,\Big|\, \int_0^1 x\,|u'|^2\,|u|^{p-2}\,dx < \infty\right\}, \\ D_{p,\mathrm{reg}} &= \big\{u \in W^{1,p}(0, 1) \,\big|\, xu'' \in L^p(0, 1),\ u(1) = 0\big\}. \end{aligned}$$

We write $D^0_{p,\mathrm{en}}$ and $D^0_{p,\mathrm{reg}}$ if we include the condition $\lim_{x\to 0^+} u(x) = 0$ in the respective space. Mainly using the explicit expression for A_p^{-1}, in Theorems 2.9, 2.12 we show that

$$\begin{aligned} &D_p = D_{p,\mathrm{reg}} = D_{p,\max} = D_{p,\mathrm{en}} && \text{if } \ b \le -1 - \tfrac{1}{p}, \\ &D_p = D_{p,\mathrm{reg}} = D_{p,\mathrm{en}} \ne D_{p,\max} && \text{if } \ -1 - \tfrac{1}{p} < b \le -1, \\ &D_p = D^0_{p,\mathrm{en}} = D^0_{p,\max} \supsetneqq D^0_{p,\mathrm{reg}} && \text{if } \ -1 < b \le -1/p, \\ &D_p = D^0_{p,\mathrm{en}} = D^0_{p,\max} = D^0_{p,\mathrm{reg}} && \text{if } \ b > -1/p. \end{aligned}$$

There are several striking features: For $b \le -1$ we cannot impose boundary conditions at $x = 0$ in the sense that any restriction of $-A$ to a proper subset of $D_{p,\max}$, respectively $D_{p,\mathrm{en}}$, cannot be a generator. Further, the regularity contained in the domain varies with b. In particular we loose the $W^{1,p}$ regularity precisely for $b \in (-1, -1/p]$ where we still keep the Dirichlet boundary condition at 0.

To see another surprising fact, let $u_\varepsilon \in W^{2,p}(\varepsilon, 1) \cap W^{1,p}_0(\varepsilon, 1)$ satisfy $Au_\varepsilon = 1$ and let $b \le -1/p$. In Proposition 3.1 we then show that the norms $\|u'_\varepsilon\|_p$ of the approximations explode as $\varepsilon \to 0$, although $D_p \subset W^{1,p}(0, 1)$ for $b \le -1$.

Motivated by these observations, in Section 3 we also study Neumann approximations of A on $(\varepsilon, 1)$, where we impose $u'_\varepsilon(\varepsilon) = 0$ instead of $u_\varepsilon(\varepsilon) = 0$. It turns out that for $b \le -1$ in the limit we obtain again A_p, but now with an approximation which is stable in $W^{1,p}$. For $b \in (-1, -1/p)$ the approximations are again stable in $W^{1,p}$, and now they give a new generator $-A_{p,N}$ of an analytic semigroup with domain $D(A_{p,N}) = D_{p,\mathrm{reg}}$. Compared to the operator

A_p constructed via Dirichlet approximations, we loose the boundary condition $u(0) = 0$, but gain the optimal regularity $u \in W^{1,p}(0,1)$. See Proposition 3.4 and Theorem 3.6 for these results. Guided by these results, we want to tackle the n–dimensional case in future work via Neumann type approximations.

We further note that the L^2 case was already studied, see [18] or [19]. The discussion in spaces of continuous functions falls into Feller's theory, which is reminiscent of the underlying diffusion model even in its terminology and is based on a classification of the endpoints. We refer to Section VI.4.c of [4] for a synthetic presentation of this theory. Concerning the operator A, the endpoint $x = 0$ is *entrance* for $b \le -1$, *regular* for $-1 < b < 0$ and *exit* for $b \ge 0$. According to Feller's theory, in order that $-A$ generates a semigroup in $C([0,1])$ no boundary conditions have to be imposed at 0 in the entrance case and the domain of the generator is the maximal one, whereas in the regular case general *elastic barrier conditions*, which include both Dirichlet and (degenerate) Neumann conditions, can be imposed and in the exit case Dirichlet boundary conditions can be imposed. We point out that our results are coherent with the case $C([0,1])$, which is formally obtained by letting $p \to \infty$.

2 The semigroup constructed via Dirichlet type approximations

After some preparations, we first recall in Proposition 2.4 the construction of the C_0-semigroup on $L^p(0,1)$ associated with A via approximations of Dirichlet type. We then study in detail the domain of its generator depending on p and b. These results finally allow to deduce the analyticity of the semigroup from know results for spaces of continuous functions.

Throughout, let $p \in (1,\infty)$ and $\varepsilon \in (0,1/2]$. For simplicity we consider real-valued functions (except when dealing with analytic semigroups where we employ the complexification of the operators on real spaces). We set

$$D_{p,\varepsilon} = W^{2,p}(\varepsilon,1) \cap W_0^{1,p}(\varepsilon,1).$$

We start by proving dissipativity estimates on $D_{p,\varepsilon}$ being independent of ε. Here and below, we put $u^* = u\,|u|^{p-2}$ for a function u.

Lemma 2.1. *Let $u \in D_{p,\varepsilon}$. Then*

$$\int_\varepsilon^1 Au\,u^*\,dx = (p-1)\int_\varepsilon^1 x\,|u'|^2\,|u|^{p-2}\,dx. \tag{2.1}$$

Proof. For $p \ge 2$, an integration by parts yields

$$\begin{aligned}\int_\varepsilon^1 Au\,u^*\,dx &= (p-1)\int_\varepsilon^1 x\,|u'|^2\,|u|^{p-2}\,dx + (b+1)\int_\varepsilon^1 u'u|u|^{p-2}\,dx \\ &= (p-1)\int_\varepsilon^1 x\,|u'|^2\,|u|^{p-2}\,dx + \frac{b+1}{p}\int_\varepsilon^1 \frac{d}{dx}\,|u|^p\,dx \\ &= (p-1)\int_\varepsilon^1 x\,|u'|^2\,|u|^{p-2}\,dx.\end{aligned}$$

For $p \in (1,2)$ one obtains this result by a regularization, see Theorem 3.1 in [15]. ☐

Lemma 2.2. *There exists a constant $\omega_p > 0$ such that for every $u \in D_{p,\varepsilon}$*

$$\omega_p \int_\varepsilon^1 |u|^p \, dx \le \int_\varepsilon^1 Au\, u^* \, dx. \tag{2.2}$$

Proof. Let $u \in D_{p,\varepsilon}$, $x \in (\varepsilon, 1)$, and $p \ge 2$. Using Hölder's inequality, we obtain

$$|u(x)|^{\frac{p}{2}} = -\int_x^1 \frac{d}{dy} |u(y)|^{\frac{p}{2}} \, dy = -\frac{p}{2} \int_x^1 u' \, u \, |u|^{\frac{p}{2}-2} \, dy \le \frac{p}{2} \left[\int_x^1 y \, |u'|^2 \, |u|^{p-2} \, dy \right]^{\frac{1}{2}} \left[\int_x^1 \frac{dy}{y} \right]^{\frac{1}{2}}.$$

Equality (2.1) now implies

$$\int_\varepsilon^1 |u(x)|^p \, dx \le \frac{p^2}{4} \int_0^1 |\log y| \, dy \int_\varepsilon^1 y(u')^2 |u|^{p-2} \, dy = \omega_p^{-1} \int_\varepsilon^1 Au\, u^* \, dx,$$

where $\omega_p^{-1} = \frac{p^2}{4(p-1)} \int_0^1 |\log y| \, dy$.

