ERROR ESTIMATES IN L^2 OF AN ADI SPLITTING SCHEME FOR THE INHOMOGENEOUS MAXWELL EQUATIONS.

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Abstract. In this paper we investigate an alternating direction implicit (ADI) time integration scheme for the linear Maxwell equations with currents, charges and conductivity. The main results establish that the scheme converges in L^2 with order two to the solution of the Maxwell system. Moreover, the divergence conditions in the system are preserved with order one. These results are based on a detailed regularity analysis of both the Maxwell system and the discrete scheme.

1. Introduction

As the foundation of the electro-magnetic theory, the Maxwell system is one of the most fundamental PDEs in physics. Already in the linear case it poses considerable difficulties to the numerical treatment since it is a coupled system of six time-depending scalar equations in three space dimensions. Around 2000 the very efficient and unconditionally stable alternating direction implicit (ADI) scheme was introduced in [18] and [23] for problems on a cuboid with linear isotropic material laws. In this scheme one splits the curl operator into the partial derivatives with a plus and a minus sign, see (1.4), and then applies the implicit-explicit Peaceman–Rachford method to the two subsystems, cf. (1.5). Astonishingly, the resulting implicit steps essentially decouple into one-dimensional problems, which makes the algorithm very fast, see [18], [23], and also [9], [12]. There are energy-conserving variants of the ADI splitting, see e.g. [3], [4], [11] and [16], not discussed here. We refer to [12], [13] and [14] for further references about the numerical treatment of the Maxwell system.

Until recently there was almost no rigorous error analysis of the ADI scheme. For a variant of the method and very regular (C^6) solutions, error estimates have been shown in [4]. For H^3–solutions, the paper [12] (co-authored by one of us) has established second order convergence of the scheme in L^2, assuming that the coefficients belong to $W^{2,3} \cap W^{1,\infty}$. These works considered the Maxwell system without conductivity, currents and charges. If one wants to incorporate these basic physical phenomena, compared to [12] one has to modify the scheme and develop a new analytical background. We started such investigations in our companion paper [9], cf. (1.4), (1.5) and (2.2) below. There we have worked

Date: December 21, 2017.

2010 *Mathematics Subject Classification.* 65M12; 35Q61, 47D06, 65J10.

Key words and phrases. Maxwell equations, splitting, efficiency, preservation of divergence conditions, convergence order.

The authors gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

with data in H^2, roughly speaking, and developed an error analysis in (variants of) H^{-1} in the case of Lipschitz coefficients. We have shown the stability of the scheme in L^2 and H^1, as well as its second order convergence in H^{-1}. Moreover, it was proved that it satisfies the Gaussian laws for the charges up to an error of first order in H^{-1}. For these results in weak norms, we needed regularity properties of the Maxwell system and the scheme in an H^1–framework only. However, error estimates in L^2 require an H^2–setting, leading to severe new difficulties, so that we have postponed them to the present paper.

In this work we do not treat the space discretization. In [13] and [14], an error analysis was given for the full discretization of the Maxwell system without conductivity, using the discontinuous Galerkin method and a locally implicit time integration scheme. We expect that one can treat the full discretization for the ADI scheme combining methods in these and our papers. In [8] one of us has done numerical experiments with finite differences on a spatial Yee grid, which confirm the results on PDE level established in the present work.

We study the linear Maxwell system with conductivity, currents and charges given by

$$\begin{aligned}
\partial_t \mathbf{E}(t) &= \frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}(t) - \frac{1}{\varepsilon}(\sigma\mathbf{E}(t) + \mathbf{J}(t)) && \text{in } Q,\ t \ge 0, && \text{(1.1a)}\\
\partial_t \mathbf{H}(t) &= -\frac{1}{\mu}\operatorname{curl}\mathbf{E}(t) && \text{in } Q,\ t \ge 0, && \text{(1.1b)}\\
\operatorname{div}(\varepsilon\mathbf{E}(t)) &= \rho(t), \qquad \operatorname{div}(\mu\mathbf{H}(t)) = 0 && \text{in } Q,\ t \ge 0, && \text{(1.1c)}\\
\mathbf{E}(t)\times\nu &= 0, \qquad \mu\mathbf{H}(t)\cdot\nu = 0 && \text{on } \partial Q,\ t \ge 0, && \text{(1.1d)}\\
\mathbf{E}(0) &= \mathbf{E}_0, \qquad \mathbf{H}(0) = \mathbf{H}_0 && \text{in } Q, && \text{(1.1e)}
\end{aligned}$$

on the cuboid Q with the unknown electric and magnetic fields $\mathbf{E}(t,x) \in \mathbb{R}^3$, resp. $\mathbf{H}(t,x) \in \mathbb{R}^3$, for $t \ge 0$ and $x \in Q$. Here, $\nu(x)$ is the outer unit normal at $x \in \partial Q$, and the initial fields in (1.1e), the current density $\mathbf{J}(t,x) \in \mathbb{R}^3$, the permittivity $\varepsilon(x) > 0$, the permeability $\mu(x) > 0$, and the conductivity $\sigma(x) \ge 0$ are given. We treat the conditions (1.1d) of a perfectly conducting boundary. As noted in Proposition 2.3 of [9], the charge density $\rho(t,x) \in \mathbb{R}$ depends on the data and (if $\sigma \ne 0$) on the solution via

$$\rho(t) = \operatorname{div}(\varepsilon\mathbf{E}(t)) = \operatorname{div}(\varepsilon\mathbf{E}_0) - \int_0^t \operatorname{div}(\sigma\mathbf{E}(s) + \mathbf{J}(s))\,\mathrm{d}s, \qquad t \ge 0. \tag{1.2}$$

Throughout, we assume that the material coefficients satisfy

$$\varepsilon,\mu,\sigma \in W^{1,\infty}(Q,\mathbb{R}) \cap W^{2,3}(Q,\mathbb{R}), \qquad \varepsilon,\mu \ge \delta > 0, \qquad \sigma \ge 0, \tag{1.3}$$

for a constant δ. For the initial fields and the current density we require regularity of third order and certain compatibility conditions in our theorems.

Section 2 briefly recalls the necessary background material from [9]. In particular, we look at the Maxwell operator M in $L^2(Q)^6$ which governs the evolution equations (1.1a) and (1.1b). In its domain one incorporates the electric boundary conditions, cf. (2.1). The solution theory of (1.1) in H^2 is presented in Section 3. Here we use the space X_2 from (3.1) which is the subspace of $D(M^2)$ containing the magnetic conditions $\mu\mathbf{H}\cdot\nu = 0$ and $\operatorname{div}(\mu\mathbf{H}) = 0$ from (1.1) as well as the regularity $\operatorname{div}(\varepsilon\mathbf{E}) \in H^1(Q)$ of the charges. Moreover, the charges have to vanish at the edges of Q in a generalized sense. For our analysis it is crucial that this space embeds into $H^2(Q)^6$ and that the part of M in X_2 is a generator, see Propositions 3.2 and 3.3. These facts are based on a rather delicate analysis and on sharp regularity results for mixed inhomogeneous boundary value problems for the Laplacian on Q, provided by Lemma 3.1. (Here one needs the trace property of the charges as a compatibility condition.) Our arguments make heavy use of trace and interpolation theory.

In the ADI method one splits curl $= C_1 - C_2$ and $M = A + B$, where we put

$$A = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -\frac{\sigma}{2\varepsilon} I & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix} \qquad \text{with}$$
$$C_1 = \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \qquad \text{and} \qquad C_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

The domains of A and B are described after (4.1). Let $\tau > 0$, $T \geq 1$, and $t_n = n\tau \leq T$ for $n \in \mathbb{N}$. The $(n+1)$–th step of the scheme is given by

$$w_{n+1} = (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\big[(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)w_n - \tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n) + \mathbf{J}(t_{n+1}), 0)\big]. \tag{1.5}$$

Here we modify an approach developed in [20] for a different situation. Note that the conductivity σ is included into the maps A and B, whereas the current density $\mathbf{J}$ is added to the scheme. It was shown in [9] that the respective parts of these operators generate (up to a shift) contraction semigroups in $L^2(Q)^6$ and a subspace of $H^1(Q)^6$, both equipped with weighted norms. These facts are recalled at the beginning of Section 4. For the preservation of the Gauss laws, we need generation properties of A and B in a suitable subspace of $H^2(Q)^6$. It has to contain the electric and magnetic boundary conditions as well as induced boundary conditions of first order, see Proposition 4.6. The higher order conditions make the corresponding proofs rather intricate. Moreover, they force us to impose in addition that the Neumann traces of the coefficients vanish.

In the final section we then combine the results of Sections 3 and 4 with the formulas in [9] for the scheme and its errors. In Theorem 5.2 we prove the second order convergence of the scheme in L^2. Assuming also the above mentioned trace condition on the coefficients, we show in Theorem 5.1 that the scheme is stable in H^2 and in Theorem 5.4 that it preserves the Gaussian laws (1.1c) and (1.2) in L^2 up to first order.

2. Auxiliary facts

The following notation and results are used throughout this paper, often without further notice. By c we denote a generic constant which may only depend on Q and on the constants from (1.3); i.e., on δ and the norms of ε, μ and σ in $W^{1,\infty}(Q) \cap W^{2,3}(Q)$. We write I for the identity operator and $v \cdot w$ for the Euclidean inner product in $\mathbb{R}^m$.

Let X and Y be real Banach spaces. On the intersection $X \cap Y$ we use the norm $\|z\|_X + \|z\|_Y$. The symbol $Y \hookrightarrow X$ means that Y is continuously embedded into X, and $X \cong Y$ that they are isomorphic. The duality pairing between X and its dual X^* is denoted by $\langle x^*, x\rangle_{X^*,X}$ for $x \in X$ and $x^* \in X^*$; and the inner product by $(\cdot \mid \cdot)_X$ if X is a Hilbert space. In the latter case, a dense embedding $Y \hookrightarrow X$ implies that $X \hookrightarrow Y^*$, where $x \in X \cong X^*$ acts on Y via $\langle x, y\rangle_{Y^*,Y} = (x \mid y)_X$ for $y \in Y \hookrightarrow X$.

Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators from X to Y, and $\mathcal{B}(X) = \mathcal{B}(X, X)$. The domain $D(L)$ of a linear operator L is always equipped with the graph norm $\|\cdot\|_L$ of L. If $Y \hookrightarrow X$, then the part L_Y of L in Y is given by $D(L_Y) = \{y \in Y \cap D(L) \mid Ly \in Y\}$ and $L_Y y = Ly$. For two operators L and G in X, the product LG is defined on $D(LG) = \{x \in D(G) \,|\, Gx \in D(L)\}$.

We employ the standard Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}_0$, $p \in [1, \infty]$ and open subsets $\Omega \subseteq \mathbb{R}^m$, where $W^{0,p}(\Omega) = L^p(\Omega)$. For $p \in [1, \infty)$, $s \in (0, \infty) \setminus \mathbb{N}$ and an integer $k > s$, we define the Slobodeckij spaces $W^{s,p}(\Omega) = (L^p(\Omega), W^{k,p}(\Omega))_{s/k,p}$ by real interpolation, see Section 7.32 in [1] or [17]. We set $W^{-s,p}(\Omega) = W_0^{s,p'}(\Omega)^*$ for $s \geq 0$ and $p \in (1, \infty)$, where $p' = p/(p-1)$ and the subscript 0 always denotes the closure of the standard test functions in the respective norm. We are mostly interested in the case $H^s(\Omega) := W^{s,2}(\Omega)$.

We work on the cuboid $Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+) \subseteq \mathbb{R}^3$ with (Lipschitz) boundary $\Gamma = \partial Q$. For $s \in [0, 1]$ and $p \in (1, \infty)$ we use the Slobodeckij spaces $W^{s,p}(\Gamma)$ on the boundary, see Section 2.5 of [19] or Sections 2 and 3 of [15]. Moreover, $W^{-s,p}(\Gamma)$ is defined as the dual space of $W^{s,p'}(\Gamma)$. We write

$$\Gamma_j^{\pm} = \{x \in \overline{Q} \mid x_j = a_j^{\pm}\} \qquad \text{and} \qquad \Gamma_j = \Gamma_j^- \cup \Gamma_j^+$$

for $j \in \{1, 2, 3\}$, and d_Q for the smallest side length of Q.

As in [9], our analysis of the Maxwell system takes place in the space $X = L^2(Q)^6$ endowed with the weighted inner product

$$\big((u, v) \mid (\varphi, \psi)\big)_X = \int_Q \big(\varepsilon u \cdot \varphi + \mu v \cdot \psi\big)\, \mathrm{d}x$$

for $(u, v), (\varphi, \psi) \in X$. The square of the induced norm $\|\cdot\|_X$ is twice the physical energy of the fields $(\mathbf{E}, \mathbf{H})$, and because of (1.3) it is equivalent to the usual L^2–norm. We further use the Hilbert spaces

$$H(\operatorname{curl}, Q) = \{u \in L^2(Q)^3 \mid \operatorname{curl} u \in L^2(Q)^3\}, \qquad \|u\|_{\operatorname{curl}}^2 = \|u\|_{L^2}^2 + \|\operatorname{curl} u\|_{L^2}^2\,,$$
$$H(\operatorname{div}, Q) = \{u \in L^2(Q)^3 \mid \operatorname{div} u \in L^2(Q)\}, \qquad \|u\|_{\operatorname{div}}^2 = \|u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2\,.$$

Theorems 1 and 2 in Section IX.A.1.2 of [6] provide the following facts. The space of restrictions to Q of test functions on $\mathbb{R}^3$ is dense in $H(\operatorname{curl}, Q)$ and $H(\operatorname{div}, Q)$. The tangential trace $u \mapsto (u \times \nu)|_\Gamma$ on $C(\overline{Q})^3 \cap H^1(Q)^3$ has a unique continuous extension $\operatorname{tr}_t : H(\operatorname{curl}, Q) \to H^{-1/2}(\Gamma)^3$, and $H_0(\operatorname{curl}, Q)$ is the kernel of tr_t in $H(\operatorname{curl}, Q)$. The normal trace $u \mapsto (u \cdot \nu)|_\Gamma$ on $C(\overline{Q})^3 \cap H^1(Q)^3$ also has a unique bounded extension $\operatorname{tr}_n : H(\operatorname{div}, Q) \to H^{-1/2}(\Gamma)$.

Let $p \in (1, \infty)$ and $s \in (\frac{1}{p}, 1 + \frac{1}{p})$. Then Section 3 of [15] or Sections 2.4 and 2.5 of [19] provide the continuous and surjective trace operator $\operatorname{tr} : W^{s,p}(Q) \to W^{s-1/p,p}(\Gamma)$, which is the extension of the map $f \mapsto f|_\Gamma$ defined on $C(\overline{Q}) \cap W^{s,p}(Q)$. Its kernel is the space $W_0^{s,p}(Q)$. Restrictions to Q of test functions on $\mathbb{R}^3$ are dense in $W^{s,p}(Q)$ for all $s > 0$, and we have $W_0^{s,p}(Q) = W^{s,p}(Q)$ for

$s \in (0, \frac{1}{p})$. By approximation one sees that the trace is multiplicative for maps in $W^{1,p}(Q)$ and $W^{1,q}(Q)$ with $p, q \in (1, \infty]$ and $\frac{1}{p} + \frac{1}{q} < 1$.

