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STRONG CONVERGENCE OF SOLUTIONS TO NONAUTONOMOUS KOLMOGOROV EQUATIONS

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ABSTRACT. We study a class of nonautonomous, linear, parabolic equations with unbounded coefficients on $\mathbb{R}^d$ which admit an evolution system of measures. It is shown that the solutions of these equations converge to constant functions as $t \to +\infty$. We further establish the uniqueness of the tight evolution system of measures and treat the case of converging coefficients.

1. Introduction

In this paper we investigate the asymptotic behaviour of a class of nonautonomous parabolic partial differential equations of second order in $\mathbb{R}^d$ with unbounded coefficients. We establish that the solutions converge to constant functions as the time t tends to $+\infty$. These limits exist both locally uniformly and in L^p spaces with respect to a time-varying family of (invariant) measures. Such convergence results have been known before only for special cases, where different more specific methods could be employed, see [2, 11, 18, 24].

The analysis of nondegenerate elliptic operators with unbounded coefficients goes back to the second half of last century with the pioneering papers by Aronson and Besala [4, 5], Bodanko [6], Feller [14] and Krzyżański [20, 21]. The interest of the mathematical community has grown considerably since the nineties because of the many applications to stochastic analysis, where they appear naturally as Kolmogorov operators of stochastic partial differential equations, to mathematical finance and also to physics (see e.g. [15]). Starting from the analysis of autonomous Ornstein–Uhlenbeck equations in [10], elliptic operators with unbounded coefficients and the associated Cauchy problems have been studied both in the space C_b of bounded continuous functions and in L^p spaces on $\mathbb{R}^d$ and on unbounded domains. In the present paper, we focus on $\mathbb{R}^d$ for simplicity.

It turned out that the usual L^p spaces with respect to the Lebesgue measure are not appropriate for these investigations. For instance, no realization of the operator $\mathcal{A}u = u'' - x|x|^\varepsilon u'$ in one spatial dimension generates a C_0–semigroup in $L^p(\mathbb{R})$ if ε is

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positive (see [27]). This example indicates that one needs rather restrictive growth conditions to develop a theory for elliptic operators with unbounded coefficients in $L^p(\mathbb{R}^d)$. The picture changes drastically if the semigroup $T(\cdot)$ associated to the elliptic operator $\mathcal{A} = \mathrm{Tr}(QD^2) + \langle b, \nabla\rangle$ on $C_b(\mathbb{R}^d)$ admits an invariant measure μ and if one works in the spaces $L^p(\mathbb{R}^d, \mu)$. A probability measure μ is called invariant if

$$\int_{\mathbb{R}^d} T(t)f\, d\mu = \int_{\mathbb{R}^d} f\, d\mu$$

for all $f \in C_b(\mathbb{R}^d)$ and $t \geq 0$. An invariant measure exists if $\mathcal{A}$ admits a so-called Lyapunov function, see Hypothesis 2.1(iii) below, which is satisfied by large classes of (possibly rapidly growing) coefficients. We stress that $T(\cdot)$ may not admit an invariant measure; but if an invariant measure exists, it is unique in our setting. We refer to e.g. [23, 26] for details on the autonomous case.

If $T(\cdot)$ admits an invariant measure, it can be extended to a strongly continuous semigroup on $L^p(\mathbb{R}^d, \mu)$ for each $p \in [1, +\infty)$. The invariant measure μ also plays an important role in the analysis of the long-time behaviour of the semigroup $T(\cdot)$. More precisely, under suitable assumptions the function $T(t)f$ tends, as $t \to +\infty$, to the average of f with respect to μ in $L^p(\mathbb{R}^d, \mu)$ if $f \in L^p(\mathbb{R}^d, \mu)$, and the convergence is locally uniform in $\mathbb{R}^d$ if $f \in C_b(\mathbb{R}^d)$, cf. [13].