If $p \in (1,2)$, we start with $|u(x)|^{\frac{p}{2}} = \lim_{\delta \to 0^+} (|u(x)^2 + \delta|^{\frac{p}{4}} - \delta^{\frac{p}{4}})$ and estimate similarly. □

Corollary 2.3. *For any $u \in D_{p,\varepsilon}$ and $\lambda > -\omega_p$ we have*

$$(\lambda + \omega_p)\|u\|_p \le \|(\lambda + A)u\|_p.$$

Moreover, $(A, D_{p,\varepsilon})$ is invertible in $L^p(\varepsilon, 1)$ and $(A - \omega_p, D_{p,\varepsilon})$ is maximally accretive.

Proof. Let $u \in D_{p,\varepsilon}$ and $\lambda > -\omega_p$. Multiplying the equation $\lambda u + Au = f$ by u^*, integrating over $(\varepsilon, 1)$ and using (2.2) and Hölder's inequality, we deduce the first statement. The second assertion is easy to check and it implies the maximal accretivity. □

Proposition 2.4. *For every $b \in \mathbb{R}$ there exists a subspace $D_p \subseteq D_{p,\max}$ such that $-A_p = (-A, D_p)$ generates a strongly continuous positive semigroup $(T(t))_{t\ge0}$ with $\|T(t)\| \le e^{-\omega_p t}$ for $t \ge 0$. In particular, (A, D_p) is invertible. Moreover, there is a sequence $\varepsilon_n \to 0$ such that for each $u \in D_p$ there are $u_n \in D_{p,\varepsilon_n}$ with $Au_n = Au$ on $(\varepsilon_n, 1)$ and the 0 extension of u_n converges to u in $L^p(0,1)$ as $n \to \infty$.*

Proof. Our reasoning differs a bit from that in Theorem 3.1 of [2] since we avoid monotonicity arguments. This has the advantage that we can use the proof given here also for Neumann boundary conditions in Proposition 3.4, where we do not have monotonicity.

Take $f \in L^p(0,1)$ and $\lambda > 0$. For $n \in \mathbb{N}$ with $n \ge 2$, let $-A_n$ be the generator in $L^p(1/n, 1)$ obtained in Corollary 2.3 with $\varepsilon_n = 1/n$. We denote by E_n the extension operator by 0 from $L^p(1/n,1)$ to $L^p(0,1)$ and by R_n the restriction operator from $L^p(0,1)$ to $L^p(1/n,1)$. We set $u_n = u_n(\lambda, f) = E_n(\lambda + A_n)^{-1} R_n f$. Observe that $\|u_n\|_\infty \le \lambda^{-1} \|f\|_\infty$ if f is bounded. Corollary 2.3 says that $(\lambda + \omega_p)\|u_n\|_p \le \|f\|_p$. One can now deduce from standard elliptic regularity results that a subsequence of u_n converges weakly to a function u in $W^{2,p}_{\mathrm{loc}}((0,1])$ and strongly in $W^{1,p}_{\mathrm{loc}}((0,1])$ and that $\lambda u + Au = f$ on $(0,1)$. It then follows that u_n converges pointwise to u and $u(1) = 0$ (we do not relabel the subsequences).

The above subsequence may depend on f and λ, but for rational $\lambda_i > 0$ and a fixed dense sequence of bounded f_j we can find a diagonal sequence such that u_n converges for all $i, j \in \mathbb{N}$. Moreover, u_n converges to u in $L^p(0,1)$ as $n \to \infty$ for these λ_i and f_j by Lebesgue's theorem. By density, it follows that $u_n \to u$ in $L^p(0,1)$ as $n \to \infty$ for all $f \in L^p(0,1)$ and $i \in \mathbb{N}$. Corollary 2.3 thus implies

$$(\lambda + \omega_p)\|u\|_p \le \|f\|_p = \|\lambda u + Au\|_p \tag{2.3}$$

for all $f \in L^p(0,1)$ and $\lambda = \lambda_i$. From Corollary 2.3 and Vitali's Theorem (see Theorem A.5 in [1]), we further deduce that $E_n(\lambda + A_n)^{-1}R_n$ converges strongly to as $n \to \infty$ for all $\lambda > -\omega_p$. Hence, (2.3) holds for all $\lambda > -\omega_p$.

Let $\lambda, \mu > 0$ and $f \in L^p(0,1)$. We set $R(\lambda)f := u$. Clearly, $\lambda R(\lambda)$ is uniformly bounded. We observe that $R(\lambda) \ge 0$ since u_n, and hence u, are positive if $f \ge 0$. The resolvent equation and $R_nE_n = I$ yield

$$E_n(\mu + A_n)^{-1}R_n - E_n(\lambda + A_n)^{-1}R_n = (\lambda - \mu)E_n(\mu + A_n)^{-1}R_nE_n(\lambda + A_n)^{-1}R_n.$$

In the strong limit $n \to \infty$, we conclude that $\{R(\lambda); \lambda > 0\}$ is a pseudo resolvent. Take $v \in C^2(0,1)$ with support in $[a,b] \subset (0,1)$ and set $g = (\lambda + A)v \in L^p(0,1)$. We then have u_n as above. For $1/n \in (0,a)$, the function $v - u_n$ belongs to the kernel of $\lambda + A_n$, and hence $v = u_n$ for all $1/n \in (0,a)$. Therefore, $R(\lambda)g = v$ and $R(\lambda)$ has dense range. Since $R(\lambda)$ is injective, Proposition III.4.6 of [4] then shows that $R(\lambda) = (\lambda + A_p)^{-1}$ for an operator $-A_p$ with dense domain D_p which generates a contraction semigroup $T(\cdot)$ on $L^p(0,1)$ by the Hille–Yosida theorem. The positivity of the resolvent implies the positivity of the semigroup. From the above results we also infer that $D_p \subset W^{2,p}_{\mathrm{loc}}((0,1])$ and $A_pu = Au$ for $u \in D_p$. Finally, (2.3) implies that $A_p - \omega_p$ is accretive, and hence maximally accretive, so that $\|T(t)\| \le e^{-\omega_p t}$ for all $t \ge 0$. $\square$

To describe D_p we use the spaces

$$\begin{aligned}
D_{p,\max} &= \big\{u \in L^p(0,1) \cap W^{2,p}_{\mathrm{loc}}((0,1]) \,\big|\, Au \in L^p(0,1), u(1) = 0\big\},\\
D_{p,\mathrm{en}} &= \left\{u \in D_{p,\max} \,\Big|\, \int_0^1 x\,|u'|^2\,|u|^{p-2}\,dx < \infty\right\},\\
D_{p,\mathrm{reg}} &= \big\{u \in W^{1,p}(0,1) \,\big|\, xu'' \in L^p(0,1),\ u(1) = 0\big\}.
\end{aligned}$$

(Here the subscript 'en' stands for 'energy'.) The superscript 0 will indicate that we include the condition $\lim_{x\to 0^+} u(x) = 0$ in the respective space. We continue with several simple observations, omitting the straightforward proof of the next lemma.