We also have to deal with cases of partial regularity. The space E_j of functions $f \in L^2(Q)$ with $\partial_j f \in L^2(Q)$ can be identified with $H^1((a_j^-, a_j^+), L^2(Q_j)) \cong L^2(Q_j, H^1(a_j^-, a_j^+))$, where $j \in \{1,2,3\}$ and e.g. $Q_1 = (a_2^-, a_2^+) \times (a_3^-, a_3^+)$. In this way, we obtain continuous trace operators $\mathrm{tr}_{\Gamma_j^\pm} : E_j \to L^2(\Gamma_j^\pm)$ to $\Gamma_j^\pm$ and $\mathrm{tr}_{\Gamma_j} : E_j \to L^2(\Gamma_j)$ to Γ_j, which coincide with the restrictions of the usual trace if f belongs to $H^1(Q)$. We usually write $u_1 = 0$ on Γ_2 instead of $\mathrm{tr}_{\Gamma_2}(u_1) = 0$ etc. For a union $\Gamma^* \subseteq \Gamma$ of some faces of Q we set

$$H^1_{\Gamma^*}(Q) = \{u \in H^1(Q) \mid \mathrm{tr}\, u = 0 \text{ on } \Gamma^*\}.$$

We recall Lemma 2.1 of [9], which is used below several times.

Lemma 2.1. *For some $j, k \in \{1,2,3\}$ with $k \neq j$, let $f \in L^2(Q)$ satisfy $\partial_j f, \partial_k f, \partial_{jk} f \in L^2(Q)$ and $f = 0$ on Γ_j. We then have $\partial_k f = 0$ on Γ_j.*

The arguments in the proof of in this lemma further yield a simple fact, needed in our approximation arguments.

Lemma 2.2. *Let $p \in [1, \infty)$ and $f \in L^p(Q)$. Assume that $\partial_j f \in L^p(Q)$ and $f = 0$ on Γ_j for some $j \in \{1,2,3\}$. Take functions $\chi_n = \chi_n^{(j)} : [a_j^-, a_j^+] \to [0,1]$ that vanish near $a_j^\pm$, are equal to 1 on $[a_j^- + 1/n, a_j^+ - 1/n]$ and satisfy $|\chi_n'| \leq c/n$ for $n > 2/d_Q$. Set $f_n(x) = \chi_n(x_j) f(x)$ for $x \in Q$. Then f_n and $\partial_j f_n$ tend to f and $\partial_j f$ in $L^p(Q)$ as $n \to \infty$, respectively.*

Following [9], we define the Maxwell operator

$$M = \begin{pmatrix} -\frac{\sigma}{\varepsilon} I & \frac{1}{\varepsilon}\,\mathrm{curl} \\ -\frac{1}{\mu}\,\mathrm{curl} & 0 \end{pmatrix}, \qquad D(M) = H_0(\mathrm{curl}, Q) \times H(\mathrm{curl}, Q), \tag{2.1}$$

at first in X. The above domain contains the electric boundary condition from (1.1). The magnetic conditions and the regularity of the charge density $\rho = \mathrm{div}(\varepsilon u)$ are included in the subspace

$$X_{\mathrm{div}} = \{(u, v) \in X \mid \mathrm{div}(\mu v) = 0, \;\; \mathrm{tr}_n(\mu v) = 0, \;\; \mathrm{div}(\varepsilon u) \in L^2(Q)\}. \tag{2.2}$$

As noted in (2.4) of [9], one can drop here ε and the second μ because of (1.3). Moreover, X_{div} is a Hilbert space with the norm given by

$$\|(u, v)\|^2_{X_{\mathrm{div}}} = \|(u, v)\|^2_X + \|\mathrm{div}(\varepsilon u)\|^2_{L^2(Q)}.$$

The part of M in X_{div} is denoted by M_{div}. We have seen in (2.5) of [9] that

$$D(M^k_{\mathrm{div}}) = D(M^k) \cap X_{\mathrm{div}} \tag{2.3}$$

for $k \in \mathbb{N}$. Proposition 2.2 in [9] yields the embedding

$$D(M_{\mathrm{div}}) \hookrightarrow H^1(Q)^6 \tag{2.4}$$

whose norm can be controlled by the constants of (1.3), and we have the traces

$$H_i = E_j = E_k = 0 \qquad \text{on } \Gamma_i \tag{2.5}$$

for all $(\mathbf{E}, \mathbf{H}) \in D(M_{\mathrm{div}})$ and $(i, j, k) = \{(1,2,3), (2,1,3), (3,1,2)\}$.

By Proposition 2.3 in [9] the operator M generates a contraction semigroup $(\mathrm{e}^{tM})_{t\geq 0}$ on X whose restrictions $\mathrm{e}^{tM_{\mathrm{div}}}$ form a linearly bounded C_0–semigroup on X_{div} generated by M_{div}. It is bounded if $\sigma = 0$ or $\sigma \geq \sigma_0$ for a constant $\sigma_0 > 0$, see Remark 2.4 in [9]. Let $w_0 = (\mathbf{E}_0, \mathbf{H}_0)$ belong to $D(M_{\mathrm{div}})$ and $(\mathbf{J}, 0)$ to $C([0,\infty), D(M_{\mathrm{div}}))+C^1([0,\infty), X_{\mathrm{div}})$. Then there is a unique solution $w = (\mathbf{E}, \mathbf{H})$ of (1.1) in $C^1([0,\infty), X_{\mathrm{div}}) \cap C([0,\infty), D(M_{\mathrm{div}}))$ given by

$$(\mathbf{E}(t), \mathbf{H}(t)) = \mathrm{e}^{tM_{\mathrm{div}}}(\mathbf{E}_0, \mathbf{H}_0) - \int_0^t \mathrm{e}^{(t-s)M_{\mathrm{div}}}\big(\tfrac{1}{\varepsilon}\mathbf{J}(s), 0\big)\,\mathrm{d}s. \tag{2.6}$$

Moreover, the charge density in (1.1c) is contained in $L^2(Q)$ and satisfies

$$\begin{aligned}\rho(t) = \operatorname{div}(\varepsilon\mathbf{E}(t)) &= \operatorname{div}(\varepsilon\mathbf{E}_0) - \int_0^t \operatorname{div}(\sigma\mathbf{E}(s) + \mathbf{J}(s))\,\mathrm{d}s \qquad (2.7)\\ &= \mathrm{e}^{-\frac{\sigma}{\varepsilon}t}\operatorname{div}(\varepsilon\mathbf{E}_0) - \int_0^t \mathrm{e}^{-\frac{\sigma}{\varepsilon}(t-s)}\Big(\nabla\big(\tfrac{\sigma}{\varepsilon}\big)\cdot\varepsilon\mathbf{E}(s) + \operatorname{div}\mathbf{J}(s)\Big)\,\mathrm{d}s, \quad t \geq 0.\end{aligned}$$

We also note that operators like $f \mapsto \varepsilon f$ are bounded on $H^2(Q)$ and $H^1(Q)$ with a norm controlled by the constants of (1.3). (Use Sobolev's embedding.)

3. H^2–solutions of the Maxwell system

In our error analysis we need solutions w of (1.1) such that Mw takes values in H^2. For the corresponding charge densities we use the space $H^1_{00}(Q)$ of all functions f in $H^1(Q)$ such that

$$\operatorname{tr}_{\Gamma'} f \in H_0^{1/2}(\Gamma') := \big(L^2(\Gamma'), H_0^1(\Gamma')\big)_{1/2,2}$$

for all faces Γ' of Q. Proposition 2.11 of [15] implies that $H_0^\alpha(\Gamma') \hookrightarrow H_0^{1/2}(\Gamma')$ for $\alpha > 1/2$. By interpolation, the space $H_0^{1/2}(\Gamma')$ is embedded into $H^{1/2}(\Gamma')$. We now define the smaller state space

$$X_2 = \{(u,v) \in D(M^2) \cap X_{\mathrm{div}} \mid \operatorname{div}(\varepsilon u) \in H^1_{00}(Q)\} \tag{3.1}$$

with the norm given by

$$\|(u,v)\|_{X_2}^2 = \|(u,v)\|_{D(M^2)}^2 + \|\operatorname{div}(\varepsilon u)\|_{H^1}^2 + \sum_{\Gamma' \text{ face of } Q} \|\operatorname{div}(\varepsilon u)\|^2_{H_0^{1/2}(\Gamma')}.$$

Observe that X_2 is a Hilbert space. It contains fields whose charge densities vanish on the edges of Q in a generalized sense. Below we show that it is embedded into $H^2(Q)^6$. To this aim, we first solve a mixed inhomogeneous boundary value problem for the Laplacian on Q, cf. Lemma 3.6 of [12] for 0 boundary data. For technical reasons, see the proof of Proposition 3.3, we also need a variant in lower regularity.

Lemma 3.1. *Let $j \in \{1,2,3\}$ and $\Gamma^* = \Gamma\backslash\Gamma_j$. Take $f \in L^2(Q)$ and $g \in L^2(\Gamma_j)$. Then the following assertions hold.*

a) There is a unique function $v \in H^1_{\Gamma^*}(Q)$ solving

$$\int_Q v\varphi\,\mathrm{d}x + \int_Q \nabla v\cdot\nabla\varphi\,\mathrm{d}x = \int_Q f\varphi\,\mathrm{d}x + \int_{\Gamma_j^+} g\varphi\,\mathrm{d}\sigma - \int_{\Gamma_j^-} g\varphi\,\mathrm{d}\sigma \tag{3.2}$$

for all $\varphi \in H^1_{\Gamma^*}(Q)$.

b) Let $g \in H_0^{1/2}(\Gamma_j)$. The solution v then belongs to $D := H^2(Q) \cap H^1_{\Gamma^*}(Q)$ and satisfies $v - \Delta v = f$ on Q, $\partial_\nu v = g$ on Γ_j, and $\|v\|_{H^2} \le c\,(\|f\|_{L^2} + \|g\|_{H_0^{1/2}(\Gamma_j)})$ with a constant only depending on Q.

c) Let $\theta \in (0, \frac{1}{2})$ and $g \in H^\theta(\Gamma_j)$. Then the solution v is contained in $D_\theta := H^{3/2+\theta}(Q) \cap H^1_{\Gamma^*}(Q)$ and bounded in $H^{3/2+\theta}(Q)$ by $c\,(\|f\|_{L^2} + \|g\|_{H^\theta(\Gamma_j)})$, and it fulfills the other assertions in part b).

Proof. The first part is a standard consequence of the Lax–Milgram lemma. Below, we take $j = 1$ for simplicity.

b1) Let $g \in H_0^{1/2}(\Gamma_1)$. We first construct an extension $w \in H^2(Q)$ with $\partial_\nu w = g$ on Γ_1 and $w = 0$ on $\Gamma_2 \cup \Gamma_3$. Let $R = (a_2^-, a_2^+) \times (a_3^-, a_3^+)$ and Δ_R be the Dirichlet Laplacian on R with domain $D(\Delta_R) = H^2(R) \cap H_0^1(R)$, which we consider also as an operator acting on Γ_1^- or Γ_1^+. Since Δ_R is self-adjoint and negative definite, we have the (self-adjoint and positive definite) fractional powers $(-\Delta_R)^\alpha$ for $\alpha \ge 0$ which generate analytic semigroups on $L^2(R)$. Moreover, $-\Delta_R$ is given by the Dirichlet form on $H_0^1(R)$ so that this space is isomorphic to the domain $D((-\Delta_R)^{1/2})$. Due to $(L^2(R), H_0^1(R))_{1/2,2} = H_0^{1/2}(R)$, Proposition 6.2 in [17] implies the crucial estimate

$$\|(-\Delta_R)^{1/2} \exp\big(\cdot\,(-\Delta_R)^{1/2}\big)h\|^2_{L^2(\mathbb{R}_+, L^2(R))} \le c\,\|h\|^2_{H_0^{1/2}(R)} \tag{3.3}$$

for $h \in H_0^{1/2}(R)$, where we also use the exponential stability of this semigroup.

Let $\chi : [0, a_1^+ - a_1^-] \to \mathbb{R}$ be a smooth function with support in $[0, (a_1^+ - a_1^-)/2]$ that is equal to 1 on $[0, (a_1^+ - a_1^-)/4]$. We denote by $g^\pm$ the restrictions of g to $\Gamma_1^\pm$, and set

$$\begin{aligned} w(x_1, x_2, x_3) = &-\big(\chi(x_1 - a_1^-)(-\Delta_R)^{-1/2} \exp\big((x_1 - a_1^-)(-\Delta_R)^{1/2}\big)g^-\big)(x_2, x_3) \\ &+ \big(\chi(a_1^+ - x_1)(-\Delta_R)^{-1/2} \exp\big((a_1^+ - x_1)(-\Delta_R)^{1/2}\big)g^+\big)(x_2, x_3) \end{aligned}$$

for $(x_1, x_2, x_3) \in \overline{Q}$. Observe that the maps $w(x_1, \cdot, \cdot)$ belong to $H^2(\{x_1\} \times R) \cap H_0^1(\{x_1\} \times R)$ for all $x_1 \in (a_1^-, a_1^+)$ because the semigroup is analytic. Using (3.3) and that $(-\Delta_R)^{-1/2} = (-\Delta_R)^{-1}(-\Delta_R)^{1/2}$, one derives that w is contained in $H^2(Q)$ with norm bounded by $c\,\|g\|_{H_0^{1/2}(\Gamma_1)}$. Taking the traces to Γ_2 and Γ_3 in $L^2((a_1^-, a_1^+), H^2(R))$, we see that w is an element of $H^1_{\Gamma_2 \cup \Gamma_3}(Q)$. On Γ_1^- we further obtain

$$\partial_\nu w(x_1, \cdot, \cdot)|_{x_1 = a_1^-} = \big(\exp\big((x_1 - a_1^-)(-\Delta_R)^{1/2}\big)g^-\big)|_{x_1 = a_1^-} = g^-,$$

and similarly on Γ_1^+. The Neumann trace of w on Γ_1 thus equals g.

b2) Set $\widetilde{f} = f - w + \Delta w \in L^2(Q)$. The case $g = 0$ was treated in Lemma 3.6 of [12], for instance, which provides a function $u \in D$ with $u - \Delta u = \widetilde{f}$ on Q, $\partial_\nu u = 0$ on Γ_1, and $\|u\|_{H^2(Q)} \le c\,\|f\|_{L^2(Q)}$. Hence, the map $v := u + w \in D$ satisfies the properties asserted in part b). The divergence theorem then yields that v also solves (3.2) for all $\varphi \in H^1_{\Gamma^*}(Q)$.

c1) To modify the extension step from part b1), we need the fractional power scale of $-\Delta_R$ as discussed in Section V.1 of [2]. For $\alpha \ge 0$ we set $V_\alpha = D((-\Delta_R)^\alpha)$ and define the space $V_{-\alpha}$ as the completion of $L^2(R)$ with respect to the norm given by $\|(-\Delta_R)^{-\alpha} f\|_{L^2}$. Since Δ_R is self-adjoint, Theorem V.1.4.12 of [2] shows that $V_{-\alpha}$ is canonically isomorphic to the dual of V_α. The power scale $\{V_\alpha \,|\, \alpha \in \mathbb{R}\}$ coincides with the real and complex interpolation scales $(\cdot,\cdot)_{\alpha,2}$ resp. $[\cdot,\cdot]_\alpha$ due to Theorem V.1.5.4 of [2] and Corollary 4.37 of [17], where we also use the Hilbert space setting. (We apply the complex interpolation method to the canonical complexification of the problem.)