In this paper we treat the nonautonomous case on time $t \geq 0$, where $\mathcal{A}$ is replaced by the elliptic operators $\mathcal{A}(t)$, $t \geq 0$, defined on smooth functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ by

$$\mathcal{A}(t)\varphi = \sum_{i,j=1}^{d} q_{ij}(t,\cdot)D_{ij}\varphi + \sum_{i=1}^{d} b_i(t,\cdot)D_i\varphi = \mathrm{Tr}(Q(t,\cdot)D_x^2\varphi) + \langle b(t,\cdot), \nabla_x\varphi\rangle, \tag{1.1}$$

under suitable conditions on its coefficients (see Hypotheses 2.1). The semigroup $T(\cdot)$ of the autonomous case is now replaced by an evolution operator $\{G(t,s) : t \geq s \geq 0\}$ in $C_b(\mathbb{R}^d)$. Its existence and its main properties have been established in [11] for nonautonomous Ornstein–Uhlenbeck operators and in [22] for the general case. For $f \in C_b(\mathbb{R}^d)$ and $s \geq 0$, the function $G(\cdot, s)f$ is defined as the unique solution u in $C_b([s, +\infty) \times \mathbb{R}^d) \cap C^{1,2}((s, +\infty) \times \mathbb{R}^d)$ of the parabolic equation $D_t u = \mathcal{A}(t)u$ on $(s, +\infty) \times \mathbb{R}^d$ satisfying $u(s, \cdot) = f$ in $\mathbb{R}^d$.

Similarly, the concept of invariant measure is replaced by the concept of evolution systems of measures (as referred to in [12]). Such a system is a one-parameter family of probability measures $\{\mu_t : t \geq 0\}$ satisfying

$$\int_{\mathbb{R}^d} G(t,s)f\, d\mu_t = \int_{\mathbb{R}^d} f\, d\mu_s \tag{1.2}$$

for all $f \in C_b(\mathbb{R}^d)$ and $t \geq s \geq 0$. As in the autonomous case, Lyapunov functions provide a convenient sufficient condition for the existence of an evolution system of measures, see Hypothesis 2.1(iii). Under such assumption, the proof of Theorem 5.4 of [22] even implies the existence of a tight evolution system of measures; i.e., for every $\varepsilon > 0$ there exists a radius $R > 0$ such that $\mu_t(B_R) \geq 1 - \varepsilon$ for all $t \geq 0$. In [17] it was shown that nonautonomous Ornstein–Uhlenbeck evolution operators admit infinitely many evolution systems of measures under reasonable assumptions. One can however derive uniqueness within certain classes of evolution systems, see [2, 17, 24] for such results in various cases.

If the evolution operator $G(t,s)$ admits an evolution system of measures, then it can be extended to a contraction evolution operator from $L^p(\mathbb{R}^d, \mu_s)$ to $L^p(\mathbb{R}^d, \mu_t)$

for every $t \geq s \geq 0$, Indeed, Proposition 2.4(i) implies that $|G(t,s)f|^p \leq G(t,s)(|f|^p)$ for all $f \in C_b(\mathbb{R}^d)$ and $t \geq s \geq 0$. Integrating this inequality with respect to μ_t, we obtain

$$\|G(t,s)f\|^p_{L^p(\mathbb{R}^d,\mu_t)} \leq \int_{\mathbb{R}^d} G(t,s)(|f|^p)d\mu_t = \int_{\mathbb{R}^d} |f|^p d\mu_s = \|f\|^p_{L^p(\mathbb{R}^d,\mu_s)}, \tag{1.3}$$

for all $t > s \geq 0$ and $p \in [1,+\infty)$. Each measure μ_r is equivalent to the Lebesgue measure λ since it has a positive density $\rho(r,\cdot)$ with respect to λ by results in Corollary 3.9 of [7]. But the spaces $L^p(\mathbb{R}^d,\mu_t)$ and $L^p(\mathbb{R}^d,\mu_s)$ differ in general for $t \neq s$. This fact causes several difficulties in the analysis and, in particular, the standard theory of evolution operators (e.g. in [8]) can not be applied to the evolution operator in the L^p spaces for μ_t. As in [2, 17, 18, 24, 25], we will use the evolution semigroup $\mathcal{T}(\cdot)$ associated with $G(t,s)$, which is defined by