Lemma 2.5. *The kernel of A on $D_{p,\max}$ is spanned by the functions $\varphi_b(x) = x^{b+1} - 1$ if $b \ne -1$ and by $\varphi_b(x) = \log x$ if $b = -1$. The limit of $\varphi_b(x)$ as $x \to 0$ exists if and only if $b > -1$, and it is then equal to -1. Moreover,*

$$\begin{aligned}
\varphi_b \in L^p(0,1) \iff \varphi_b \in D_{p,\max} &\iff b > -1 - \frac{1}{p},\\
\varphi_b \in D_{p,\mathrm{en}} &\iff b > -1,\\
\varphi_b \in D_{p,\mathrm{reg}} &\iff b > -\frac{1}{p}.
\end{aligned}$$

Lemma 2.6. *Let $b \in \mathbb{R}$. We have $D_p \subseteq D_{p,\mathrm{en}} \subseteq D_{p,\max}$.*

Proof. We only have to show the first inclusion. Let $u \in D_p$. From Proposition 2.4 we know that u is the limit in $L^p(0,1)$ and in $W^{1,p}_{\mathrm{loc}}((0,1])$ of (0 extensions of) functions $u_n \in D_{p,\varepsilon_n}$ with $Au_n = f$. Equality (2.1), Hölder's inequality and Corollary 2.3 yield

$$(p-1)\int_{\varepsilon_n}^1 x\,|u_n'|^2\,|u_n|^{p-2}\,dx = \int_{\varepsilon_n}^1 Au_n\,u_n^*\,dx \le \|Au_n\|_p\,\|u_n\|_p^{p-1} \le \omega_p^{1-p}\,\|f\|_p^p.$$

Letting $n \to \infty$, we deduce from Fatou's Lemma that

$$(p-1)\int_0^1 x\,|u'|^2|u|^{p-2}\,dx \le \omega_p^{1-p}\,\|f\|_p^p < +\infty. \qquad \square$$

Lemma 2.5 implies that for certain values of b some of the above inclusions are in fact equalities.

Lemma 2.7. *We have $D_p = D_{p,\mathrm{en}} = D_{p,\max}$ for $b \le -1-\frac{1}{p}$, and $D_p = D_{p,\mathrm{en}} \ne D_{p,\max}$ for $-1-\frac{1}{p} < b \le -1$.*

Proof. Let $b \le -1-\frac{1}{p}$. Then A is injective on $D_{p,\max}$ by Lemma 2.5. Since it is bijective on D_p by Proposition 2.4, the asserted equalities now follow from Lemma 2.6. Similarly, A is injective on $D_{p,\mathrm{en}}$ for $-1-\frac{1}{p} < b \le -1$ and hence $D_p = D_{p,\mathrm{en}}$. On the other hand, $\varphi_b \in D_{p,\max}\setminus D_{p,\mathrm{en}}$ for these values of b. $\square$

Since (A, D_p) is invertible for any b, we obtain an explicit representation of the functions in D_p solving the ordinary differential equation $Au = f$ for $f \in L^p(0,1)$. This formula will be crucial for our further results. To this aim, we first observe that the function

$$u_c(x) = \begin{cases} \dfrac{c}{b+1}(x^{b+1}-1) - \dfrac{1}{b+1}\displaystyle\int_x^1 f(y)\,dy + \dfrac{x^{b+1}}{b+1}\displaystyle\int_x^1 \frac{f(y)}{y^{b+1}}\,dy, & \text{if } b \ne -1,\\ c\log x - \displaystyle\int_x^1 f(y)\log\frac{y}{x}\,dy, & \text{if } b = -1, \end{cases} \tag{2.4}$$

where $c \in \mathbb{R}$, is the general solution of the equation $Au = f$ satisfying $u(1) = 0$. In the following results we find the value of c that gives the solution in D_p. We treat the cases $b \le -1$ and $b > -1$ separately.

Proposition 2.8. *Let $b \le -1$. For every $f \in L^p(0,1)$, the unique solution in D_p of $Au = f$ is given by formula (2.4) with c replaced by*

$$\bar{c} = -\int_0^1 \frac{f(x)}{x^{b+1}}\,dx.$$

Moreover, $D_p \subset W^{1,p}(0,1)$ and for every $u \in D_p$ one has $\lim\limits_{x\to 0^+} x^{-b}u'(x) = 0$.

Proof. Assume first that $b < -1$. Let $f \in L^p(0,1)$ be fixed. By Proposition 2.4 the unique solution $u \in D_p$ of $Au = f$ is the limit of the solutions $v_\varepsilon \in D_{p,\varepsilon}$ of $Av_\varepsilon = f$. (We write $\varepsilon \to 0$ instead of $\varepsilon_n \to 0$, for simplicity.) Using (2.4) and imposing $u_c(\varepsilon) = 0$, we find

$$c_\varepsilon := \frac{1}{\varepsilon^{b+1}-1}\left(\int_\varepsilon^1 f(x)\,dx - \varepsilon^{b+1}\int_\varepsilon^1 \frac{f(x)}{x^{b+1}}\,dx\right), \tag{2.5}$$

so that $v_\varepsilon = u_{c_\varepsilon}$. We thus have to show that $\lim\limits_{\varepsilon\to 0} c_\varepsilon = \bar{c}$. It holds

$$c_\varepsilon - \bar{c} = \frac{1}{\varepsilon^{b+1}-1}\left(\int_\varepsilon^1 f(x)\,dx + \varepsilon^{b+1}\int_0^\varepsilon \frac{f(x)}{x^{b+1}}\,dx - \int_0^1 \frac{f(x)}{x^{b+1}}\,dx\right).$$

Moreover, ε^{b+1} tends to infinity as $\varepsilon \to 0$ since $b+1<0$, and

$$\varepsilon^{b+1}\int_0^\varepsilon \frac{|f(x)|}{x^{b+1}}\,dx \le \int_0^1 |f(x)|\,dx.$$

So we derive the claim, implying

$$u(x) = -\frac{x^{b+1}}{b+1}\int_0^x \frac{f(y)}{y^{b+1}}\,dy - \frac{1}{b+1}\int_x^1 f(y)\,dy + \frac{1}{b+1}\int_0^1 \frac{f(y)}{y^{b+1}}\,dy.$$

It follows that

$$u'(x) = -x^b\int_0^x \frac{f(y)}{y^{b+1}}\,dy = -\int_0^1 \frac{f(sx)}{s^{b+1}}\,ds, \tag{2.6}$$

and hence

$$\|u'\|_p \le \int_0^1 \frac{1}{s^{b+1}}\left(\int_0^1 |f(sx)|^p\,dx\right)^{\frac{1}{p}} ds \le \|f\|_p \int_0^1 \frac{1}{s^{b+1+\frac{1}{p}}}\,ds < +\infty$$

because of $b+1+\frac{1}{p} < \frac{1}{p} < 1$. From (2.6) we further deduce that $\lim\limits_{x\to 0^+} x^{-b}u'(x) = 0$.