Let $\alpha \in (0,1)\backslash\{1/4, 3/4\}$. Interpolating the embeddings $H_0^2(R) \hookrightarrow D(\Delta_R) \hookrightarrow H^2(R)$ and $L^2(R) \to L^2(R)$, we derive

$$H_0^{2\alpha}(R) = (L^2(R), H_0^2(R))_{\alpha,2} \hookrightarrow V_\alpha \hookrightarrow H^{2\alpha}(R).$$

Here the equality follows from Proposition 2.11 in [15]. Proposition 3.5 and Remark 2.7 in [15] further imply that $H_0^{2\alpha}(R) = H^{2\alpha}(R)$ for $\alpha \in (0, \frac{1}{4})$. As a result, V_α is equal to $H_0^{2\alpha}(R)$, and thus $V_{-\alpha}$ coincides with $H^{-2\alpha}(R)$, for $\alpha \in (0, \frac{1}{4})$.

c2) Let $\theta \in (0, \frac{1}{2})$ and $g \in H^\theta(\Gamma_1)$. We define the extension $w \in L^2(Q)$ as in step b1). To show that it belongs to $H^{\theta+3/2}(Q)$, we make use of the semigroup on $H^{\theta-1/2}(R) = V_{\theta/2-1/4}$ generated by $(-\Delta_R)^{1/2}$ with domain $V_{\theta/2+1/4}$. Proposition 6.2 in [17] and step c1) yield

$$\begin{aligned} w &\in L^2((a_1^-, a_1^+), V_{\theta/2+3/4}) \hookrightarrow L^2((a_1^-, a_1^+), H^{\theta+3/2}(R)), \\ \partial_1 w &\in L^2((a_1^-, a_1^+), V_{\theta/2+1/4}) \hookrightarrow L^2((a_1^-, a_1^+), H^{\theta+1/2}(R)), \\ \partial_{11} w &\in L^2((a_1^-, a_1^+), V_{\theta/2-1/4}) = L^2((a_1^-, a_1^+), H^{\theta-1/2}(R)); \end{aligned}$$

so that $\partial_j w$ belongs to $L^2((a_1^-, a_1^+), H^{\theta+1/2}(R))^3$ for each $j \in \{1,2,3\}$. Theorem V.1.5.4 of [2] and Corollary 4.37 of [17] show that

$$V_\alpha = [V_{\theta/2-1/4}, V_{\theta/2+1/4}]_\eta = (V_{\theta/2-1/4}, V_{\theta/2+1/4})_{\eta,2}$$

for $\eta \in (0,1)$ and $\alpha = \frac{\theta}{2} - \frac{1}{4} + \frac{\eta}{2}$. From Theorem 14 in [7] with $p = 2$ and $\varepsilon = \theta + 1/2$ we then deduce that $\partial_j w =: f_j$ is contained in $H^{\theta+1/2}((a_1^-, a_1^+), L^2(R))$, also using that $\partial_j(-\Delta_R)^{-1/2}$ is bounded in $L^2(R)$ for $j \in \{2,3\}$.

We next apply Stein's extension operator from Theorem 5.24 of [1] to (a_1^-, a_1^+) and R to obtain an extension $\hat{f}_j$ of f_j in the space $L^2(\mathbb{R}, H^{\theta+1/2}(\mathbb{R}^2)) \cap H^{\theta+1/2}(\mathbb{R}, L^2(\mathbb{R}^2))$. By means of e.g. the characterization of H^s via the Fourier transform, we conclude that $\hat{f}_j$ is contained in $H^{\theta+1/2}(\mathbb{R}^3)$ and hence f_j in $H^{\theta+1/2}(Q)$, cf. Paragraph 7.62 of [1] and Section 2 of [15]. In view of Proposition 2.18 of [15], the function w thus belongs to $H^{\theta+3/2}(Q)$. The above arguments also allow us to bound it in this norm by $c\,\|g\|_{H^\theta(\Gamma_1)}$.

Since $V_{\theta/2+3/4} \hookrightarrow V_{1/2} = H_0^1(R)$, the map w also is contained in $H^1_{\Gamma^*}(Q)$. As in part b1), one obtains the trace $\partial_\nu w = g$ on Γ_1.

c3) Let $L = \Delta$ be endowed with the domain

$$D(L) = \{v \in H^2(Q) \cap H^1_{\Gamma^*}(Q) \mid \partial_\nu v = 0 \text{ on } \Gamma_1\}.$$

in $L^2(Q)$. In the proof of Lemma 3.6 of [12] it was shown that the operator L is self-adjoint and $I - L$ is positive definite. It is induced by the shifted Dirichlet form on $H^1_{\Gamma^*}(Q)$. Exactly as in part c1) we thus obtain the power scale X_α^L for $I - L$ with $X_{1/2}^L = H^1_{\Gamma^*}(Q)$, $H_0^{2\alpha}(Q) \hookrightarrow X_\alpha^L \hookrightarrow H^{2\alpha}(Q)$ and $X_{-\alpha}^L = \big(X_\alpha^L\big)^*$ for

$\alpha \in (0,1]$, as well as $H_0^{2\alpha}(Q) = X_\alpha^L = H^{2\alpha}(Q)$ and $X_{-\alpha}^L = H^{-2\alpha}(Q)$ for $\alpha \in (0,1/4)$. The map $(I-L_{-1})^{-1} : X_{-\alpha}^L \to X_{1-\alpha}^L$ is continuous by Corollary V.1.3.9 in [2].

Following step b2), we set $\tilde f = f - w + \Delta w$ which is an element of $H^{\theta-1/2}(Q) = X_{-\beta}^L$ for $\beta := 1/4 - \theta/2 \in (0,1/4)$. We next take maps $g_n \in C_c^\infty(\Gamma_1)$ which converge to g in $H^\theta(\Gamma_1)$ as $n \to \infty$. We define w_n as above for these boundary data and $\tilde f_n = f - w_n + \Delta w_n$ for $n \in \mathbb{N}$. By part b), the functions w_n belong to D and $\tilde f_n$ to $L^2(Q)$. Step c2) shows that w_n tends to w in $H^{\theta+3/2}(Q) \cap H^1_{\Gamma^*}(Q)$, and hence $\tilde f_n$ to $\tilde f$ in $H^{\theta-1/2}(Q)$. We then define $u_n = (I-L)^{-1}\tilde f_n \in D$. These maps also fulfill $u_n - \Delta u_n = \tilde f_n$ and $\partial_\nu u_n = 0$ on Γ_1 by part b2). The observations in the previous paragraph imply that (u_n) tends to $u := (I - L_{-1})^{-1}\tilde f$ in $X_{1-\beta}^L \hookrightarrow H^{\theta+3/2}(Q) \cap H^1_{\Gamma^*}(Q)$ as $n \to \infty$.

Finally, we set $v_n = u_n + w_n \in D$ for $n \in \mathbb{N}$. These functions satisfy $v_n - \Delta v_n = f$, $v_n = 0$ on Γ^*, and $\partial_\nu v_n = g_n$ on Γ_1. Moreover, they converge to $v := u + w$ in $H^{\theta+3/2}(Q) \cap H^1_{\Gamma^*}(Q)$ as $n \to \infty$, so that v fulfills the assertions. □

We can now establish the desired embedding of the space X_2 and deduce its trace properties. In the case $\operatorname{div}(\varepsilon \mathbf{E}) = 0$ without charges such a result was shown in Lemma 3.7 of [12]. See [5] for a much more detailed study of regularity properties of the Maxwell operator on polygonal domains. We further compute the domain of the part M_2 of M in X_2.

Proposition 3.2. *Let* (1.3) *hold. The space X_2 is continuously embedded into $H^2(Q)^6$ with an embedding constant only depending on Q and the constants in* (1.3). *Fields in $(\boldsymbol{E}, \boldsymbol{H}) \in X_2$ have the traces*

$$\begin{aligned} E_j = E_k &= 0, & \partial_j E_j = \partial_k E_j = \partial_k E_j = \partial_k E_k &= 0 \qquad \text{on } \Gamma_i, \\ H_i &= 0, & \partial_j H_i = \partial_k H_i &= 0 \qquad \text{on } \Gamma_i \end{aligned}$$

for all permutations (i,j,k) of $(1,2,3)$. Moreover, the part M_2 of M in X_2 possesses the domain $D(M_2) = D(M^3) \cap X_2$.

Proof. 1) Let $(\mathbf{E}, \mathbf{H}) \in X_2$. Formulas (2.3) and (2.4) show the embedding $D(M) \cap X_{\mathrm{div}} \hookrightarrow H^1(Q)^6$ with a constant only depending on (1.3). Together with (2.5) we see the asserted zero-order traces. The claimed first-order ones will follow from Lemma 2.1 after we have established the claim $X_2 \hookrightarrow H^2(Q)^6$. We first observe that

$$M^2(\mathbf{E}, \mathbf{H}) = \begin{pmatrix} \frac{\sigma^2}{\varepsilon^2}\mathbf{E} - \frac{1}{\varepsilon}\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbf{E}\right) - \frac{\sigma}{\varepsilon^2}\operatorname{curl}\mathbf{H} \\ \frac{1}{\mu}\nabla\left(\frac{\sigma}{\varepsilon}\right) \times \mathbf{E} + \frac{\sigma}{\mu\varepsilon}\operatorname{curl}\mathbf{E} - \frac{1}{\mu}\operatorname{curl}\left(\frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}\right) \end{pmatrix}.$$

To exploit that $M^2(\mathbf{E}, \mathbf{H})$ belongs to X, we compute

$$\begin{aligned} \operatorname{curl}\left(\tfrac{1}{\mu}\operatorname{curl}\mathbf{E}\right) &= \left(\nabla\tfrac{1}{\mu}\right) \times \operatorname{curl}\mathbf{E} + \frac{1}{\mu}\operatorname{curl}\operatorname{curl}\mathbf{E} \\ &= -\frac{1}{\mu^2}(\nabla\mu) \times \operatorname{curl}\mathbf{E} + \frac{1}{\mu}(-\Delta\mathbf{E} + \nabla\operatorname{div}\mathbf{E}) \\ &= -\frac{1}{\mu^2}(\nabla\mu) \times \operatorname{curl}\mathbf{E} - \frac{1}{\mu}\Delta\mathbf{E} + \frac{1}{\mu}\nabla\left(\tfrac{1}{\varepsilon}\operatorname{div}(\varepsilon\mathbf{E}) - \tfrac{1}{\varepsilon}\nabla\varepsilon \cdot \mathbf{E}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\mu^2}(\nabla\mu)\times\operatorname{curl}\mathbf{E} - \frac{1}{\mu}\Delta\mathbf{E} - \frac{1}{\mu\varepsilon^2}\operatorname{div}(\varepsilon\mathbf{E})\nabla\varepsilon + \frac{1}{\mu\varepsilon}\nabla\operatorname{div}(\varepsilon\mathbf{E})\\
&\quad + \frac{1}{\mu\varepsilon^2}(\nabla\varepsilon\cdot\mathbf{E})\nabla\varepsilon - \frac{1}{\mu\varepsilon}\nabla(\nabla\varepsilon\cdot\mathbf{E})
\end{aligned}
$$

in $H^{-1}(Q)^3$, which yields the equation

$$
\begin{aligned}
\Delta\mathbf{E} = {}& \mu\varepsilon\big(M^2(\mathbf{E},\mathbf{H})\big)_1 - \frac{\mu\sigma^2}{\varepsilon}\mathbf{E} + \frac{\mu\sigma}{\varepsilon}\operatorname{curl}\mathbf{H}\\
&- \frac{1}{\mu}(\nabla\mu)\times\operatorname{curl}\mathbf{E} - \frac{1}{\varepsilon^2}\operatorname{div}(\varepsilon\mathbf{E})\nabla\varepsilon + \frac{1}{\varepsilon}\nabla\operatorname{div}(\varepsilon\mathbf{E})\\
&+ \frac{1}{\varepsilon^2}(\nabla\varepsilon\cdot\mathbf{E})\nabla\varepsilon - \frac{1}{\varepsilon}\Big(\sum_{j=1}^{3}\big((\partial_{jk}\varepsilon)E_j + (\partial_j\varepsilon)\partial_k E_j\big)\Big)_{k=1}^{3},
\end{aligned}
\tag{3.4}
$$

where $\big(M^2(\mathbf{E},\mathbf{H})\big)_1$ is the first component of $M^2(\mathbf{E},\mathbf{H})$. Since $M^2(\mathbf{E},\mathbf{H}) \in L^2(Q)^6$ and $(\mathbf{E},\mathbf{H}) \in H^1(Q)^3$, the assumption (1.3) and the Sobolev embedding $H^1(Q)^3 \hookrightarrow L^6(Q)^3$ imply the estimate $\|\Delta\mathbf{E}\|_{L^2} \le c\,\|(\mathbf{E},\mathbf{H})\|_{X_2}$. In the same way one bounds $\Delta\mathbf{H}$. Standard interior elliptic regularity then shows that the fields belong to $H^2_{\mathrm{loc}}(Q)^6$.