$$(\mathcal{T}(t)h)(s,x) = (G(s,s-t)h(s-t,\cdot))(x), \qquad (s,x) \in \mathbb{R}^{1+d}, \tag{1.4}$$

for $t \geq 0$ and $h \in C_b(\mathbb{R}^{1+d})$. Here we extend the given coefficients constantly to $t < 0$ to obtain an evolution operator $G(t,s)$ for $t \geq s$ on $\mathbb{R}$, as explained in Remark 2.3.

In the papers [2, 18, 24] for several special cases it was established that $G(t,s)f$ converges to the average $m_s(f) := \int_{\mathbb{R}^d} f d\mu_s$ as $t \to +\infty$. For bounded diffusion coefficients and time-periodic coefficients, Corollary 3.8 of [24] shows that $\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}$ tends to 0 as $t \to +\infty$ for $f \in L^p(\mathbb{R}^d,\mu_s)$ and $s \in \mathbb{R}$. The proof of this result relies on the fact that one can employ the evolution semigroup on the *compact* time interval $[0,T]$, for the period T.

The non-periodic case was addressed in [2], but only for diffusion coefficients q_{ij} which are constant in the spatial variables and under an additional strict dissipativity assumption on the drift term (namely that $r_0 < 0$ in Hypothesis 2.1(iv) below). These extra conditions yield the exponentially decaying gradient estimate $|(\nabla_x G(t,s)f)(x)| \leq ce^{r_0(t-s)}(G(t,s)|\nabla f|)(x)$ for all $t > s$, $x \in \mathbb{R}^d$ and $f \in C^1_b(\mathbb{R}^d)$. This decay property is crucial for the proofs in [2]. In turn, it implies the cyclic condition $D_i q_{jk} + D_j q_{ki} + D_k q_{ij} = 0$ in $\mathbb{R} \times \mathbb{R}^d$ for all $i,j,k \in \{1,\dots,d\}$ by Theorem 3.1 in [1], which explains the restriction to space independent diffusion coefficients q_{ij} in [2]. On the other hand, Corollary 5.4 of [2] even establishes the exponential decay of $\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}$ with rate $e^{r_0 t}$ (recall that $r_0 < 0$). To the best of our knowledge, this is the only available result on the long-time behaviour of the function $\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}$ for non-periodic coefficients (besides [18] for the special case of Ornstein–Uhlenbeck operators).

For non-periodic coefficients, our main result Theorem 3.2 shows that $\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}$ tends to 0 as $t \to +\infty$ if $f \in L^p(\mathbb{R}^d,\mu_s)$ and that $G(t,s)f$ converges to $m_s(f)$ locally uniformly if $f \in C(\mathbb{R}^d)$ vanishes at infinity, where $s \geq 0$ and $p \in [1,+\infty)$. This theorem then implies the uniqueness of tight evolution systems of measures. Compared to [2], we allow for space dependent and possibly unbounded diffusion coefficients and we do not need the strict dissipativity assumption $r_0 < 0$ in Hypothesis 2.1(iv). To use certain estimates on Green's functions, we require additional bounds on the coefficients which are global in time but only local in space, see Hypothesis 2.1(i).