Let $b=-1$. Then $c_\varepsilon = -\int_\varepsilon^1 f(y)\,dy + (\log\varepsilon)^{-1}\int_\varepsilon^1 f(y)\log y\,dy$. Hölder's inequality yields

$$\left|\int_\varepsilon^1 f(x)\log x\,dx\right| \le \|f\|_p\left(\int_0^1 |\log x|^{p'}\,dx\right)^{\frac{1}{p'}} < +\infty$$

so that again c_ε tends to $\bar{c}$. Equation (2.6) now also holds for $b=-1$, and the remaining assertions are consequences of the previous computations with $b=-1$. □

We point out that for $b \le -1$ the solutions in D_p of the elliptic equation $Au=f$ enjoy the best regularity one may expect, thanks to $D_p \subset W^{1,p}(0,1)$. However, D_p contains functions which do not vanish at 0 though each $u \in D_p$ is limit of solutions to Dirichlet problems. We summarize the results obtained so far in the following theorem.

Theorem 2.9. *If $b \le -1$, then $D_p = D_{p,\mathrm{reg}}$. Moreover, $x^{-b}u'(x)$ converges to 0 as $x\to 0^+$ for $u \in D_p$. We further have*

$$\begin{aligned} &D_p = D_{p,\mathrm{reg}} = D_{p,\max} = D_{p,\mathrm{en}} \quad \text{if } \ b \le -1-\tfrac{1}{p},\\ &D_p = D_{p,\mathrm{reg}} = D_{p,\mathrm{en}} \ne D_{p,\max} \quad \text{if } \ -1-\tfrac{1}{p} < b \le -1. \end{aligned}$$

No restriction of $-A$ to a proper subspace of $D_{p,\max}$ (if $b \le -1-\frac{1}{p}$), resp. of $D_{p,\mathrm{en}}$ (if $-1-\frac{1}{p} < b \le -1$) is a generator. In this sense, we cannot impose boundary conditions at $x=0$ if $b \le -1$.

Proof. Proposition 2.8 yields $D_p \subset D_{p,\mathrm{reg}}$. On the other hand, A is injective on $D_{p,\mathrm{reg}}$ by Lemma 2.5 and $b \leq -1$, so that $D_p = D_{p,\mathrm{reg}}$. The other results have been shown before or follow easily. □

To treat the case $b > -1$, we need further preparations.

Lemma 2.10. *For $u \in D_{p,\max}$ and $b > -1$, the following assertions hold.*

(i) If $-1 < b < -1/p$, then $x^{-b}u'(x)$ converges as $x \to 0^+$.

(ii) If $b = -1/p$ then $x^{-b}(\log x)^{-1}u'$ is bounded on $(0, 1/2)$.

(iii) If $b > -\frac{1}{p}$, then $u \in D_{p,\mathrm{reg}}$.

Proof. Let $f = Au$. For a suitable $c \in \mathbb{R}$, equation (2.4) yields

$$x^{-b}u'(x) = c + \int_x^1 f(y)y^{-b-1}\,dy. \tag{2.7}$$

If $-1 < b < -1/p$, by Hölder's inequality the last integral converges as $x \to 0$ and therefore

$$\lim_{x\to 0} x^{-b}u'(x) = c + \int_0^1 f(y)y^{-b-1}\,dy.$$

This proves (i). For $b = -1/p$, the integral in (2.7) diverges at most logarithmically and (ii) follows. If $b > -1/p$, then the functions x^b and v belong to $L^p(0,1)$ where $v(x) = x^b \int_x^1 f(y)y^{-b-1}\,dy$. In fact, extending f to zero outside $[0,1]$, we derive $v(x) = \int_1^\infty f(sx)s^{-b-1}\,ds$ and then

$$\|v\|_p \leq \int_1^\infty s^{-b-1}\left(\int_0^\infty |f(sx)|^p\,dx\right)^{\frac{1}{p}} ds \leq \|f\|_p \int_1^\infty s^{-b-1-\frac{1}{p}}\,ds < \infty,$$

using the integral version of Minkowski's inequality. Hence, $u' = cx^b + v \in L^p(0,1)$. □

For $b > -1$ we can now show that the generator $-A_p = (-A, D_p)$ incorporates Dirichlet boundary conditions at $x = 0$.

Proposition 2.11. *Let $b > -1$. For every $f \in L^p(0,1)$, the unique solution u in D_p of $Au = f$ is given by formula (2.4) with c replaced by*

$$\hat{c} = -\int_0^1 f(x)\,dx.$$

Moreover, $u(x)$ tends to 0 as $x \to 0$.

Proof. We proceed as in the proof of Proposition 2.8 and show that c_ε tends to $\hat{c}$, cf. (2.5). To this aim, we observe that

$$x^{b+1}\int_x^1 \frac{|f(y)|}{y^{b+1}}\,dy \leq x^{b+1}\|f\|_p \left(\int_x^1 y^{-(b+1)p'}\,dy\right)^{\frac{1}{p'}} = \|f\|_p \left(x\int_x^1 t^{(b+1)p'-2}\,dt\right)^{\frac{1}{p'}},$$

which implies $\lim_{x\to 0} x^{b+1} \int_x^1 \frac{|f(y)|}{y^{b+1}}\, dy = 0$. Therefore $\lim_{\varepsilon\to 0} c_\varepsilon = \hat{c}$ by (2.5), and (2.4) yields

$$u(x) = -\frac{x^{b+1}}{b+1} \int_0^1 f(y)\, dy + \frac{1}{b+1} \int_0^x f(y)\, dy + \frac{x^{b+1}}{b+1} \int_x^1 \frac{f(y)}{y^{b+1}}\, dy. \tag{2.8}$$

All terms in this representation tend to 0 as $x \to 0$. $\square$

We can now complete the description of D_p for the case $b > -1$.

Theorem 2.12. *If $b \in (-1, -1/p]$, we have $D_p = D^0_{p,\mathrm{en}} = D^0_{p,\max} \supsetneqq D^0_{p,\mathrm{reg}}$. If $b > -1/p$, we have $D_p = D^0_{p,\mathrm{en}} = D^0_{p,\max} = D^0_{p,\mathrm{reg}}$.*

Proof. Let $b > -1$. Proposition 2.11 and Lemma 2.6 imply that $D_p \subset D^0_{p,\mathrm{en}} \subset D^0_{p,\max}$, and the equalities follow from the injectivity of A on $D^0_{p,\max}$, see Lemma 2.5. It is clear that $D^0_{p,\mathrm{reg}} \subset D^0_{p,\max}$. For $b > -1/p$, the converse holds due to Lemma 2.10(iii). Finally, let $-1 < b \le -\frac{1}{p}$. We choose $f = 1$ in (2.8). Then $u'(x) = -x^b + x^b(x^{-b} - 1)/b$ which does not belong to $L^p(0,1)$. $\square$

We finally show that the semigroup generated by $(-A, D_p)$ is analytic in $L^p(0,1)$, using the corresponding result in spaces of continuous functions and an interpolation argument.