2) We set $\Gamma^* = \Gamma_2\cup\Gamma_3$. From the identity $\Delta(\varepsilon E_1) = E_1\Delta\varepsilon + 2\nabla\varepsilon\cdot\nabla E_1 + \varepsilon\Delta E_1$, the embedding $H^1(Q)\hookrightarrow L^6(Q)$ and the assumption (1.3), we infer that $(I-\Delta)(\varepsilon E_1)$ belongs to $L^2(Q)$. Similarly, one sees that εE_1 is contained in $H^2_{\mathrm{loc}}(Q)$. The function εE_1 vanishes on Γ^* by step 1). We fix a smooth map ψ on $\overline{Q}$ having support in

$$
Q^{(\eta)} = [a_1^-, a_1^+]\times[a_2^- + \eta, a_2^+ - \eta]\times[a_3^- + \eta, a_3^+ - \eta] \tag{3.5}
$$

for a number $\eta = \eta(\psi)\in(0, d_Q/2)$. For each $\kappa\in(0,d_Q/2)$ we define

$$
Q_\kappa = (a_1^- + \kappa, a_1^+ - \kappa)\times(a_2^- + \kappa, a_2^+ - \kappa)\times(a_3^- + \kappa, a_3^+ - \kappa).
$$

We take $\kappa\in(0,\eta)$ and denote by $\Gamma_1^\pm(\kappa)$ those open faces of Q_κ that contain the points of the form $(a_1^\mp \pm \kappa, x_2, x_3)$. Dominated convergence and integration by parts imply that

$$
\begin{aligned}
\int_Q \varepsilon E_1\psi\,\mathrm{d}x + \int_Q \nabla(\varepsilon E_1)\cdot\nabla\psi\,\mathrm{d}x &= \lim_{\kappa\to0}\int_{Q_\kappa}\big(\varepsilon E_1\psi + \nabla(\varepsilon E_1)\cdot\nabla\psi\big)\,\mathrm{d}x\\
&= \lim_{\kappa\to0}\Big[\int_{Q_\kappa}\psi(I-\Delta)(\varepsilon E_1)\,\mathrm{d}x + \int_{\partial Q_\kappa}\psi\operatorname{tr}_n\nabla(\varepsilon E_1)\,\mathrm{d}\sigma\Big]\\
&= \int_Q \psi(I-\Delta)(\varepsilon E_1)\,\mathrm{d}x \pm \lim_{\kappa\to0}\int_{\Gamma_1^\pm(\kappa)}\psi\partial_1(\varepsilon E_1)\,\mathrm{d}\sigma
\end{aligned}
$$

by the support of ψ. We set $\rho = \operatorname{div}(\varepsilon\mathbf{E})\in H^1_{00}(Q)$. Integrating by part once more, we then deduce

$$
\begin{aligned}
&\int_Q \varepsilon E_1\psi\,\mathrm{d}x + \int_Q \nabla(\varepsilon E_1)\cdot\nabla\psi\,\mathrm{d}x\\
&= \int_Q \psi(I-\Delta)(\varepsilon E_1)\,\mathrm{d}x \pm \lim_{\kappa\to0}\int_{\Gamma_1^\pm(\kappa)}\Big(\psi\operatorname{div}(\varepsilon\mathbf{E}) - \psi\big(\partial_2(\varepsilon E_2) + \partial_3(\varepsilon E_3)\big)\Big)\,\mathrm{d}\sigma
\end{aligned}
$$

$$= \int_Q \psi(I-\Delta)(\varepsilon E_1)\,\mathrm{d}x \pm \lim_{\kappa\to 0}\int_{\Gamma_1^{\pm}(\kappa)} \Big(\psi\rho + \varepsilon(E_2\partial_2\psi + E_3\partial_3\psi)\Big)\,\mathrm{d}\sigma,$$

$$= \int_Q \psi(I-\Delta)(\varepsilon E_1)\,\mathrm{d}x + \int_{\Gamma_1^+} \psi\rho\,\mathrm{d}\sigma - \int_{\Gamma_1^-} \psi\rho\,\mathrm{d}\sigma, \tag{3.6}$$

using that ψ vanishes on the boundary of $\Gamma_1^{\pm}(\kappa)$, as well as E_2 and E_3 on Γ_1.

As in step 3) of the proof of Lemma 3.3 in [9], one approximates in $H^1(Q)$ each function from $H^1_{\Gamma^*}(Q)$ by maps ψ as above. Equation (3.6) is thus true for all $\psi \in H^1_{\Gamma^*}(Q)$. Lemma 3.1b) now implies that εE_1, and hence E_1, belong to $H^2(Q)$. Our reasoning also yields the asserted estimate. The components E_2 and E_3 are treated similarly, whereas $\mathbf{H}$ is handled as in Lemma 3.7 of [12].

3) It is clear that $D(M_2)$ is contained in $D(M^3)\cap X_2$. Let $(\mathbf{E},\mathbf{H}) \in D(M^3)\cap X_2$. Then $M(\mathbf{E},\mathbf{H}) =: (f,g)$ belongs to $D(M^2)$ and to X_{div}, see (2.3). To check that $\operatorname{div}(\varepsilon f)$ is an element of $H^1_{00}(Q)$, we compute

$$\operatorname{div}(\varepsilon f) = \operatorname{div}(\tfrac{\sigma}{\varepsilon}\varepsilon\mathbf{E}) = \nabla\sigma\cdot\mathbf{E} - \tfrac{\sigma}{\varepsilon}\nabla\varepsilon\cdot\mathbf{E} + \tfrac{\sigma}{\varepsilon}\operatorname{div}(\varepsilon\mathbf{E}). \tag{3.7}$$

Because of $\operatorname{div}(\varepsilon\mathbf{E}) \in H^1_{00}(Q)$, (1.3), $\mathbf{E} \in H^2(Q)^3$ and Sobolev's embedding, the last summand on the right-hand side of (3.7) belongs to $H^1(Q)$ and the other two even to $W^{1,3}(Q)$.

To treat the boundary condition, we first note that the map $f \mapsto \frac{\sigma}{\varepsilon} f$ is contained in $\mathcal{B}(L^2(\Gamma'))$ and in $\mathcal{B}(H^1_0(\Gamma'))$, and hence in $\mathcal{B}(H^{1/2}_0(\Gamma'))$ by interpolation, for each face Γ' of Q. As a result, $\frac{\sigma}{\varepsilon}\operatorname{div}(\varepsilon\mathbf{E})$ is an element of $H^1_{00}(Q)$.

The other two terms on the right-hand side of (3.7) have traces in $W^{2/3,3}(\Gamma)$. We first look at the summands $\varphi = (\partial_1\sigma)E_1 - \frac{\sigma}{\varepsilon}(\partial_1\varepsilon)E_1$. This function vanishes on $\Gamma_2\cup\Gamma_3$ by part 1), and in particular it is contained in $H^{1/2}_0(\Gamma')$ for the faces Γ' in $\Gamma_2\cup\Gamma_3$. As in step 2) of the proof of Lemma 3.3 in [9] and using Lemma 2.2 above, we construct smooth functions φ_n tending to φ in $W^{1,3}(Q)$ with support in the set $Q^{(1/n)}$, see (3.5). Their traces to Γ_1 thus converge in $W^{2/3,3}(\Gamma_1)$ as $n\to\infty$, and hence in $H^\theta(\Gamma_1)$ for each $\theta\in(1/2,1)$ by Sobolev's embedding. Since φ_n vanishes near the boundary of Γ_1 in Γ, the function $\operatorname{tr}_{\Gamma_1}\varphi$ belongs to the closed subspace $H^\theta_0(\Gamma_1)$ of $H^\theta(\Gamma_1)$, which is contained in $H^{1/2}_0(\Gamma_1)$ as noted above. The remaining summands are treated in the same way. We conclude that $M(\mathbf{E},\mathbf{H}) \in X_2$ as needed. $\square$

We can now prove the desired regularity result for the Maxwell system (1.1). If $g = g_1+g_2$ belongs to the space E below, we write $\|g\|_E = \|g_1\|_{L^1([0,T],D(M_2))} + \|g_2\|_{C^1([0,T],X_2)}$.

Proposition 3.3. *Let* (1.3) *hold. Then the following assertions are true.*

a) The restrictions of e^{tM} *to* X_2 *form a* C_0*–semigroup* $(e^{tM_2})_{t\ge 0}$ *generated by* M_2 *which is bounded by* $c(1+t)^3$.

b) Let $w_0 = (\boldsymbol{E}_0,\boldsymbol{H}_0)$ *belong to* $D(M_2)$, $g = (\frac{1}{\varepsilon}\boldsymbol{J},0) : [0,T]\to X_2$ *be continuous, and* g *be an element of* $E := L^1([0,T],D(M_2)) + W^{1,1}([0,T],X_2)$ *for some* $T\ge 1$. *Then the solution* $w = (\boldsymbol{E},\boldsymbol{H})$ *of* (1.1) *from* (2.6) *is contained in* $C([0,T],D(M_2))\cap C^1([0,T],X_2)$, *and* Mw *is bounded in* X_2 *by* $cT^3(\|w_0\|_{D(M_2)} + \|g\|_E)$.

The constant $c>0$ *only depends on the constants from* (1.3) *and on* Q.

Proof. In view of standard semigroup theory and Proposition 3.2, we only have to show that e^{tM} is strongly continuous and bounded in X_2 by $c(1+t)^3$. See Paragraph II.2.3 of [10] and Theorem 8.1.4 in [22], for instance, whereas the final estimate is an easy consequence of Duhamel's formula (2.6).

1) We set $w(t) = (\mathbf{E}(t), \mathbf{H}(t)) = \mathrm{e}^{tM} w_0$ for $w_0 \in X_2$ and $t \geq 0$. Proposition 2.3 of [9] implies that w is continuous in $X_{\mathrm{div}} \cap D(M^2)$ as well as bounded in X and $D(M^2)$ and linearly bounded in X_{div}. Because of Proposition 2.2 of [9], w is also continuous and linearly bounded in $H^1(Q)^6$. It remains to check that the charge density $\mathrm{div}(\varepsilon\mathbf{E}(t))$ is continuous in t and bounded by $c(1+t)^3\|w_0\|_{X_2}$ in $H^1_{00}(Q)$. The continuity and the linear boundedness is already known in $L^2(Q)$. We differentiate the second line of (2.7) with $\mathbf{J} = 0$, obtaining

$$\begin{aligned}\nabla \,\mathrm{div}(\varepsilon\mathbf{E}(t)) = {} & -t\mathrm{e}^{-\frac{\sigma}{\varepsilon}t}\nabla\big(\tfrac{\sigma}{\varepsilon}\big)\,\mathrm{div}(\varepsilon\mathbf{E}_0) + \mathrm{e}^{-\frac{\sigma}{\varepsilon}t}\nabla\,\mathrm{div}(\varepsilon\mathbf{E}_0)\\ & + \int_0^t \mathrm{e}^{-\frac{\sigma}{\varepsilon}(t-s)}\Big((s-t)\nabla\big(\tfrac{\sigma}{\varepsilon}\big)\,\nabla\big(\tfrac{\sigma}{\varepsilon}\big)\cdot\varepsilon\mathbf{E}(s) - D^2\big(\tfrac{\sigma}{\varepsilon}\big)\varepsilon\mathbf{E}(s)\\ & \qquad - \nabla\big(\tfrac{\sigma}{\varepsilon}\big)(\nabla\varepsilon)\mathbf{E}(s) - \varepsilon\big(\nabla\mathbf{E}(s)\big)^T\nabla\big(\tfrac{\sigma}{\varepsilon}\big)\Big)\,\mathrm{d}s\end{aligned}$$

at first in $H^{-1}(Q)^3$ for all $t \geq 0$. Set $\gamma_k = \|\,\mathrm{div}(\varepsilon\mathbf{E}_0)\|_{H^1} + \|w_0\|_{D(M^k_{\mathrm{div}})}$ for $k \in \{1,2\}$. Using (1.3), Sobolev's embedding and the above mentioned properties of $\mathbf{E}$, we deduce that the map $t \mapsto \mathrm{div}(\varepsilon\mathbf{E}(t))$ is continuous in $H^1(Q)$ and bounded by $c(1+t)^2\gamma_1$.

2) We next show an intermediate regularity result for $\mathbf{E}(t)$ in order to take traces of $\mathbf{E}(t)$ on the edges of Q. Let $t \geq 0$. One checks that $\Delta(\varepsilon E_1(t))$ belongs to $L^2(Q)$ as after (3.4) and that $\varepsilon E_1(t)$ satisfies (3.6) for all $\psi \in H^1_{\Gamma_2\cup\Gamma_3}(Q)$ as in step 2) of the proof of Proposition 3.2. The boundary inhomogeneity $\rho(t) = \mathrm{div}(\varepsilon E_1(t))$ in (3.6) belongs to $H^{1/2} \hookrightarrow H^\theta(\Gamma)$ for all $t \geq 0$ and $\theta \in (0, 1/2)$. Lemma 3.1c) hence yields that $\varepsilon E_1(t)$ is contained in $H^{\theta+3/2}(Q)$. By (1.3) and interpolation, the operator $f \mapsto \frac{1}{\varepsilon}f$ is continuous in $H^{\theta+3/2}(Q)$. As a result, $E_1(t)$ is an element of $H^{\theta+3/2}(Q) \hookrightarrow H^{\alpha+1/2}(Q)$ for all $\alpha \in (1/2, 1)$, and analogously for $E_2(t)$ and $E_3(t)$.

The same reasoning also shows that $\mathbf{E}(t)$ is bounded in $H^{\theta+3/2}(Q)^3$ by $c(1+t)^2\gamma_2$ for each $\theta \in (0, 1/2)$. Interpolating with its boundedness in $L^2(Q)$, we can estimate the norm of $\mathbf{E}(t)$ in $H^{\alpha+1/2}(Q)^3$ by $c(1+t)^{\alpha'+1/2}\gamma_2$ for any $\alpha' \in (\alpha, 1)$. (Here the constant also depends on α', but this number will be fixed at the end of the proof.)

3) We still have to treat the map $t \mapsto \rho(t)$ in $H^{1/2}_0(\Gamma')$ for the faces Γ' of Q. We first infer from part 2) that $\mathrm{tr}\, E_1(t)$ belongs to $H^\alpha(\Gamma)$ so that $\mathrm{tr}_{\Gamma_j} E_1(t)$ has traces on the edges forming the boundary of Γ_j within Γ. Let $j \in \{2,3\}$. The functions $\mathrm{tr}_{\Gamma_j} E_1(t)$ vanish due to (2.5). For continuous functions f in $H^\alpha(\Gamma)$ the traces of $\mathrm{tr}_{\Gamma_j} f$ and of $\mathrm{tr}_{\Gamma_1} f$ coincide on common edges, and by approximation the same is true for $E_1(t)$. Therefore, $\mathrm{tr}_{\Gamma_1} E_1(t)$ is an element of $H^\alpha_0(\Gamma_1) \hookrightarrow H^{1/2}_0(\Gamma_1)$. The other components can be treated analogously so that $\mathbf{E}(t)$ is contained in $H^1_{00}(Q)^3$. We see by similar arguments that the map $t \mapsto \mathbf{E}(t)$ is continuous and bounded in this space by $c(1+t)^{\alpha'+1/2}\gamma_2$.