As in [18, 24], our approach relies on the decay to 0 of $|\nabla_x \mathcal{T}(t)h|$ as $t \to +\infty$ in $L^p(\mathbb{R}^{1+d}, \nu)$ for all $h \in L^p(\mathbb{R}^{1+d}, \nu)$, where ν is defined by

$$\nu(A \times B) = \int_A \mu_s(B) ds, \tag{1.5}$$

on the product of a Borel set $A \subset \mathbb{R}$ and a Borel set $B \subset \mathbb{R}^d$, and canonically extended to the σ-algebra of all the Borel sets of $\mathbb{R}^{1+d}$, see Proposition 2.6. This decay is proved by means of a "carré du champs" type inequality for the generator of $\mathcal{T}(\cdot)$, which we recall in Proposition 2.4. To exploit the decay in $L^p(\mathbb{R}^{1+d}, \nu)$, we need lower bounds on the density of μ_t which are local in space, but uniform in time. We show such estimates in Lemma 3.1 using known lower bounds of Green's functions solving the Dirichlet problem on balls, [3]. Still it is rather delicate to pass from the strong convergence of $\nabla_x \mathcal{T}(t)$ in $L^p(\mathbb{R}^{1+d}, \nu)^d$ to that of $G(t,s)$ in the proof of Theorem 3.2.

As we have already noticed, the spaces $L^p(\mathbb{R}^d, \mu_t)$ differ from each other. If the coefficients of the operators $\mathcal{A}(t)$ converge as $t \to +\infty$, we establish that the solution $G(t,s)f$ tends to the mean $m_s(f)$ as $t \to +\infty$ in $L^p(\mathbb{R}^d, \mu_\infty)$ for all $f \in C_b(\mathbb{R}^d)$ (which is dense in $L^p(\mathbb{R}^d, \mu_s)$), $s \geq 0$ and $p \in [1, +\infty)$, see Theorem 4.4. Here μ_∞ is the invariant measure of the semigroup associated to the limiting autonomous operator $\mathcal{A}_\infty$. The main step in the proof is the convergence result of Proposition 4.3 for the densities of the invariant measures, where we use the regularity properties of these densities proved in [7]. In Section 5 we exhibit a class of operators that satisfy all our assumptions.

Notation. We consider the usual spaces $C^{k+\alpha}(\Omega)$ when Ω is an open set or the closure of an open set, $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in [0,1)$. We use the subscript "b" (resp., "c") for the subspaces of the above spaces consisting of functions which are bounded together with all their derivatives up to the order k(resp., are compactly supported). We also consider the spaces $C^{1,2}(J \times \Omega)$ and $C^{k+\alpha/2,2k+\alpha}(J \times \Omega)$ for an interval J, and use the subscripts "b" and "c" with the same meaning as above. For $\alpha \in (0,1)$ the subscript "loc" means that the derivatives of order k are α-Hölder continuous in each compact set contained in Ω or $J \times \mathbb{R}^d$.

For a Borel measure μ on Ω and $p \in [1, +\infty)$, we denote by $L^p(\Omega, \mu)$ the usual Lebesgue space (omitting μ if it is the Lebesgue measure). For an open set $\Omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, the standard Sobolev space with respect to the Lebesgue measure is denoted by $W^{k,p}(\Omega)$. Similarly, $W^{1,2}_p(J \times \Omega)$ is the usual parabolic Sobolev space with respect to the Lebesgue measure for an interval J.

Given a family of measures $\{\mu_t : t \geq 0\}$, we denote by $m_t(f)$ the average of the function f with respect to the measure μ_t. Finally, B_R designates the open ball centered at 0 with radius R and $\mathbb{R}_+ := [0, +\infty)$.

2. Assumptions and background material

Throughout this paper, we assume the following conditions on the operator $\mathcal{A}$ in (1.1):

Hypotheses 2.1. (i) *For some $\alpha \in (0,1)$ and every $i,j \in \{1,\dots,d\}$ q_{ij}, b_i belong to $C^{\alpha/2,1+\alpha}_{\rm loc}(\mathbb{R}_+ \times \mathbb{R}^d)$. Moreover, $q_{ij} \in C_b(\mathbb{R}_+ \times B_R)$ and $D_k q_{ij}, b_j \in C_b(\mathbb{R}_+; L^p(B_R))$ for all $i,j,k \in \{1,\dots,d\}$, all $R > 0$ and some $p > d+2$.*

(ii) *The matrix* $Q(t,x)$ *is symmetric and* $\langle Q(t,x)\xi,\xi\rangle \geq \eta(t,x)|\xi|^2$ *for all* $t \geq 0$, $x,\xi \in \mathbb{R}^d$ *and a function* $\eta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ *with* $\inf_{\mathbb{R}_+\times\mathbb{R}^d} \eta =: \eta_0 > 0$.