Theorem 2.13. *The semigroup $(T(t))_{t\ge 0}$ constructed in Proposition 2.4 with generator $(-A, D_p)$ is analytic in $L^p(0,1)$.*

Proof. **Case $b \le -1$.** Let

$$D_\infty = \{u \in C^1([0,1]) \cap C^2(]0,1]) \,\big|\, u(1) = 0, \lim_{x\to 0} x u''(x) = 0\}. \tag{2.9}$$

Proposition 3.1 and Theorem 3.5 of [14] imply that $(-A, D_\infty)$ generates an analytic semigroup $T(z)$, $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : |\mathrm{Arg}\, z| < \pi/2\}$, in $C([0,1])$. Since $D_\infty \subset D_p$ by Theorem 2.9, the resolvents of A in $L^p(0,1)$ and $C([0,1])$, hence the semigroups, coincide on $C([0,1])$. The assertion follows from the Stein interpolation theorem, as stated in Theorem 6.5 and the subsequent observations of [13], once we have proved that $T(z)$ can be extended to an analytic semigroup in $L^\infty(0,1)$ which is bounded near 0. Let $t > 0$, $f \in L^\infty(0,1)$ and (f_n) be a sequence of continuous functions such that $f_n \to f$ in $L^p(0,1)$ and $\|f_n\|_\infty \le \|f\|_\infty$. Then $\|T(t)f_n\|_\infty \le \|f\|_\infty$ and $T(t)f_n$ is bounded in $C^1([0,1])$, since $T(t)$ is analytic in $C([0,1])$ and D_∞ embeds continuously into $C^1([0,1])$. Because $T(t)f_n \to T(t)f$ in $L^p(0,1)$, we infer that $T(t)f \in C([0,1])$. This shows that for every $t > 0$, $T(t)$ maps $L^\infty(0,1)$ into $C([0,1])$ and that $\|T(t)f\|_\infty \le \|f\|_\infty$. Therefore $T(z) = T(z-\varepsilon)T(\varepsilon)$, $Re\, z \ge 2\varepsilon$, defines an analytic semigroup in $L^\infty(0,1)$ for $z \in \mathbb{C}_+$ which is bounded in a neighbourhood of 0 .

Case $b > -1$. Here we impose Dirichlet boundary conditions at $x = 0$ and therefore we work in $C_0(0,1)$. We set

$$D^0_\infty = \{u \in C_0(0,1) \cap C^2(]0,1]) \,\big|\, Au \in C_0(0,1)\}.$$

Theorem 3.1 of [3] implies that $(-A, D^0_\infty)$ generates an analytic semigroup in $C_0(0,1)$. (We remark that though Theorem 3.1 in [3] states the result in $C([0,1])$, the corresponding result in $C_0(0,1)$ is an immediate consequence.) Since $D^0_\infty \subset D_p$ by Theorem 2.12, it follows that the resolvents of A in $L^p(0,1)$ and $C([0,1])$ coincide on $C([0,1])$, hence the semigroups too. Then the assertion is a consequence of the Stein interpolation theorem, as above, using the compactness of the embedding of D^0_∞ into $C_0(0,1)$ which follows from Lemma 3.2 of [3]. $\square$

3 The semigroup constructed via Neumann type approximations

By Theorems 2.9 and 2.12 the semigroup $T(\cdot)$ constructed via the Dirichlet approximation solves the parabolic problem for A with full space regularity (i.e., $T(t)u_0 \in D_{p,\mathrm{reg}}$ for all $t > 0$ and $u_0 \in L^p(0,1)$) if and only if $b \le -1$ or $b > -1/p$. For the intermediate range $b \in (-1, -1/p]$ we have less regularity, but Dirichlet boundary conditions. Hence, there could exist another semigroup solving the parabolic problem for A with full space regularity but different boundary conditions at $x = 0$ if $b \in (-1, -1/p]$. In fact, we shall construct such a semigroup for $b \in (-1, -1/p)$ by means of a *Neumann* type approximation. For such b this approximation procedure thus behaves differently, and in some sense better than the Dirichlet type approximation of the previous section. The next result also indicates another drawback of the Dirichlet approximation: It is unstable in $W^{1,p}$ for $b \le -\frac{1}{p}$ although all the approximations $u_n = A_n^{-1} f$ converge in $L^p(0,1)$ to functions in $W^{1,p}(0,1)$.

Proposition 3.1. *Let $u_\varepsilon \in W^{2,p}(\varepsilon, 1)$ solve $Au_\varepsilon = f$ with $u_\varepsilon(\varepsilon) = u_\varepsilon(1) = 0$ for $\varepsilon \in (0,1)$. These functions are uniformly bounded in $W^{1,p}(0,1)$ (after extension by 0) for each $f \in L^p(0,1)$ if and only if $b > -1/p$.*

Proof. Let f and u_ε be given as in the statement. We first take $b \neq -1$. Formula (2.4) yields

$$u_\varepsilon'(x) = c_\varepsilon x^b + x^b \int_x^1 \frac{f(y)}{y^{b+1}}\, dy.$$

The constant c_ε is determined by the boundary condition $u_\varepsilon(\varepsilon) = 0$ and is given by (2.5), so that we obtain

$$u_\varepsilon'(x) = \frac{x^b}{\varepsilon^{b+1} - 1} \left(\int_\varepsilon^1 f(y)\, dy - \varepsilon^{b+1} \int_\varepsilon^1 \frac{f(y)}{y^{b+1}}\, dy \right) + x^b \int_x^1 \frac{f(y)}{y^{b+1}}\, dy \tag{3.1}$$

for $x \in (\varepsilon, 1)$. We now take $f = 1$. For $b \neq 0$, it follows

$$u_\varepsilon'(x) = \frac{1}{b} + x^b \, \frac{b - b\varepsilon - \varepsilon + 1}{b(\varepsilon^{b+1} - 1)} =: \frac{1}{b} + \beta(\varepsilon) x^b.$$

Observe that $\beta(\varepsilon)$ tends to $-\frac{b+1}{b}$ as $\varepsilon \to 0$ if $b > -1$ and that it behaves like $\frac{b+1}{b}\, \varepsilon^{-b-1}$ as $\varepsilon \to 0$ if $b < -1$. Moreover,

$$\|\beta(\varepsilon) x^b\|_p = \begin{cases} |bp+1|^{-\frac{1}{p}} \beta(\varepsilon) |1 - \varepsilon^{bp+1}|^{\frac{1}{p}}, & \text{if } b \neq -\frac{1}{p}, \\ \beta(\varepsilon) |\log \varepsilon|^{\frac{1}{p}}, & \text{if } b = -\frac{1}{p}. \end{cases}$$

As a result, for $b < -1$ the norms $\|u_\varepsilon'\|_p$ behave like $\varepsilon^{-b-1} \varepsilon^{b+1/p}$ as $\varepsilon \to 0$, and hence tend to infinity. For $b \in (-1, -1/p)$ or $b = -1/p$, we have $\|u_\varepsilon'\|_p \sim \varepsilon^{b+1/p}$ or $\|u_\varepsilon'\|_p \sim |\log \varepsilon|^{1/p}$, respectively, and the norms again tend to infinity as $\varepsilon \to 0$.

We next consider the case $b = -1$. For $f = 1$, we deduce from (2.4)

$$u_\varepsilon'(x) = \frac{c_\varepsilon}{x} + \frac{1}{x} \int_x^1 \log y \, dy = \frac{c_\varepsilon}{x} - \frac{1}{x} + 1 - \log x,$$

$$0 = u_\varepsilon(\varepsilon) = c_\varepsilon \log \varepsilon - \int_\varepsilon^1 \log \frac{y}{\varepsilon}\, dy = c_\varepsilon \log \varepsilon + 1 - \varepsilon + \log \varepsilon,$$

arriving at

$$u_\varepsilon'(x) = \frac{1}{x} \frac{\varepsilon - 1 - 2\log \varepsilon}{\log \varepsilon} + 1 - \log x$$

for $x \in (\varepsilon, 1)$. Hence, $\|u_\varepsilon'\|_p \sim \varepsilon^{\frac{1}{p}-1}$ tends to infinity as $\varepsilon \to 0$, also for $b = -1$.