To extend these properties to ρ, we again use the second line of (2.7) with $\mathbf{J} = 0$. The function $\mathrm{e}^{-\frac{\sigma}{\varepsilon}t}$ is Lipschitz on Q and hence induces a multiplication operator on $H_0^1(\Gamma')$ and $L^2(\Gamma')$, which is linearly bounded on $H_0^1(\Gamma')$ for $t \geq 0$ and bounded on $L^2(\Gamma')$. By interpolation, it acts on $H_0^{1/2}(\Gamma')$ with norm less or equal $c(1+t)^{1/2}$ so that the summand $\mathrm{e}^{-\frac{\sigma}{\varepsilon}t} \operatorname{div}(\varepsilon \mathbf{E}_0)$ is continuous and bounded by $c(1+t)^{1/2}\|w_0\|_{X_2}$ in $H_0^{1/2}(\Gamma')$. The coefficient $\varepsilon\nabla(\frac{\sigma}{\varepsilon})$ belongs to $W^{1,3}(Q)^3 \cap L^\infty(Q)^3$ by (1.3). In view of the above observations with $\alpha = 4/5$ and $\alpha' = 9/10$, the fields $\mathbf{E}(s)$ and $\nabla\mathbf{E}(s)$ are contained in $H^{\alpha+1/2}(Q)^3 \hookrightarrow L^{15}(Q)^3$ and $H^{\alpha-1/2}(Q)^3 \hookrightarrow L^{5/2}(Q)^3$, respectively, where we use Sobolev's embedding. The integrand $\varphi(s)$ in the second line of (2.7) is thus continuous in $W^{1,5/2}(Q)$ and bounded by $c(1+s)^{\alpha'+1/2}\gamma_2$. As at the end of the proof of Proposition 3.2, we then deduce that $\operatorname{tr}_{\Gamma'} \varphi(s)$ has the analogous properties in $H_0^{1/2}(\Gamma')$. Summing up, the map $s \mapsto \operatorname{div}(\varepsilon\mathbf{E}(s))$ from $[0,t]$ to $H_0^{1/2}(\Gamma')$ is continuous and bounded by $c(1+t)^{\alpha'+2}\|w_0\|_{X_2}$. $\square$

Remark 3.4. If we also assume in Proposition 3.3 that $\sigma = 0$ or $\sigma \geq \sigma_0$ for a constant $\sigma_0 > 0$, then Remark 2.4 of [9] and an inspection of the above proof shows that we can omit the factors $(1+t)^3$ and T^3. If $\sigma \geq \sigma_0$, then the constant c also depends on $1/\sigma_0$.

4. The split operators

We first recall results from [9] about the operators

$$A = \begin{pmatrix} -\frac{\sigma}{2\varepsilon}I & \frac{1}{\varepsilon}C_1 \\ \frac{1}{\mu}C_2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\frac{\sigma}{2\varepsilon}I & -\frac{1}{\varepsilon}C_2 \\ -\frac{1}{\mu}C_1 & 0 \end{pmatrix} \quad \text{with}$$
$$C_1 = \begin{pmatrix} 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_1 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & \partial_3 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

of the splitting scheme. In X these operators are endowed with the domains

$$D(A) = \{(u,v) \in X \mid (C_1 v, C_2 u) \in X,\ \ \operatorname{tr}_{\Gamma_2} u_1 = 0,\ \ \operatorname{tr}_{\Gamma_3} u_2 = 0,\ \ \operatorname{tr}_{\Gamma_1} u_3 = 0\},$$
$$D(B) = \{(u,v) \in X \mid (C_2 v, C_1 u) \in X,\ \ \operatorname{tr}_{\Gamma_3} u_1 = 0,\ \ \operatorname{tr}_{\Gamma_1} u_2 = 0,\ \ \operatorname{tr}_{\Gamma_2} u_3 = 0\}.$$

Each of it contains one half of the electric boundary conditions in (1.1), but the magnetic ones and the divergence conditions are not included. Clearly, $A + B = M$ on $D(A) \cap D(B) \hookrightarrow D(M)$ and $D(M_{\mathrm{div}}) \hookrightarrow D(A) \cap D(B)$ by (2.4) and (2.5).

From (3.3) in [9] we recall the following crucial integration by parts formula. Let $u, \varphi \in L^2(Q)^3$ satisfy $C_1\varphi \in L^2(Q)^2$, $C_2 u \in L^2(Q)^3$, and

$$\operatorname{tr}_{\Gamma_3} u_2 \cdot \operatorname{tr}_{\Gamma_3} \varphi_1 = 0, \qquad \operatorname{tr}_{\Gamma_1} u_3 \cdot \operatorname{tr}_{\Gamma_1} \varphi_2 = 0, \qquad \operatorname{tr}_{\Gamma_2} u_1 \cdot \operatorname{tr}_{\Gamma_2} \varphi_3 = 0.$$

(For instance, take $(u,\varphi) \in D(A)$ or $(\varphi, u) \in D(B)$.) We then have

$$(C_2 u \mid \varphi)_{L^2} = (u \mid -C_1\varphi)_{L^2}. \tag{4.2}$$

For our error analysis we need the restrictions of the above operators to a suitable subset of $H^2(Q)^6$. In [9] we have already discussed the space

$$Y = \{(u,v) \in H^1(Q)^6 \mid u_j = 0 \text{ on } \Gamma \setminus \Gamma_j,\ \ v_j = 0 \text{ on } \Gamma_j \text{ for all } j \in \{1,2,3\}\}.$$

It is a Hilbert space for the weighted inner product

$$\big((u,v) \mid (\varphi,\psi)\big)_Y = \int_Q \Big(\varepsilon u\cdot\varphi + \mu v\cdot\psi + \varepsilon\sum_{j=1}^3 \partial_j u\cdot\partial_j\varphi + \mu\sum_{j=1}^3 \partial_j v\cdot\partial_j\psi\Big)\,\mathrm{d}x$$

whose the induced norm $\|\cdot\|_Y$ is equivalent to the usual H^1–norm. We denote by A_Y and B_Y the parts of A and B in Y, respectively. By Lemma 3.2 of [9] they have the domains

$$\begin{aligned}
D(A_Y) = \{(u,v)\in H^1(Q)^6 \mid{} & u_j = 0 \text{ on } \Gamma\setminus\Gamma_j,\ \ v_j = 0 \text{ on } \Gamma_j \text{ for } j\in\{1,2,3\},\\
& \partial_2 u_1, \partial_3 u_2, \partial_1 u_3, \partial_3 v_1, \partial_1 v_2, \partial_2 v_3 \in H^1(Q),\\
& \partial_3 v_1 = 0 \text{ on } \Gamma_3,\ \ \partial_1 v_2 = 0 \text{ on } \Gamma_1,\ \ \partial_2 v_3 = 0 \text{ on } \Gamma_2\},\\
D(B_Y) = \{(u,v)\in H^1(Q)^6 \mid{} & u_j = 0 \text{ on } \Gamma\setminus\Gamma_j,\ v_j = 0 \text{ on } \Gamma_j \text{ for } j\in\{1,2,3\},\\
& \partial_3 u_1, \partial_1 u_2, \partial_2 u_3, \partial_2 v_1, \partial_3 v_2, \partial_1 v_3 \in H^1(Q),\\
& \partial_2 v_1 = 0 \text{ on } \Gamma_2,\ \partial_3 v_2 = 0 \text{ on } \Gamma_3,\ \ \partial_1 v_3 = 0 \text{ on } \Gamma_1\}.
\end{aligned}$$

Moreover, $A_Y - \kappa_Y I$ and $B_Y - \kappa_Y I$ generate contraction semigroups on Y for a number $\kappa_Y \ge 0$ which only depends on the constants in (1.3) and vanishes if ε, μ and σ are constant. See Proposition 3.6 in [9].

We next extend crucial embeddings from Proposition 4.4 in [12] to the case of non-zero conductivity and charges.

Proposition 4.1. *Let* (1.3) *hold. Then X_2 is embedded into $D(A^2)\cap D(B^2)\cap D(AB)\cap D(BA)$ with constants only depending on those in* (1.3) *and on Q.*

Proof. Let $(u,v)\in X_2$. The space X_2 is embedded into $H^2(Q)^6$ by Proposition 3.2 so that the result will follow as soon as we know that (u,v) satisfies the respective boundary conditions. The first component of $A(u,v)$ and $B(u,v)$ have the traces

$$-\frac{\sigma}{2\varepsilon}u_1 + \frac{1}{\varepsilon}\partial_2 v_3 = 0 \quad \text{on } \Gamma_3 \qquad \text{and} \qquad -\frac{\sigma}{2\varepsilon}u_1 - \frac{1}{\varepsilon}\partial_3 v_2 = 0 \quad \text{on } \Gamma_2,$$

respectively, due to (2.5) and Lemma 2.1. These are the boundary conditions for the first component in $D(B)$ and $D(A)$, respectively. Since $M(u,v)$ belongs to $D(M)$, we similarly obtain

$$\begin{aligned}
-\frac{\sigma}{2\varepsilon}u_1 + \frac{1}{\varepsilon}\partial_2 v_3 &= (M(u,v))_1 + \frac{\sigma}{2\varepsilon}u_1 + \frac{1}{\varepsilon}\partial_3 v_2 = 0 \qquad \text{on } \Gamma_2,\\
-\frac{\sigma}{2\varepsilon}u_1 - \frac{1}{\varepsilon}\partial_3 v_2 &= (M(u,v))_1 + \frac{\sigma}{2\varepsilon}u_1 - \frac{1}{\varepsilon}\partial_2 v_3 = 0 \qquad \text{on } \Gamma_3,
\end{aligned}$$

respectively, where $(M(u,v))_1$ is the first component of $M(u,v)$. So these maps also fulfill the conditions for the first component in $D(A)$ and $D(B)$, respectively. The second and third components are treated analogously. $\square$

For our error analysis we have to show that the operators A and B behave well in H^2. To this aim, we introduce the space

$$\begin{aligned}
Z = \{(u,v)\in H^2(Q)^6 \mid{} & u_i = 0 \text{ on } \Gamma\setminus\Gamma_i,\ \ v_i = 0 \text{ on } \Gamma_i,\ \ \partial_i u_i = 0 \text{ on } \Gamma_i,\\
& \partial_j v_k = 0 \text{ on } \Gamma_j \text{ for all } i,j,k\in\{1,2,3\} \text{ with } j\neq k\}
\end{aligned}$$

endowed with the weighted inner product

$$\big((u,v)\mid(\varphi,\psi)\big)_Z=\int_Q\Big[\varepsilon u\cdot\varphi+\mu v\cdot\psi+\sum_{j,k=1}^{3}\big(\varepsilon\partial_{jk}u\cdot\partial_{jk}\varphi+\mu\partial_{jk}v\cdot\partial_{jk}\psi\big)\Big]\,\mathrm{d}x.$$

Because of (1.3) and the continuity of the traces, Z is a closed subspace of $H^2(Q)^6$ with an equivalent induced norm $\|\cdot\|_Z$. Clearly, Z is embedded into $D(A_Y)\cap D(B_Y)$. We define the restrictions A_Z and B_Z of A and B, respectively, on the domains

$$\begin{aligned}D(A_Z)=\{(u,v)\in Z\mid{}&\partial_2u_1,\partial_3u_2,\partial_1u_3,\partial_3v_1,\partial_1v_2,\partial_2v_3\in H^2(Q),\\ &\partial_{22}u_1=0\text{ on }\Gamma_2,\ \ \partial_{33}u_2=0\text{ on }\Gamma_3,\ \ \partial_{11}u_3=0\text{ on }\Gamma_1\}\\ D(B_Z)=\{(u,v)\in Z\mid{}&\partial_3u_1,\partial_1u_2,\partial_2u_3,\partial_2v_1,\partial_3v_2,\partial_1v_3\in H^2(Q),\\ &\partial_{33}u_1=0\text{ on }\Gamma_3,\ \ \partial_{11}u_2=0\text{ on }\Gamma_1,\ \ \partial_{22}u_3=0\text{ on }\Gamma_2\}.\end{aligned}$$

These operators are not defined as the parts of A and B in Z (in contrast to A_Y and B_Y) because of certain technical problems in later proofs. To enforce that A_Z and B_Z map into Z, we have to impose a Neumann boundary condition on the coefficients.

Lemma 4.2. *Assume that* (1.3) *holds and that* $\partial_\nu\varepsilon=\partial_\nu\mu=\partial_\nu\sigma=0$ *on* Γ. *Then the operators* A_Z *and* B_Z *map their domains into* Z.

Proof. Let $(u,v)\in D(A_Z)$ and set $(f,g)=A(u,v)$. By the definition of $D(A_Z)$, (1.3) and $D(A_Z)\subseteq D(A_Y)$, the functions (f,g) belong to $H^2(Q)^6\cap Y$. It thus remains to check the boundary conditions of first order in the definition of Z. We compute

$$\partial_1f_1=-\partial_1\big(\tfrac{\sigma}{2\varepsilon}u_1\big)+\partial_1\big(\tfrac{1}{\varepsilon}\partial_2v_3\big)=-\tfrac{\partial_1\sigma}{2\varepsilon}u_1+\tfrac{\sigma\partial_1\varepsilon}{2\varepsilon^2}u_1-\tfrac{\sigma}{2\varepsilon}\partial_1u_1-\tfrac{\partial_1\varepsilon}{\varepsilon^2}\partial_2v_3+\tfrac{1}{\varepsilon}\partial_2\partial_1v_3.$$

Our regularity assumptions imply that each summand has a trace. The third and fifth ones vanish on Γ_1 because of $(u,v)\in D(A_Z)$ and Lemma 2.1. The other summands have zero traces on Γ_1 thanks to the extra assumptions on ε and σ. Similarly, we obtain

$$\partial_jg_1=\partial_j\big(\tfrac{1}{\mu}\partial_3u_2\big)=-\tfrac{\partial_j\mu}{\mu^2}\partial_3u_2+\tfrac{1}{\mu}\partial_3\partial_ju_2=0$$

on Γ_j for $j\in\{2,3\}$. Treating the other components of (f,g) analogously, we see that these functions are contained in Z. The operator B_Z is handled in the same way. □

In the next lemmas we establish the basic properties of A_Z and B_Z.

Lemma 4.3. *Assume that* (1.3) *holds and that* $\partial_\nu\varepsilon=\partial_\nu\mu=\partial_\nu\sigma=0$ *on* Γ. *Then the operators* A_Z *and* B_Z *are closed and densely defined in* Z.

Proof. As above we only treat A_Z since B_Z can be handled analogously.

1) To show the closedness of A_Z, take a sequence $((u_n,v_n))$ in $D(A_Z)$ such that $(u_n,v_n)\to(u,v)$ and $A_Z(u_n,v_n)=A(u_n,v_n)\to(f,g)$ in Z as $n\to\infty$. We then have $A(u,v)=(f,g)$ because A is continuous from Z to $H^1(Q)^6$. Assumption (1.3) yields the limit

$$w_n:=\begin{pmatrix}C_1v_n\\C_2u_n\end{pmatrix}=\begin{pmatrix}\varepsilon&0\\0&\mu\end{pmatrix}A(u_n,v_n)+\begin{pmatrix}\frac{\sigma}{2}u_n\\0\end{pmatrix}\ \longrightarrow\ \begin{pmatrix}\varepsilon f+\frac{\sigma}{2}u\\\mu g\end{pmatrix}$$

in $H^2(Q)^6$. Since also $w_n \to w := (C_1 v, C_2 u)$ in $H^1(Q)^6$, the map w belongs to $H^2(Q)^6$, which is the extra regularity demanded by $D(A_Z)$. By this limit also the second-order boundary conditions in $D(A_Z)$ for u_n transfer to u. As a result, (u, v) is an element of $D(A_Z)$ and A_Z is thus closed in Z.