(iii) *There exist a function* $0 < V \in C^2(\mathbb{R}^d)$ *and constants* $a \geq 0$, $\kappa > 0$ *such that* $V(x)$ *tends to* $+\infty$ *as* $|x| \to +\infty$ *and* $(\mathcal{A}(t)V)(x) \leq a - \kappa V(x)$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

(iv) *There exist constants* $c_0 \geq 0$ *and* $r_0 \in \mathbb{R}$ *such that* $|\nabla_x Q(t,x)| \leq c_0\eta(t,x)$ *and* $\langle \nabla_x b(t,x)\xi,\xi\rangle \leq r_0\,|\xi|^2$ *for all* $t \geq 0$ *and* $x,\xi \in \mathbb{R}^d$.

(v) *There exists a constant* $c > 0$ *such that either* $|Q(t,x)| \leq c(1+|x|)V(x)$ *and* $\langle b(t,x),x\rangle \leq c(1+|x|^2)V(x)$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *or* $|Q(t,x)| \leq c$ *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Except for the second part in (i), assumptions (i)–(iii) are needed to construct the evolution operator and the evolution system of measures $\{\mu_t : t \geq 0\}$. Condition (iv) leads to the gradient estimate (2.2). The second part of (i) is needed to obtain uniform lower bounds of the density of the measures, see Lemma 3.1. On the last condition we comment in Remark 2.5.

In the next proposition we collect several basic properties of the evolution operator $G(t,s)$.

Proposition 2.2. *The following properties are satisfied.*

(i) *Let* $D = \{(t,s,x,y) \in \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d : t > s\}$. *Then there exists a Green's function* $g : D \to (0,+\infty)$ *such that*

$$G(t,s)f = \int_{\mathbb{R}^d} g(t,s,\cdot,y)f(y)\,dy \tag{2.1}$$

in $\mathbb{R}^d$ *for* $f \in C_b(\mathbb{R}^d)$ *and* $t > s \geq 0$. *For every* $t > s \geq 0$ *and* $x \in \mathbb{R}^d$, *the function* $g(t,s,x,\cdot)$ *belongs to* $L^1(\mathbb{R}^d)$ *and* $\|g(t,s,x,\cdot)\|_{L^1(\mathbb{R}^d)} = 1$. *Each operator* $G(t,s)$ *is a contraction on* $C_b(\mathbb{R}^d)$ *and* $G(t,s)\mathbb{1} = \mathbb{1}$.

(ii) *For every* $f \in C_c(\mathbb{R}^d)$ *and* $t > 0$, *the function* $s \mapsto G(t,s)f$ *is continuous from* $[0,t]$ *to* $C_b(\mathbb{R}^d)$. *If* $f \in C_c^2(\mathbb{R}^d)$, *then for every* $(t,x) \in (0,+\infty) \times \mathbb{R}^d$ *the function* $(G(t,\cdot)f)(x)$ *is differentiable in* $[0,t]$ *and* $(D_sG(t,s)f)(x) = -(G(t,s)\mathcal{A}(s)f)(x)$.