To treat the case $b > -1/p$, we go back to (3.1) for any given $f \in L^p(0,1)$. We first note that the norms of x^b in $L^p(\varepsilon, 1)$ converge to $(bp+1)^{-1}$ and $\varepsilon^{b+1} \to 0$ as $\varepsilon \to 0$. Moreover,

$$\left| \varepsilon^{b+1} \int_\varepsilon^1 \frac{f(y)}{y^{b+1}}\, dy \right| \le \|f\|_1 \le \|f\|_p.$$

The p-norm of last summand in (3.1) can be estimated as follows. Extend f by 0 to $\mathbb{R}_+$ and write

$$x^b \int_x^\infty \frac{f(y)}{y^{b+1}}\, dy = \int_1^\infty \frac{f(sx)}{s^{b+1}}\, ds.$$

Using Minkowski' inequality and Fubini's theorem, we further compute

$$\begin{aligned}\Big(\int_0^\infty \Big| \int_1^\infty \frac{f(sx)}{s^{b+1}}\, ds \Big|^p dx \Big)^{1/p} &\le \int_1^\infty \frac{ds}{s^{b+1}} \Big(\int_0^\infty |f(sx)|^p\, dx \Big)^{1/p} \\ &= \|f\|_p \int_1^\infty \frac{ds}{s^{b+1+1/p}} = \frac{\|f\|_p}{b+1/p}.\end{aligned}$$

As a consequence, $\|u_\varepsilon'\|_p \le c\,\|f\|_p$ for a constant $c > 0$ and all $\varepsilon \in (0, 1/2)$ and $f \in L^p(0,1)$. Since $u_\varepsilon(1) = 0$, we further have $|u_\varepsilon(x)| \le \|u_\varepsilon'\|_1 \le \|u_\varepsilon'\|_p$ for all $x \in (\varepsilon, 1)$, so that u_ε is bounded in $W^{1,p}(\varepsilon, 1)$ by $c\,\|f\|_p$ if $b > -1/p$, as asserted. □

Motivated by the above observations we now study approximating problems with Neumann boundary conditions, employing the domains

$$D_{p,\varepsilon}^N = \{u \in W^{2,p}(\varepsilon, 1) \,|\, u'(\varepsilon) = 0,\ u(1) = 0\}.$$

Lemma 3.2. *Let $u \in D_{p,\varepsilon}^N$ and $b \in \mathbb{R}$. We have*

$$\int_\varepsilon^1 Au\, u^*\, dx = (p-1) \int_\varepsilon^1 x\, |u'|^2\, |u|^{p-2}\, dx - \frac{b+1}{p}\, |u(\varepsilon)|^p. \tag{3.2}$$

Let $b \le -1$. Then $(A, D_{p,\varepsilon}^N)$ is accretive and

$$\int_\varepsilon^1 x\, |u'|^2\, |u|^{p-2}\, dx \le \frac{\|f\|_p\, \|u\|_p^{p-1}}{p-1}. \tag{3.3}$$

Proof. The estimate (3.2) can be shown exactly as (2.1), and it implies the remaining results for $b \le -1$. □

The next result shows that the Neumann approximation is stable in $W^{1,p}$ if $b < -1/p$.

Lemma 3.3. *Let $b < -1/p$ and $f \in L^p(0,1)$. There exists $u_\varepsilon \in D^N_{p,\varepsilon}$ such that $Au_\varepsilon = f$. It then holds*

$$\|u_\varepsilon\|_p \le \frac{1}{-b-\frac{1}{p}}\,\|f\|_p \quad \text{and} \quad \|u_\varepsilon'\|_p \le \frac{1}{-b-\frac{1}{p}}\,\|f\|_p. \tag{3.4}$$

Moreover, for $b \in (-1,-1/p)$ we have

$$\int_\varepsilon^1 x\,|u'|^2\,|u|^{p-2}\,dx \le \frac{1}{p-1}\Big(\|f\|_p\,\|u\|_p^{p-1} + \frac{b+1}{p(-b-\frac{1}{p})^p}\,\|f\|_p^p\Big). \tag{3.5}$$

Proof. Let $b < -1/p$ and $f \in L^p(0,1)$. The first assertion is clear. We drop the subscript ε in the rest of the proof. We multiply the equation $Au = f$ by $(u')^* = |u'|^{p-2}u'$ and integrate over $(\varepsilon,1)$. An integration by parts then yields

$$\begin{aligned}\int_\varepsilon^1 fu'\,|u'|^{p-2}\,dx &= -\int_\varepsilon^1 xu''\,u'\,|u'|^{p-2}\,dx + b\int_\varepsilon^1 u'\,u'\,|u'|^{p-2}\,dx\\ &= -\frac{1}{p}\int_\varepsilon^1 x\,\frac{d}{dx}\,|u'|^p\,dx + b\,\|u'\|_p^p\\ &= \Big(b+\frac{1}{p}\Big)\,\|u'\|_p^p - \frac{1}{p}\,|u'(1)|^p\end{aligned}$$

since $u'(\varepsilon) = 0$. We thus obtain

$$-\Big(b+\frac{1}{p}\Big)\,\|u'\|_p^p \le -\int_\varepsilon^1 fu'\,|u'|^{p-2}\,dx \le \|f\|_p\,\|u'\|_p^{p-1}$$

by means of Hölder's inequality. Because of $u(1) = 0$, we also have

$$|u(x)| \le \|u'\|_1 \le \|u'\|_p \le \frac{1}{-b-\frac{1}{p}}\,\|f\|_p$$

for all $x \in (\varepsilon,1)$. Therefore, (3.4) holds. Combining the above estimate with (3.2) and Hölder's inequality, we arrive at (3.5). □

Proposition 3.4. *a) Let $b < -1/p$. Then $A_{p,N} = (A, D_{p,\mathrm{reg}})$ is invertible. Moreover, $u = A_{p,N}^{-1}f$ satisfies the estimates (3.3)–(3.5) (depending on b) for $\varepsilon = 0$ and each $f \in L^p(0,1)$. In particular, the Neumann type approximation of $A_{p,N}$ is stable in $W^{1,p}$, cf. Proposition 3.1. Finally, for $b \le -1$ the operator $A_{p,N}$ is maximally accretive.*

b) If $b \ge -1/p$, there are $f \in L^p(0,1)$ such that $\|u_\varepsilon\|_p \to \infty$ as $\varepsilon \to 0$ for the functions $u_\varepsilon \in D^N_{p,\varepsilon}$ with $Au_\varepsilon = f$. Hence, the Neumann approximation does not work in this case.