2) Let $(u, v) \in Z$. Take $n \in \mathbb{N}$ with $n > 2/d_Q =: \ell$. Let $\rho_n^{(j)} : \mathbb{R} \to [0, \infty)$ be symmetric and smooth with integral 1 and $\operatorname{supp}(\rho_n^{(j)}) \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right]$. The superscript indicates that it acts on the j-th variable for $j \in \{1, 2, 3\}$. We employ the antisymmetric extension

$$\widetilde{u}_1(x_1, x_2, x_3) = \begin{cases} -u_1(x_1, 2a_2^- - x_2, x_3), & x_2 \in (2a_2^- - a_2^+, a_2^-), \\ u_1(x_1, x_2, x_3), & x_2 \in [a_2^-, a_2^+], \\ -u_1(x_1, 2a_2^+ - x_2, x_3), & x_2 \in (a_2^+, 2a_2^+ - a_2^-), \end{cases}$$

of u_1 to the enlarged cuboid $\widetilde{Q} = (a_1^-, a_1^+) \times (2a_2^- - a_2^+, 2a_2^+ - a_2^-) \times (a_3^-, a_3^+)$. Using that $u_1 \in H^2(Q)$ vanishes on Γ_2, one can check that $\widetilde{u}_1$ belongs to $H^2(\widetilde{Q})$. Moreover, $\widetilde{u}_1 = 0$ on $\Gamma_2 \cup \Gamma_3$ and $\partial_1 \widetilde{u}_1 = 0$ on Γ_1 by the definition of Z. We extend $\widetilde{u}_1$ by 0 to $\mathbb{R}^3$ and set

$$\varphi_n = (\rho_n^{(2)} * \widetilde{u}_1)|_Q.$$

Clearly, the functions $\partial_2^k \varphi_n$ belong to $H^2(Q) \hookrightarrow C(\overline{Q})$ for all $k \in \mathbb{N}_0$, and (φ_n) tends to u_1 in $H^2(Q)$ as $n \to \infty$. The properties of $\widetilde{u}_1$ imply that $\varphi_n = 0$ on Γ_3 and $\partial_1 \varphi_n = \rho_n^{(2)} * \partial_1 \widetilde{u}_1 = 0$ on Γ_1. (Use that $\partial_1 \widetilde{u}_1 \in L^2(\mathbb{R}, H_0^1((a_1^-, a_1^+) \times (a_3^-, a_3^+)))$, for instance.) By means of the symmetry of $\rho_n^{(2)}$, we further compute

$$\begin{aligned} \varphi_n(x_1, a_2^-, x_3) &= \int_{-\frac{1}{n}}^{\frac{1}{n}} \rho_n^{(2)}(t) \widetilde{u}_1(x_1, a_2^- - t, x_3) \, \mathrm{d}t \\ &= \int_{-\frac{1}{n}}^{0} \rho_n^{(2)}(t) u_1(x_1, a_2^- - t, x_3) \, \mathrm{d}t - \int_0^{\frac{1}{n}} \rho_n^{(2)}(t) u_1(x_1, a_2^- + t, x_3) \, \mathrm{d}t \\ &= \int_0^{\frac{1}{n}} \rho_n^{(2)}(-s) u_1(x_1, a_2^- + s, x_3) \, \mathrm{d}s - \int_0^{\frac{1}{n}} \rho_n^{(2)}(t) u_1(x_1, a_2^- + t, x_3) \, \mathrm{d}t \\ &= 0 \end{aligned}$$

for all $(x_1, x_3) \in (a_1^-, a_1^+) \times (a_3^-, a_3^+)$, and analogously at $x_2 = a_2^+$. Therefore, $\varphi_n = 0$ vanishes on Γ_2. Let $\eta > 0$. We can then fix an index $m > \ell$ with $\|\varphi_m - u_1\|_{H^2} \le \eta$ and set $\widehat{u}_1 = \varphi_m$.

3) As $\partial_{22} \widehat{u}_1$ does not necessarily vanish on Γ_2, we define

$$\begin{aligned} u_1^n(x_1, x_2, x_3) = \widehat{u}_1(x_1, x_2, x_3) &+ \alpha(x_2) \int_{a_2^-}^{x_2} \int_{a_2^-}^{t} \big(\chi_n(s) - 1\big) \partial_{22} \widehat{u}_1(x_1, s, x_3) \, \mathrm{d}s \, \mathrm{d}t \\ &+ \beta(x_2) \int_{x_2}^{a_2^+} \int_t^{a_2^+} \big(\chi_n(s) - 1\big) \partial_{22} \widehat{u}_1(x_1, s, x_3) \, \mathrm{d}s \, \mathrm{d}t \end{aligned}$$

for $x \in Q$, where α, β and χ_n are smooth functions from $[a_2^-, a_2^+]$ to $[0, 1]$ satisfying $\alpha + \beta = 1$, $\alpha = 1$ near a_1^-, $\beta = 1$ near a_2^+, $\chi_n = 0$ near $a_2^\pm$, and $\chi_n = 1$ on $[a_2^- + 1/n, a_2^+ - 1/n]$. By dominated convergence, we conclude that u_1^n tends to $\widehat{u}_1$ in $H^2(Q)$ as $n \to \infty$. Choosing a sufficiently large n, we thus obtain the bound $\|u_1^n - u_1\|_{H^2} \le 2\eta$.

Since $\widehat{u}_1 = 0$ on $\Gamma_2 \cup \Gamma_3$ and $\partial_1 \widehat{u}_1 = 0$ on Γ_1, Lemma 2.1 shows that $\partial_{22}\widehat{u}_1$ vanishes on Γ_3 and $\partial_{22}\partial_1\widehat{u}_1$ on Γ_1. We then deduce that $u_1^n = 0$ on Γ_3 and $\partial_1 u_1^n = 0$ on Γ_1. Taking also into account the behavior of α, β and χ_n, we see that u_1^n has zero trace on Γ_2 and

$$\partial_{22}u_1^n = \partial_{22}\widehat{u}_1 + \alpha(\chi_n - 1)\partial_{22}\widehat{u}_1 + \beta(\chi_n - 1)\partial_{22}\widehat{u}_1 = \partial_{22}\widehat{u}_1 - \partial_{22}\widehat{u}_1 = 0$$

near Γ_2. Hence, u_1^n satisfies the properties of a first component of a function in $D(A_Z)$. The components u_2 and u_3 are treated similarly.

4) To deal with v_1, we redefine the cuboids as $\widetilde{Q} = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (2a_3^- - a_3^+, 2a_3^+ - a_3^-)$. Because of $\partial_3 v_1 = 0$ on Γ_3, the symmetric extension

$$\widetilde{v}_1(x_1, x_2, x_3) = \begin{cases} v_1(x_1, x_2, 2a_3^- - x_3), & x_3 \in (2a_3^- - a_3^+, a_3^-), \\ v_1(x_1, x_2, x_3), & x_3 \in [a_3^-, a_3^+], \\ v_1(x_1, x_2, 2a_3^+ - x_3), & x_3 \in (2a_3^+ - a_3^-, a_3^+), \end{cases}$$

of $v_1 \in H^2(Q)$ is contained in $H^2(\widetilde{Q})$. Since (u, v) belongs to Z, we obtain the traces $\widetilde{v}_1 = 0$ on Γ_1, $\partial_2 \widetilde{v}_1 = 0$ on Γ_2 and $\partial_3 \widetilde{v}_1 = 0$ on Γ_3. For $n > \ell$ we extend $\widetilde{v}_1$ by 0 to $\mathbb{R}^3$ and set

$$v_1^n(x) = \big(\rho_n^{(3)} * \widetilde{v}_1\big)(x) = \int_{-1/n}^{1/n} \rho_n^{(3)}(t)\widetilde{v}_1(x_1, x_2, x_3 - t)\,\mathrm{d}t$$

for $x \in Q$. Then v_1^n and $\partial_3 v_1^n$ are elements of $H^2(Q)$, and v_1^n tends to v_1 in $H^2(Q)$ as $n \to \infty$. The properties of $\widetilde{v}_1$ imply that $v_1^n = 0$ on Γ_1 and $\partial_2 v_1^n = 0$ on Γ_2. We further compute

$$\begin{aligned} &(\partial_3 v_1^n)(x_1, x_2, a_3^-) \\ &= \int_{-1/n}^{0} \rho_n^{(3)}(t)(\partial_3 v_1)(x_1, x_2, a_3^- - t)\,\mathrm{d}t - \int_0^{1/n} \rho_n^{(3)}(t)(\partial_3 v_1)(x_1, x_2, a_3^- + t)\,\mathrm{d}t \\ &= \int_0^{1/n} \rho_n^{(3)}(-s)(\partial_3 v_1)(x_1, x_2, a_3^- + s)\,\mathrm{d}s - \int_0^{1/n} \rho_n^{(3)}(t)(\partial_3 v_1)(x_1, x_2, a_3^- + t)\,\mathrm{d}t \\ &= 0 \end{aligned}$$

due to the symmetry of $\rho_n^{(3)}$, and analogously $\partial_3 v_1^n(x_1, x_2, a_3^+) = 0$. Hence, the trace of $\partial_3 v_1$ on Γ_3 vanishes. As a result, v_1^n fits to $D(A_Z)$. The remaining two components of v are treated similarly. Altogether we have approximated (u, v) in $H^2(Q)^6$ by functions (u^n, v^n) in $D(A_Z)$. $\square$

We show below the dissipativity of A_Z and B_Z up to a shift by

$$\begin{aligned} \kappa_Z := \max\big\{c_0\big(&\|\nabla\sigma\|_{L^\infty} + \|\nabla\varepsilon\|_{L^\infty} + \|\nabla\mu\|_{L^\infty} \\ &+ \|D^2\sigma\|_{L^3} + \|D^2\varepsilon\|_{L^3} + \|D^2\mu\|_{L^3}\big), \kappa_Y\big\}, \end{aligned} \tag{4.3}$$

with κ_Y from (3.6) in [9]. The constant c_0 only depends on Q and the constants in (1.3), and it is determined by the next proof. Observe that $\kappa_Z = 0$ if all coefficients are constant.

Lemma 4.4. *Assume that* (1.3) *holds and that* $\partial_\nu\varepsilon = \partial_\nu\mu = \partial_\nu\sigma = 0$ *on* Γ. *Then the operators* $A_Z - \kappa_Z I$ *and* $B_Z - \kappa_Z I$ *are dissipative on* Z.

Proof. Let $(u,v) \in D(A_Z)$. In view of (4.2) we want to show that $\partial_{jk}u_2\,\partial_{jk}v_1 = 0$ on Γ_3, for instance. Note that these traces exist by $C_2u, C_1v \in H^2(Q)^3$. Lemma 2.1 implies this trace equality for $\{j,k\} \in \{1,2\}$ since $u_2 = 0$ on Γ_3 by the definition of Z. The desired equality also follows from $\partial_3 v_1 = 0$ on Γ_3 in Z if either j or k is equal to 3. Finally, $\partial_{33}u_2$ vanishes on Γ_3 because of $(u,v) \in D(A_Z)$. Treating the other components in the same way, we derive from (4.2) the identity

$$\sum_{j,k=1}^{3} \int_Q \big(\partial_{jk}C_1v\cdot\partial_{jk}u + \partial_{jk}C_2u\cdot\partial_{jk}v\big)\,\mathrm{d}x = 0.$$

Using this formula, (4.2), assumption (1.3), Sobolev's embedding and Young's inequality we arrive at

$$\begin{aligned}
&\operatorname{Re}\big(A(u,v)\mid(u,v)\big)_Z\\
&= \int_Q \Big[-\frac{\sigma\varepsilon}{2\varepsilon}|u|^2 + \frac{\varepsilon}{\varepsilon}C_1v\cdot u + \frac{\mu}{\mu}C_2u\cdot v + \sum_{j,k=1}^{3}\Big(-\varepsilon\partial_{jk}\big(\tfrac{\sigma}{2\varepsilon}u\big)\cdot\partial_{jk}u\\
&\qquad + \varepsilon\partial_{jk}\big(\tfrac{1}{\varepsilon}C_1v\big)\cdot\partial_{jk}u + \mu\partial_{jk}\big(\tfrac{1}{\mu}C_2u\big)\cdot\partial_{jk}v\Big)\Big]\mathrm{d}x\\
&\le \sum_{j,k=1}^{3}\Big[-\int_Q\Big(\tfrac{\sigma}{2}|\partial_{jk}u|^2 + \big(\tfrac{\partial_j\sigma}{2} - \tfrac{\sigma\partial_j\varepsilon}{2\varepsilon}\big)\partial_k u\cdot\partial_{jk}u + \big(\tfrac{\partial_k\sigma}{2} - \tfrac{\sigma\partial_k\varepsilon}{2\varepsilon}\big)\partial_j u\cdot\partial_{jk}u\Big)\mathrm{d}x\\
&\quad - \int_Q\big(\tfrac{\partial_{jk}\sigma}{2} - \tfrac{\partial_j\sigma\partial_k\varepsilon}{2\varepsilon} - \tfrac{\partial_k\sigma\partial_j\varepsilon}{2\varepsilon} - \tfrac{\sigma\partial_{jk}\varepsilon}{2\varepsilon} + \tfrac{\sigma(\partial_j\varepsilon)\partial_k\varepsilon}{\varepsilon^2}\big)u\cdot\partial_{jk}u\,\mathrm{d}x\\
&\quad + \int_Q\Big(\big(-\tfrac{\partial_{jk}\varepsilon}{\varepsilon} + \tfrac{2(\partial_j\varepsilon)\partial_k\varepsilon}{\varepsilon^2}\big)C_1v - \tfrac{\partial_j\varepsilon}{\varepsilon}\partial_k C_1v - \tfrac{\partial_k\varepsilon}{\varepsilon}\partial_j C_1v\Big)\cdot\partial_{jk}u\,\mathrm{d}x\\
&\quad + \int_Q\Big(\big(-\tfrac{\partial_{jk}\mu}{\mu} + \tfrac{2(\partial_j\mu)\partial_k\mu}{\mu^2}\big)C_2u - \tfrac{\partial_j\mu}{\mu}\partial_k C_2u - \tfrac{\partial_k\mu}{\mu}\partial_j C_2u\Big)\cdot\partial_{jk}v\,\mathrm{d}x\Big]\\
&\le c\kappa_Z\,\|(u,v)\|_{H^2} \le \kappa_Z\,\|(u,v)\|_Z^2.
\end{aligned}$$

The statement for B_Z is established in the same way. □

Lemma 4.5. *Assume that* (1.3) *holds and that* $\partial_\nu\varepsilon = \partial_\nu\mu = \partial_\nu\sigma = 0$ *on* Γ. *Then the operators* $(1+\kappa_Z)I - A_Z$ *and* $(1+\kappa_Z)I - B_Z$ *have a dense range in* Z.