Proof. a) Let $b < -1/p$. As in the proof of Proposition 2.4 we obtain a function $u \in W^{2,p}_{\mathrm{loc}}((0,1])$ with $u(1) = 0$ satisfying $Au = f$ on $(0,1)$. It is the limit in $W^{1,p}_{\mathrm{loc}}((0,1])$ of sequence of functions u_ε as in Lemma 3.2 and 3.3. (We note that the uniform bound needed in the proof of Proposition 2.4 here follows from (3.4) since $|u_\varepsilon(x)| \le \|u_\varepsilon'\|_p \le c\,\|f\|_p$.) Fatou's lemma then implies that the estimates (3.3)–(3.5) hold for u on $(0,1)$. In particular, u belongs to $W^{1,p}(0,1)$ and hence to $D_{p,\mathrm{reg}}$. Since A is injective on $D_{p,\mathrm{reg}}$ by Lemma 2.5, we obtain the invertibility of $A_{p,N} = (A, D_{p,\mathrm{reg}})$. Moreover, for $b \le -1$ the operator $A_{p,N}$ is accretive due to (3.2).

b) Let $b > -1/p$. The function $f(x) = x^b$ then belongs to $L^p(0,1)$. We consider again the function $u_\varepsilon \in D^N_{p,\varepsilon}$ such that $Au_\varepsilon = f$. Due to formula (2.4), this function is given by

$$\begin{aligned} u_\varepsilon(x) &= \frac{c_\varepsilon}{b+1}(x^{b+1}-1) - \frac{1}{b+1}\int_x^1 y^b\,dy + \frac{x^{b+1}}{b+1}\int_x^1 \frac{y^b}{y^{b+1}}\,dy \\ &= \frac{c_\varepsilon}{b+1}(x^{b+1}-1) + \frac{x^{b+1}-1}{(1+b)^2} - \frac{x^{b+1}\log x}{b+1} \end{aligned}$$

Since $0 = u_\varepsilon'(\varepsilon) = c_\varepsilon \varepsilon^b - \varepsilon^b \log\varepsilon$, we infer that $c_\varepsilon = \log\varepsilon$ and

$$u_\varepsilon(x) = \frac{\log\varepsilon}{b+1}(x^{b+1}-1) + \frac{x^{b+1}-1}{(1+b)^2} - \frac{x^{b+1}\log x}{b+1}.$$

As a result, the norms $\|u_\varepsilon\|_p \ge c\,(1 + |\log\varepsilon|)$ explode as $\varepsilon \to 0$.

Finally, let $b = -1/p$. We now consider $f(x) = x^{-\frac{1}{p}}(-\log\frac{x}{2})^{-1}$. Observe that f belongs to $L^p(0,1)$. Proceeding as for $b > -1/p$, we obtain that $c_\varepsilon = C + \frac{1}{2}\log(-\log(\frac{\varepsilon}{2}))$ and that $\|u_\varepsilon\|_p$ behaves like c_ε and thus tends to infinity as $\varepsilon \to 0$. $\square$

Remark 3.5. The case $b = -1/p$ in some sense borderline. Indeeed, Dirichlet conditions at 0 can be imposed, but the domain D_p of the generator is larger than $D^0_{p,\mathrm{reg}}$ for $b = -1/p$ and it is equal to $D^0_{p,\mathrm{reg}}$ for $b > -1/p$. Moreover, for $b = -1/p$ the Neumann approximation looses its stability which holds for $b < -1/p$.

Theorem 3.6. *For $b < -1/p$, the operator $-A_{p,N} = (-A, D_{p,\mathrm{reg}})$ generates a positive analytic C_0–semigroup $T_N(\cdot)$ on $L^p(0,1)$. For $b \le -1$, the operator $-A_{p,N}$ coincides with the generator $-A_p$ from Theorem 2.9, and hence $T_N(\cdot) = T(\cdot)$. For $b \in (-1,-1/p)$, the operator $A_{p,N}$ differs from A_p, hence $T_N(\cdot) \ne T(\cdot)$, and we have $D_{p,\mathrm{reg}} \subsetneq D_{p,\mathrm{en}} = D_{p,\max}$.*

Proof. 1) Theorem 2.9 says that $D_p = D_{p,\mathrm{reg}}$ if $b \le -1$ and hence $A_p = A_{p,N}$ in this case. So the asserted generation results for $b \le -1$ were already shown in Proposition 2.4 and in Theorem 2.13.

2) Let $b \in (-1,-1/p)$. Then there are functions in $D_p \setminus D_{p,\mathrm{reg}}$ by Theorem 2.12, so that $A_{p,N} \ne A_p$. We first give a short proof of the generation result for the special case $p = 2$, where $b \in (-1,-1/2)$. Here we treat $A_{2,N}$ as the perturbation $A_{2,N} = A_0 + (b+1)D_x$ of the operator $A_0 = -xD_x^2 - D_x$ with domain $D(A_0) = D_{2,\mathrm{reg}}$ that corresponds to $b = -1$. Take $u \in D_{2,\mathrm{reg}}$ and $Au = f$. It is straightforward to check that $A_{0,\varepsilon} = (A_0, D_{2,\varepsilon})$ is self adjoint on $L^2(\varepsilon,1)$. Hence, the resolvents $(\lambda + A_{0,\varepsilon})^{-1}$, and by approximation also $(\lambda + A_0)^{-1}$, are symmetric for $\lambda > 0$, cf. Proposition 2.4. As a result, A_0 is self adjoint so that

$$\begin{aligned} \|A_0(\lambda + A_0)^{-1}\| &\le \sup_{\tau\ge 0}\frac{\tau}{|\tau+\lambda|} \le \sup_{\tau\ge 0}\frac{\tau}{\tau + \operatorname{Re}\lambda} \le 1, \\ \|\lambda(\lambda + A_0)^{-1}\| &\le \sup_{\tau\ge 0}\frac{|\lambda|}{|\tau+\lambda|} \le 1 \end{aligned}$$

for all $\lambda \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Due to Proposition 3.4, estimate (3.4) holds for A_0 leading to

$$\|(b+1)D_x(\lambda + A_0)^{-1}\| \le \frac{b+1}{1-\frac{1}{2}}\,\|A_0(\lambda + A_0)^{-1}\| \le 2(b+1) =: \beta < 1.$$

Since $A = A_0 + (b+1)D_x$, these estimates easily imply that $\lambda \in \rho(-A)$ and $\|\lambda(\lambda + A)^{-1}\| \leq 1/(1-\beta)$ for all $\lambda \in \mathbb{C}_+$. Thus $-A_{2,N} = (-A, D_{2,\mathrm{reg}})$ generates an analytic semigroup. We remark that this argument can be extended to $p \in (1, \infty)$ if $b \in (-1, -1/p)$ is sufficiently close to -1.

3a) We next prove the generation result for $p \in (1, \infty)$ and $b \in (-1, -\frac{1}{p})$. Proposition 3.1 and Theorem 3.5 of [14] imply that $-A_\infty = (-A, D_\infty)$, defined in (2.9), on $D_\infty \subset D_{p,\mathrm{reg}} \cap C^1([0,1])$ generates a bounded, positive, analytic semigroup of angle $\frac{\pi}{2}$ on $L^\infty(0,1)$, cf. the proof of Theorem 2.13, case $b \leq -1$. In view of Lemma 2.5, the operator A_∞ is injective and thus invertible. Observe that its inverse is the restriction of the operator $A_{p,N}^{-1}$ obtained in Proposition 3.4. Let $\lambda \in \mathbb{C}_+$ and $0 \leq f \in L^\infty(0,1)$. From e.g. Corollary 3.11.3 of [1] we deduce that

$$|(\lambda + A_\infty)^{-1} f| \leq (\operatorname{Re}\lambda + A_\infty)^{-1} f \leq A_\infty^{-1} f = A_{p,N}^{-1} f.$$

As a result, $(\lambda + A_\infty)^{-1} = R(\lambda, -A_\infty)$ can be extended to a uniformly bounded pseusoresolvent $\{R(\lambda); \lambda \in \mathbb{C}_+\}$ on $L^p(0,1)$ for all $p \in (-\frac{1}{b}, \infty)$. It holds $R(0) = A_{p,N}^{-1}$, and thus $R(\lambda)$ has the range $D_{p,\mathrm{reg}} = A_{p,N}^{-1} L^p(0,1)$, due to e.g. Lemma III.4.5 of [4]. We can now compute

$$\begin{aligned}(\lambda + A_{p,N})R(\lambda) &= (\lambda + A_{p,N})[A_{p,N}^{-1} - \lambda A_{p,N}^{-1} R(\lambda)] \\ &= I + \lambda[A_{p,N}^{-1} - R(\lambda) - \lambda A_{p,N}^{-1} R(\lambda)] = I.\end{aligned}$$

for all $\lambda \in \mathbb{C}_+$. In the same way, one sees that $R(\lambda)(\lambda + A_{p,N}) = I$. It follows that $\mathbb{C}_+ \subset \rho(-A_{p,N})$ and $(\lambda + A_{p,N})^{-1} = R(\lambda)$ for all $\lambda \in \mathbb{C}_+$ is uniformly bounded and positive for $\lambda > 0$ and all $p \in (-\frac{1}{b}, \infty)$.

Fix now $p \in (1, \infty)$, $b \in (-1, -\frac{1}{p})$, and $q \in (-\frac{1}{b}, p)$. Interpolating between L^q and L^∞ we derive

$$\|(\lambda + A_{r,N})^{-1}\|_r \leq c\,|\lambda|^{-3/4} \tag{3.6}$$

for all $\lambda \in \mathbb{C}_+$ and the operator norm in $L^r(0,1)$ with $r = 4q$. In the next step we will improve this estimate to $\|(\lambda + A_{r,N})^{-1}\|_r \leq c\,|\lambda|^{-1}$. We can then repeat the procedure, interpolating between L^q and L^{4q} , to obtain the sectoriality estimate for $r = 16q/7$. In finitely many steps we arrive at $\|(\lambda + A_{p,N})^{-1}\| \leq c\,|\lambda|^{-1}$ for all $\lambda \in \mathbb{C}_+$. The operator $-A_{p,N}$ thus generates a positive analytic C_0–semigroup.

For later use, we first derive another estimate. Let $V = (1/2, 1)$, $r = 4q$, $\lambda \in \mathbb{C}_+$ and $f \in L^r(0,1)$. Set $\eta = |\lambda|^{-1/2}$. Interpolation yields

$$\begin{aligned}&\|D_x(\lambda + A_{r,N})^{-1} f\|_{L^r(V)} \\ &\quad\leq c_0\eta\,\|D_x^2(\lambda + A_{r,N})^{-1} f\|_{L^r(V)} + c_1\eta^{-1}\,\|(\lambda + A_{r,N})^{-1} f\|_{L^r(V)} \\ &\quad\leq 2c_0\eta\,\|A(\lambda + A_{r,N})^{-1} f\|_{L^r(V)} + c_0|b|\,\eta\,\|D_x(\lambda + A_{r,N})^{-1} f\|_{L^r(V)} \\ &\qquad + c_1\eta^{-1}\,\|(\lambda + A_{r,N})^{-1} f\|_{L^r(V)}\end{aligned}$$

for some constants $c_0, c_1 > 0$. Choosing $\eta \leq \eta_0 := (2c_0)^{-1} \leq (2c_0|b|)^{-1}$, we derive from the above estimates

$$\begin{aligned}&\|D_x(\lambda + A_{r,N})^{-1} f\|_{L^r(V)} \\ &\quad\leq 4c_0\,|\lambda|^{-1/2}\,\|f - \lambda(\lambda + A_{r,N})^{-1} f\|_{L^r(0,1)} + 2c_1\,|\lambda|^{1/2}\,\|(\lambda + A_{r,N})^{-1} f\|_{L^r(0,1)}\end{aligned} \tag{3.7}$$

$$\le c\,|\lambda|^{-1/2}\,\|f\|_{L^r(0,1)} + c\,|\lambda|^{-1/4}\,\|f\|_{L^r(0,1)} \le c\,|\lambda|^{-1/4}\,\|f\|_{L^r(0,1)}$$

for some constants $c > 0$ and all $\lambda \in \mathbb{C}_+$ with $|\lambda| \ge 4c_0^2$.

3b) We define the operator $\tilde{A} = -xD_x^2 + bD_x$ on $\tilde{D}_{p,\mathrm{reg}} = \{v \in W^{1,p}(0,\infty) \,|\, xv'' \in L^p(0,\infty)\}$. To construct and estimate the resolvent of $\tilde{A}$, we use our operator $A_{p,N}$ and the restriction $\tilde{A}^+$ of $\tilde{A}$ to the domain $\tilde{D}^+_{p,\mathrm{reg}} = \{v \in W^{1,p}(1/2,\infty) \,|\, v(1/2) = 0,\ xv'' \in L^p(1/2,\infty)\}$. One shows that $-\tilde{A}^+$ generates an analytic semigroup on $L^p(1/2,\infty)$ as in Proposition 6.1 of [16], using Theorem 2.7 of [7] for the spatial domain $\mathbb{R}$. As in (3.7), one can then derive

$$\|D_x(\lambda + \tilde{A})^{-1} f\|_{L^r(V)} \le c\,|\lambda|^{-1/2}\,\|f\|_{L^r(1/2,\infty)} \tag{3.8}$$

for all $\lambda \in \mathbb{C}_+$ with $|\lambda| \ge r_0$ for some $r_0 > 0$. Thanks to (3.6), (3.7), (3.8) and the sectoriality of $\tilde{A}^+$, (a slight variant of) Proposition 5.5 of [16] implies that (3.6) holds for $\tilde{A}$ and all $\lambda \in \mathbb{C}_+$ with $|\lambda| \ge r_1$ for some $r_1 > 0$.

We next use a scaling argument from [16]. Let $s > 0$. The map $J_s : L^p(0,\infty) \to L^p(0,\infty)$ given by $(J_s u)(x) = u(sx)$ is linear, has the inverse $J_{s^{-1}}$ and satisfies $\|J_s u\|_p = s^{-1/p}\,\|u\|_p$ for all $u \in L^p(0,\infty)$. Observe that $sJ_s\tilde{A}J_s^{-1} = \tilde{A}$. Let $\omega \in \mathbb{C}_+$ with $|\omega| = r_1$ and $\lambda = s\omega$. It follows $\lambda + \tilde{A} = sJ_s(\omega + \tilde{A})J_s^{-1}$ and thus $\|(\lambda + \tilde{A})^{-1}\| \le |\lambda|^{-1}\,\|\omega(\omega + \tilde{A})^{-1}\| \le c\,|\lambda|^{-1}$. for all $\lambda \in \mathbb{C}_+$. This sectoriality estimate again implies the version of (3.8) for $\tilde{A}$. As in Proposition 5.7 of [16], we then deduce that $-A_{r,N}$ generates an analytic semigroup on $L^r(0,1)$. This fact was missing to complete the proof in step 3a). □

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