Proof. As above we only treat the operator $(1+\kappa_Z)I - A_Z$. We take (f,g) from the dense subspace $D(A_Z)$, see Lemma 4.3, which is contained in $D(A_Y)$. If we replace in Lemma 3.4 of [9] the number κ_Y by κ_Z, we obtain fields $(u,v) \in D(A_Y)$ solving $\big((1+\kappa_Z)I - A\big)(u,v) = (f,g)$. The first and last of these equations read as

$$\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)u_1 - \partial_2 v_3 = \varepsilon f_1, \qquad (1+\kappa_Z)\mu v_3 - \partial_2 u_1 = \mu g_3. \tag{4.4}$$

It follows

$$Lu_1 := \big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)u_1 - \tfrac{1}{1+\kappa_Z}\partial_2\big(\tfrac{1}{\mu}\partial_2 u_1\big) = \varepsilon f_1 + \tfrac{1}{1+\kappa_Z}\partial_2 g_3 =: h_1. \tag{4.5}$$

We have to show that u_1, v_3, $\partial_2 u_1$ and $\partial_2 v_3$ belong to $H^2(Q)$, and not just to $H^1(Q)$. Moreover, we need the traces $\partial_1 u_1 = 0$ and $\partial_1 v_3 = 0$ on Γ_1 as well as $\partial_{22} u_1 = 0$ on Γ_2. The trace condition $\partial_2 v_3 = 0$ on Γ_2 is fulfilled since $(u, v) \in D(A_Y)$.

We first note that by the definition of $D(A_Z)$ the functions f_1, g_3, $\partial_2 f_1$ and $\partial_2 g_3$ belong to $H^2(Q)$, $f_1 = 0$ on Γ_2 and $\partial_j g_3 = 0$ on Γ_j for $j \in \{1, 2\}$. Hence, h_1 is contained in $H^2(Q)$ and satisfies $h_1 = 0$ on Γ_2, so that also $\partial_1 h_1 = 0$ on Γ_2 by Lemma 2.1.

As in Lemma 3.4 of [9] we use the domain $D(\partial_2) = \{\varphi \in L^2(Q) \,|\, \partial_2\varphi \in L^2(Q),\ \varphi = 0 \text{ on } \Gamma_2\}$. Let $j, k \in \{1, 2, 3\}$. Since $\partial_2\partial_{jk} u_1 = \partial_{jk}\partial_2 u_1$ in $H^{-2}(Q)$ and $\partial_2 u_1 \in H^1(Q)$, the function $\partial_2(\mu^{-1}\partial_2\partial_{jk} u_1)$ belongs to $H^{-2}(Q)$. Let $\varphi \in H^3_0(Q)$. Using (1.3) and integrating by parts, we can thus compute

$$
\begin{aligned}
&\langle L\partial_{jk} u_1, \varphi\rangle_{H^{-2}\times H^2_0} \\
&= \big\langle \partial_{jk} u_1, \big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\varphi\big\rangle_{H^{-1}\times H^1_0} - \frac{1}{1+\kappa_Z}\Big\langle \partial_2 \tfrac{1}{\mu}\partial_2\partial_{jk} u_1, \varphi\Big\rangle_{H^{-2}\times H^2_0} \\
&= \int_Q u_1 \Big(\big(\partial_{jk}\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\big)\varphi + \big(\partial_j\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\big)\partial_k\varphi \\
&\qquad + \big(\partial_k\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\big)\partial_j\varphi + \big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\partial_{jk}\varphi\Big)\,\mathrm{d}x \\
&\quad + \frac{1}{1+\kappa_Z}\int_Q \partial_2 u_1 \Big(\big(\partial_{jk}\tfrac{1}{\mu}\big)\partial_2\varphi + \big(\partial_j\tfrac{1}{\mu}\big)\partial_k\partial_2\varphi + \big(\partial_k\tfrac{1}{\mu}\big)\partial_j\partial_2\varphi + \tfrac{1}{\mu}\partial_2\partial_{jk}\varphi\Big)\,\mathrm{d}x \\
&= \int_Q \varphi\partial_{jk} L u_1\,\mathrm{d}x + \int_Q u_1\varphi\partial_{jk}\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\,\mathrm{d}x \\
&\quad - \int_Q \Big(\partial_k\big(u_1(\partial_j\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\big) + \partial_j\big(u_1\partial_k\big((1+\kappa_Z)\varepsilon + \tfrac{\sigma}{2}\big)\big)\Big)\varphi\,\mathrm{d}x \\
&\quad + \frac{1}{1+\kappa_Z}\Big\langle \partial_2\Big(\partial_k\big((\partial_2 u_1)\big(\partial_j\tfrac{1}{\mu}\big)\big) + \partial_j\big((\partial_2 u_1)\big(\partial_k\tfrac{1}{\mu}\big)\big) \\
&\qquad\qquad - \partial_2 u_1\big(\partial_{jk}\tfrac{1}{\mu}\big)\Big), \varphi\Big\rangle_{D(\partial_2)^*\times D(\partial_2)}.
\end{aligned}
$$

Arguing as in the proof of Lemma 3.3 in [9], one can check that $H^3_0(Q)$ is dense in $D(\partial_2)$. With some more calculations we thus obtain the identity

$$
\begin{aligned}
L\partial_{jk} u_1 = \psi_1(h_1) := {}& \partial_{jk} h_1 - \big((1+\kappa_Z)\partial_{jk}\varepsilon + \tfrac{\partial_{jk}\sigma}{2}\big)u_1 - \big((1+\kappa_Z)\partial_j\varepsilon + \tfrac{\partial_j\sigma}{2}\big)\partial_k u_1 \\
&- \big((1+\kappa_Z)\partial_k\varepsilon + \tfrac{\partial_k\sigma}{2}\big)\partial_j u_1 + \tfrac{1}{1+\kappa_Z}\partial_2\big(\big(\partial_{jk}\tfrac{1}{\mu}\big)\partial_2 u_1\big) \\
&+ \tfrac{1}{1+\kappa_Z}\partial_2\big(\big(\partial_j\tfrac{1}{\mu}\big)\partial_{2k} u_1\big) + \tfrac{1}{1+\kappa_Z}\partial_2\big(\big(\partial_k\tfrac{1}{\mu}\big)\partial_{2j} u_1\big)
\end{aligned}
\tag{4.6}
$$

in $D(\partial_2)^*$. We have seen in the proof of Lemma 3.5 in [9] that $L : D(\partial_2) \to D(\partial_2)^*$ is invertible and hence $\partial_{jk} u_1$ belongs to $D(\partial_2)$ as required. Because of $f_1, g_3 \in H^2(Q)$ and (1.3), equations (4.4) then imply that v_3 and $\partial_2 v_3$ are also contained $H^2(Q)$.

The identities (4.5) and $u_1 = h_1 = 0$ on Γ_2 yield the trace

$$\partial_2\big(\tfrac{1}{\mu}\partial_2 u_1\big) = (1+\kappa_Z)\big(\varepsilon(1+\kappa_Z) + \tfrac{\sigma}{2}\big)u_1 - (1+\kappa_Z)h_1 = 0$$

on Γ_2. With our assumption $\partial_\nu \mu = 0$ on Γ, we thus infer the desired condition $\partial_{22} u_1 = 0$ on Γ_2.

It remains to prove that $\partial_1 u_1 = 0$ and $\partial_1 v_3 = 0$ on Γ_1. To this aim, we use a variant of (4.6) for $w := \partial_1 u_1$. Setting $b = (1+\kappa_Z)\varepsilon + \frac{\sigma}{2}$, we obtain as in formula (3.9) of [9] the identity

$$\varphi := Lw = \partial_1 h_1 - u_1 \partial_1 b + \tfrac{1}{1+\kappa_Z}\, \partial_2\big((\partial_1 \tfrac{1}{\mu})\partial_2 u_1\big).$$

Let $k \in \{1,2,3\}$. Equation (4.6) with $j=1$ now reads as

$$L\partial_k w = \partial_k \varphi - w\partial_k b + \tfrac{1}{1+\kappa_Z}\, \partial_2\big((\partial_k \tfrac{1}{\mu})\partial_2 w\big)$$

in $D(\partial_2^*)$. We pick cut-off maps χ_n as in Lemma 2.2 for $j=1$, and set $\varphi_n(x) = \chi_n(x_1)\varphi(x)$ and $w_n(x) = \chi_n(x_1) w(x)$ on Q. These functions vanish near Γ_1, and (φ_n) tends to φ in $L^2(Q)$. Since $Lw_n = \chi_n Lw = \varphi_n$, we obtain $w_n = L^{-1}\varphi_n$. The maps w_n thus converge to w in $D(\partial_2)$. As above the equation $Lw_n = \varphi_n$ yields

$$L\partial_k w_n = \partial_k \varphi_n - w_n\partial_k b + \tfrac{1}{1+\kappa_Z}\, \partial_2\big((\partial_k \tfrac{1}{\mu})\partial_2 w_n\big).$$

The invertibility of $L : D(\partial_2) \to D(\partial_2)^*$ and (1.3) thus imply

$$\|\partial_k(w_n - w)\|_{L^2} \le c\,\|w - w_n\|_{D(\partial_2)} + c\,\|\partial_k(\varphi - \varphi_n)\|_{D(\partial_2^*)}.$$

Using Kronecker's delta, we calculate

$$\begin{aligned}\partial_k(\varphi_n - \varphi) = \delta_{1k}\chi_n' \Big[\partial_1 h_1 - u_1\partial_1 b + \tfrac{1}{1+\kappa_Z}\, \partial_2\big((\partial_1 \tfrac{1}{\mu})\partial_2 u_1\big)\Big] + (\chi_n - 1)\Big[\partial_{1k} h_1 \\ - u_1 \partial_{1k} b + (\partial_1 b)\partial_k u_1 + \tfrac{1}{1+\kappa_Z}\, \partial_2\big((\partial_{1k}\tfrac{1}{\mu})\partial_2 u_1 + (\partial_1 \tfrac{1}{\mu})\partial_{2k} u_1\big)\Big].\end{aligned}$$

In $D(\partial_2^*)$ we can commute ∂_2 with χ_n and χ_n'. Hence, the outer derivatives ∂_2 can be absorbed by the norm of $D(\partial_2^*)$. The remaining terms in the second bracket all belong to $L^2(Q)$ so that the second summand tends to 0 in $D(\partial_2^*)$ as $n \to \infty$. Omitting the outer ∂_2, our assumptions and the above observations imply that the terms in the first bracket are contained in $H^1(Q)$ and vanish on Γ_1. By Lemma 2.2 these terms tend to 0 in $L^2(Q)$ as $n \to \infty$, and so the first summand converges to 0 in $D(\partial_2^*)$. Summing up, we have shown that $w = \partial_1 u_1$ is the limit of (w_n) in $H^1(Q)$ and thus has 0 trace on Γ_1.

Lemma 2.1 then implies that $\partial_{21} u_1 = 0$ on Γ_1. Since we also have $\partial_1 \mu = 0$ and $\partial_1 g_3 = 0$ on Γ_1 by our assumptions, formula (4.4) implies that $\partial_1 v_3$ vanishes on Γ_1 as desired. The other components of (u,v) are treated in the same way. □

The above lemmas and the Lumer–Phillips theorem (see e.g. Section II.3.b in [10]) now yield the basic properties of our split operators on Z. We omit the proof which follows the lines of that of Proposition 3.6 in [9] for the space Y. Recall the definition of κ_Z in (4.3). We set $\gamma_\tau(L) = (I+\tau L)(I-\tau L)^{-1}$ for $\tau \in (0, 1/\kappa)$ and an operator L on a Banach space such that $L - \kappa I$ generates a contraction semigoup.

Proposition 4.6. *Assume that* (1.3) *holds and that* $\partial_\nu \varepsilon = \partial_\nu \mu = \partial_\nu \sigma = 0$ *on* Γ. *The operators* A_Z *and* B_Z *generate* C_0*–semigroups on* Z *bounded by* $\mathrm{e}^{\kappa_Z t}$.

The resolvents $(I-\tau A_Z)^{-1}$ and $(I-\tau B_Z)^{-1}$ are the restrictions of $(I-\tau A_Y)^{-1}$ and $(I-\tau B_Y)^{-1}$, respectively, and satisfy

$$\left\|(I-\tau A_Z)^{-1}\right\|_{\mathcal{B}(Z)}, \left\|(I-\tau B_Z)^{-1}\right\|_{\mathcal{B}(Z)} \leq \frac{1}{1-\tau\kappa_Z}$$

for all $0<\tau<\frac{1}{\kappa_Z}$, so that $\left\|(I-\tau A_Z)^{-1}\right\|_{\mathcal{B}(Z)}, \left\|(I-\tau B_Z)^{-1}\right\|_{\mathcal{B}(Z)} \leq 2$ for all $0<\tau\leq\frac{1}{2\kappa_Z}$. The Cayley transforms are dominated by

$$\|\gamma_\tau(A_Z)\|_{\mathcal{B}(Z)}, \|\gamma_\tau(B_Z)\|_{\mathcal{B}(Z)} \leq \mathrm{e}^{3\kappa_Z\tau}$$

for all $0<\tau\leq\tau_Z$ and a constant $\tau_Z\in(0,(2\kappa_Z)^{-1}]$ only depending on κ_Z.

5. Error analysis

Let $T,\tau>0$ and $t_n=n\tau\leq T$ with $n\in\mathbb{N}_0$. We assume that $w_0=(\mathbf{E}_0,\mathbf{H}_0)\in D(B)$ and that $(\mathbf{J}(t),0)\in D(A)$ for all $t\in[0,T]$. The ADI splitting scheme is iteratively given by

$$\begin{aligned} w_{n+\frac{1}{2}} &= (\mathbf{E}_{n+\frac{1}{2}},\mathbf{H}_{n+\frac{1}{2}}) = (I-\tfrac{\tau}{2}A)^{-1}(I+\tfrac{\tau}{2}B)w_n\,, \qquad (5.1)\\ w_{n+1} &= (\mathbf{E}_{n+1},\mathbf{H}_{n+1}) = (I-\tfrac{\tau}{2}B)^{-1}(I+\tfrac{\tau}{2}A)\Big[w_{n+\frac{1}{2}}-\tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_n)+\mathbf{J}(t_{n+1}),0)\Big]. \end{aligned}$$

Observe that w_n belongs to $D(B)$, as well as $w_{n+1/2}$ and $(\frac{1}{\varepsilon}\mathbf{J}(t),0)$ to $D(A)$. The efficiency and stability of the scheme in X and Y were discussed in Section 4 of [9]. Formula (4.5) of this paper provides the closed expression

$$\begin{aligned} w_n &= (I-\tfrac{\tau}{2}B)^{-1}\gamma_\tau(A)[\gamma_\tau(B)\gamma_\tau(A)]^{n-1}(I+\tfrac{\tau}{2}B)w_0\\ &\quad -(I-\tfrac{\tau}{2}B)^{-1}\sum_{k=1}^{n}[\gamma_\tau(A)\gamma_\tau(B)]^{n-k}(I+\tfrac{\tau}{2}A)\big(\tfrac{\tau}{2\varepsilon}(\mathbf{J}(t_{k-1})+\mathbf{J}(t_k)),0\big). \end{aligned}$$

Proposition 4.6 easily yields the unconditional stability of the scheme in Z, cf. Theorem 4.2 of [9]. Recall the definition of $\kappa_Z\geq0$ in (4.3) and that of $\tau_Z>0$ in Proposition 4.6. Both depend only on the constants in (1.3) and on Q.

Theorem 5.1. *Let (1.3) hold, $n\in\mathbb{N}$, $0<\tau\leq\min\{\tau_Z,1\}$ and $T\geq n\tau$. Take $w_0\in D(B_Z)$ and $(\varepsilon^{-1}\boldsymbol{J},0)\in C([0,T],D(A_Z))$. We then have $w_n\in D(B_Z)$, $w_{n+1/2}]\in D(A_Z)$, and*

$$\begin{aligned} \|w_n\|_{H^2} &\leq c\mathrm{e}^{6\kappa_Z T}\big(\|w_0\|_{B_Z}+T\max_{t\in[0,T]}\|(\tfrac{1}{\varepsilon}\boldsymbol{J}(t),0)\|_{A_Z}\big),\\ \|(I-\tfrac{\tau}{2}B)w_n\|_Z &\leq \mathrm{e}^{6\kappa_Z T}\big(\|(I+\tfrac{\tau}{2}B_Z)w_0\|_{H^2}+T\max_{t\in[0,T]}\|(I+\tfrac{\tau}{2}A_Z)(\tfrac{1}{\varepsilon}\boldsymbol{J}(t),0)\|_{H^2}\big). \end{aligned}$$

The constant $c>0$ only depend on the constants from (1.3).

To study the convergence of the scheme, we use the operators

$$\Lambda_{j+1}(\tau)=\frac{1}{j!\tau^{j+1}}\int_0^\tau s^j\mathrm{e}^{(\tau-s)M}\,\mathrm{d}s$$

in X for $j\in\mathbb{N}_0$ $\tau\in(0,1]$, and set $\Lambda_0(\tau)=\mathrm{e}^{\tau M}$. These operators and their restrictions to X_{div} and X_2 are uniformly bounded in the respective spaces. We show the desired second order convergence in L^2 of the scheme.

Theorem 5.2. *Let* (1.3) *hold,* $T \geq 1$, $\tau \in (0,1]$, $w_0 = (\boldsymbol{E}_0, \boldsymbol{H}_0) \in D(M_2)$, *and* $(\varepsilon^{-1}\boldsymbol{J}, 0)$ *belong to* $E := W^{1,1}([0,T], X_2) \cap W^{2,1}([0,T], D(M_{\mathrm{div}}))$. *Let* $w = (\boldsymbol{E}, \boldsymbol{H})$ *be the solution of* (1.1) *and* w_n *be its approximation from* (5.1). *For all* $n\tau \leq T$ *we then have*

$$\|w_n - w(n\tau)\|_{L^2} \leq c\tau^2 T^4 \big(\|w_0\|_{D(M_2)} + \big\|(\varepsilon^{-1}\boldsymbol{J}, 0)\big\|_E \big).$$

The constant $c > 0$ *only depends on the constants from* (1.3) *and on* Q.

If also $\sigma = 0$ *or* $\sigma \geq \sigma_0$ *for a constant* $\sigma_0 > 0$, *then we can replace the factor* T^4 *by* T, *where* c *depends on* $1/\sigma_0$ *in the case* $\sigma \geq \sigma_0$.

Remark 5.3. If the solution w belongs to $C([0,T], D(M_2))$ with norm smaller than C, one can replace E by $F = C([0,T], X_2) \cap W^{2,1}\big([0,T], D(M_{\mathrm{div}})\big)$. One then obtains the bound

$$\|w_n - w(n\tau)\|_{L^2} \leq c\tau^2 T \big(C + \big\|(\varepsilon^{-1}\mathbf{J}, 0)\big\|_F \big).$$

Proof. We use the two remainder terms

$$R_n(\tau) = \int_0^\tau \mathrm{e}^{(\tau-s)M} \Big(\int_{n\tau}^{n\tau+s} (n\tau + s - r)\big(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0\big)\, \mathrm{d}r \Big)\, \mathrm{d}s,$$

$$r_n(\tau) = \frac{\tau}{2} \int_{n\tau}^{(n+1)\tau} ((n+1)\tau - r)\big(-\tfrac{1}{\varepsilon}\mathbf{J}''(r), 0\big)\, \mathrm{d}r.$$

Take $y \in Y$ and $0 < \tau \leq \min\{1, \tau_0\}$ for the number τ_0 from Proposition 3.6 of [9], which only depends on the constants in (1.3). For the inner product of X, equation (5.6) of [9] yields

$$\begin{aligned}
&(w_n - w(n\tau) \,|\, y) \\
&= \tau^3 \sum_{k=0}^{n-1} \Big(\big[-M^2\Lambda_3(\tau) + \tfrac{1}{2}M^2\Lambda_2(\tau)\big] w(k\tau) \,\Big|\, \\
&\qquad\qquad M^*(I - \tfrac{\tau}{2}A^*)^{-1} \big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-1-k} (I - \tfrac{\tau}{2}B^*)^{-1} y \Big) \\
&\quad - \tau^3 \sum_{k=0}^{n-1} \Big(\tfrac{1}{4}BM\Lambda_1(\tau) w(k\tau) \,\Big|\, A^*(I - \tfrac{\tau}{2}A^*)^{-1} \big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-1-k} \\
&\qquad\qquad \cdot (I + \tfrac{\tau}{2}B^*)^{-1} y \Big) \\
&\quad - \tau^3 \sum_{k=0}^{n-1} \Big(\big[\tfrac{1}{4}A + \tfrac{1}{4}B\Lambda_1(\tau)\big] \big(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0\big) \,\Big|\, \\
&\qquad\qquad A^*(I - \tfrac{\tau}{2}A^*)^{-1} \big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-1-k} (I + \tfrac{\tau}{2}B^*)^{-1} y \Big) \\
&\quad + \tau^3 \sum_{k=0}^{n-1} \Big(\big[\tfrac{1}{2}M\Lambda_2(\tau) - M\Lambda_3(\tau)\big] \big(-\tfrac{1}{\varepsilon}\mathbf{J}(k\tau), 0\big) \,\Big|\, M^*(I - \tfrac{\tau}{2}A^*)^{-1} \\
&\qquad\qquad \cdot \big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-1-k} (I - \tfrac{\tau}{2}B^*)^{-1} y \Big) \\
&\quad + \tau^3 \sum_{k=0}^{n-1} \Big(\big(-\tfrac{1}{\varepsilon}\mathbf{J}'(k\tau), 0\big) \,\Big|\, \big[\tfrac{1}{4}A^* - \Lambda_3(\tau)^*M^* + \tfrac{1}{2}\Lambda_2(\tau)^*B^*\big]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \cdot \big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-1-k}(I-\tfrac{\tau}{2}B^*)^{-1}y\Big)\\
&+\sum_{k=0}^{n-1}\Big(r_k(\tau)\,\big|\,(I+\tfrac{\tau}{2}A^*)\big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-1-k}(I-\tfrac{\tau}{2}B^*)^{-1}y\Big)\\
&+\sum_{k=0}^{n-2}\Big(R_k(\tau)\,\big|\,(I+\tfrac{\tau}{2}B^*)\gamma_{\tau/2}(A)^*\big[\gamma_{\tau/2}(B)^*\gamma_{\tau/2}(A)^*\big]^{n-2-k}(I-\tfrac{\tau}{2}B^*)^{-1}y\Big)\\
&+\big(R_{n-1}(\tau)\,\big|\,y\big).
\end{aligned}
$$

(The adjoints are taken with respect to X.) Propositions 3.2 and 3.3 as well as the properties of A and B in X from Proposition 3.1 in [9] then imply the asserted estimate for $\tau \le \tau_0$ as in the proof of Theorem 5.1 in [9].

For $\tau \ge \tau_0$ the assertion is a direct consequence of Proposition 3.1 in [9] and Proposition 3.3 since

$$
\begin{aligned}
\|w_n - w(n\tau)\|_{L^2} &\le c(1+n\tau)^3\big(\|w_0\|_{D(M_2)} + \big\|(\varepsilon^{-1}\mathbf{J},0)\big\|_E\big)\\
&\le c\tau^2T^4\big(\|w_0\|_{D(M_2)} + \big\|(\varepsilon^{-1}\mathbf{J},0)\big\|_E\big).
\end{aligned}
$$

due to $T \ge n\tau \ge \tau_0$ and $T \ge 1$. The addendum follows from Remark 3.4. ☐

We finally show that the ADI scheme (5.1) satisfies a discrete version of the divergence conditions (1.1c) and (2.7) up to an error of first order in L^2. We recall that the numbers $\kappa_Z \ge 0$ and $\tau_Z > 0$ from (4.3) and Proposition 4.6 only depend on the constants in (1.3) and on Q, and that $\kappa_Z = 0$ if the coefficients are constant.

Theorem 5.4. *Assume that* (1.3) *holds and that $\partial_\nu\varepsilon = \partial_\nu\mu = \partial_\nu\sigma = 0$ on Γ. Let $T > 0$, $\tau \in (0, \min\{1, \tau_Z\}]$, $n \in \mathbb{N}_0$, and $n\tau \le T$. Take $w_0 = (\boldsymbol{E}_0, \boldsymbol{H}_0)$ in $D(B_Z)$ and $(\frac{1}{\varepsilon}\boldsymbol{J}, 0)$ in $C([0,T], D(A_Z)) \cap C^1([0,T], H(\mathrm{div}))$. Let $w_n = (\boldsymbol{E}_n, \boldsymbol{H}_n)$ be given by* (5.1)*. We then have*

$$
\begin{aligned}
&\Big\|\big(\mathrm{div}(\varepsilon\boldsymbol{E}_n), \mathrm{div}(\mu\boldsymbol{H}_n)\big) - \big(\mathrm{div}(\varepsilon\boldsymbol{E}_0), 0\big)\\
&\qquad + \sum_{k=0}^{n-1}\tfrac{\tau}{2}\big(\mathrm{div}(\tfrac{\sigma}{2}\boldsymbol{E}_{k+1} + \sigma\boldsymbol{E}_{k+1/2} + \tfrac{\sigma}{2}\boldsymbol{E}_k), 0\big) + \int_0^{n\tau}\big(\mathrm{div}(\boldsymbol{J}(s)), 0\big)\,\mathrm{d}s\Big\|_{L^2}\\
&\le c\tau e^{6\kappa_Z T}\Big[\|w_0\|_{H^2} + \tau\|B_Z w_0\|_{H^2} + T\max_{t\in[0,T]}\big(\|(\boldsymbol{J}(t),0)\|_{H^2} + \tau\|A_Z(\tfrac{1}{\varepsilon}\boldsymbol{J}(t),0)\|_{H^2}\big)\Big]\\
&\qquad + c\tau\int_0^T\big\|(\boldsymbol{J}'(s),0)\big\|_{H(\mathrm{div})}\,\mathrm{d}s
\end{aligned}
$$

for constants $c \ge 0$ only depending on the constants in (1.3).

Proof. We first recall formula (6.3) from Theorem 6.1 in [9]

$$
\begin{aligned}
&\begin{pmatrix}\mathrm{div}(\varepsilon\mathbf{E}_n)\\ \mathrm{div}(\mu\mathbf{H}_n)\end{pmatrix} - \begin{pmatrix}\mathrm{div}(\varepsilon\mathbf{E}_0)\\ 0\end{pmatrix} &(5.2)\\
&\qquad + \frac{\tau}{2}\sum_{k=0}^{n-1}\left[\begin{pmatrix}\mathrm{div}[\frac{\sigma}{2}\mathbf{E}_{k+1} + \sigma\mathbf{E}_{k+\frac12} + \frac{\sigma}{2}\mathbf{E}_k]\\ 0\end{pmatrix} + \begin{pmatrix}\mathrm{div}[\mathbf{J}(t_k) + \mathbf{J}(t_{k+1})]\\ 0\end{pmatrix}\right]
\end{aligned}
$$

$$
= \frac{\tau^2}{4}\begin{pmatrix}\operatorname{div}(D_\mu^{(2)}\mathbf{E}_k)\\ \operatorname{div}(D_\varepsilon^{(1)}\mathbf{H}_k)\end{pmatrix} - \frac{\tau^2}{4}\begin{pmatrix}\operatorname{div}(D_\mu^{(2)}\mathbf{E}_0)\\ \operatorname{div}(D_\varepsilon^{(1)}\mathbf{H}_0)\end{pmatrix} + \frac{\tau^2}{8}\begin{pmatrix}0\\ \operatorname{div}\big[C_1\frac{\sigma}{\varepsilon}(\mathbf{E}_k-\mathbf{E}_0)\big]\end{pmatrix}
$$
$$
+\sum_{n=0}^{n-1}\Big[\frac{\tau^3}{16}\begin{pmatrix}\operatorname{div}\big[D_\mu^{(1)}\frac{2}{\varepsilon}(\mathbf{J}(t_k)+\mathbf{J}(t_{k+1}))\big]\\ -\operatorname{div}\big[C_1\frac{\sigma}{\varepsilon^2}(\mathbf{J}(t_k)+\mathbf{J}(t_{k+1}))\big]\end{pmatrix} + \frac{\tau^2}{8}\begin{pmatrix}\operatorname{div}\big[\frac{\sigma}{\varepsilon}(\mathbf{J}(t_k)+\mathbf{J}(t_{k+1}))\big]\\ 0\end{pmatrix}\Big]
$$

for $n \le T/\tau$, which was proved under weaker assumptions in $H^{-1}(Q)^6$. Here we use the operators $D_\lambda^{(1)} = C_1\lambda^{-1}C_2$ and $D_\lambda^{(2)} = C_2\lambda^{-1}C_1$. As in [9] we can rewrite the main term as

$$
\frac{\tau^2}{4}\begin{pmatrix}D_\mu^{(2)}\mathbf{E}_n\\ D_\varepsilon^{(1)}\mathbf{H}_n\end{pmatrix}
$$
$$
= \tfrac{\tau}{2}K(B+S)\tfrac{\tau}{2}(B_Z+S)(I-\tfrac{\tau}{2}B_Z)^{-1}\Big[\gamma_{\frac{\tau}{2}}(A_Z)\big[\gamma_{\frac{\tau}{2}}(B_Z)\gamma_{\frac{\tau}{2}}(A_Z)\big]^{n-1}
$$
$$
\cdot (I+\tfrac{\tau}{2}B_Z)w_0 - \sum_{k=0}^{n-1}\big[\gamma_{\frac{\tau}{2}}(A_Z)\gamma_{\frac{\tau}{2}}(B_Z)\big]^k(I+\tfrac{\tau}{2}A_Z)\tfrac{\tau}{2\varepsilon}\big(\mathbf{J}(t_{n-k-1})+\mathbf{J}(t_{n-k}),0\big)\Big]
$$

in $L^2(Q)^6$, where we have put

$$
K = \begin{pmatrix}\varepsilon I & 0\\ 0 & \mu I\end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix}\frac{\sigma}{2\varepsilon}I & 0\\ 0 & 0\end{pmatrix}.
$$

Proposition 4.6 then yields the desired bound. The other summands in (5.2) are treated similarly, see the proof of Theorem 6.1 in [9]. $\square$

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