(iii) *There exists a constant* $C_1 > 0$ *such that for all* $p \geq 2$ *and* $s \in \mathbb{R}_+$

$$|\nabla_x G(s+t,s)f|^p \leq C_1^p(t^{-p/2} \vee 1)G(s+t,s)(|f|^p), \qquad t > 0,\ f \in C_b(\mathbb{R}^d). \tag{2.2}$$

Proof. Statement (i) and (ii) come from Proposition 2.4 (and its proof) and Lemma 3.1 of [22]. Also statement (iii) is a consequence of the results in [22] although it was not explicitly stated there. To prove it, for every $n \in \mathbb{N}$ and $s \geq 0$ we denote by u_n the unique classical solution to Cauchy-Neumann problem

$$\begin{aligned} &D_t u_n(t,x) = \mathcal{A}(t)u_n(t,x), \qquad (t,x) \in (s,+\infty)\times B_n,\\ &D_\nu u_n(t,x) = 0, \qquad (t,x) \in (s,+\infty)\times\partial B_n,\\ &u_n(s,t) = f(x), \qquad x \in B_n. \end{aligned}$$

Let w_n solve the same boundary value problem with initial condition $w_n(s,\cdot) = f^2$. In the proof of Theorem 4.1 of [22] it is shown that the function

$$(t,x) \mapsto z_n(t,x) = (u_n(t,x))^2 + C_1^{-2}(t-s)|\nabla_x u_n(t,x)|^2$$

satisfies the inequality $D_t z_n - \mathcal{A}z_n \leq 0$ in $(s,s+1]\times B_n$, $D_\nu z_n \leq 0$ on $(s,s+1]\times\partial B_n$ and $z_n(s,\cdot) = f^2$ in B_n for each $n \in \mathbb{N}$. The constant C_1 only depends on η_0, d,

c_0 and r_0 from Hypothesis 2.1. The classical maximum principle now implies that $z_n \le w_n$ in $(s, s+1] \times B_n$.

By Remark 2.3 of [22], the functions u_n and w_n converge to $G(\cdot, s)f$ and $G(\cdot, s)f^2$, respectively, in $C^{1,2}((s, s+R) \times B_R)$ for every $R > 0$. Taking the limit as $n \to +\infty$, the inequality $z_n \le w_n$ thus yields formula (2.2) with $p = 2$ for all $s \ge 0$ and $t \in (0, 1]$. Let $p > 2$. Using (2.2) with $p = 2$, Hölder's inequality and $\|g(t, s, x, \cdot)\|_{L^1(\mathbb{R}^d)} = 1$ from (i), we derive

$$\begin{aligned}|\nabla_x G(s+t, s)f|^p &= (|\nabla_x G(s+t, s)f|^2)^{p/2} \le (C_1^2 t^{-1} G(s+t, s)(|f|^2))^{p/2}\\ &\le C_1^p t^{-p/2} G(s+t, s)(|f|^p)\end{aligned}$$

for all $s \ge 0$, $t \in (0, 1]$ and $f \in C_b(\mathbb{R}^d)$. To extend this estimate to $t > 1$, one finally uses the evolution law and splits $\nabla_x G(s+t, s) = \nabla_x G(s+t, s+t-1)G(s+t-1, s)f$. □

Remark 2.3. Setting $\mathcal{A}(t) := \mathcal{A}(0)$ for $t < 0$, we extend the coefficients q_{ij} and b_i to $t \in \mathbb{R}$ in such a way that Hypotheses 2.1 hold with $\mathbb{R}_+$ replaced by $\mathbb{R}$. Hence, Proposition 2.2 is valid on $\mathbb{R}$ instead of $\mathbb{R}_+$, and for $t \in \mathbb{R}$ and $(-\infty, t]$ instead of $[0, t]$ in part (ii). The extended evolution operator is also denoted by $G(t, s)$, for $t \ge s$ in $\mathbb{R}$. Moreover, we set $\mu_s := G(0, s)^* \mu_0$ for $s < 0$, where $G(0, s)^*$ is the adjoint of the operator $G(0, s)$ in $C_b(\mathbb{R}^d)$. Using formula (2.1) to extend $G(0, s)$ to characteristic functions, it is easy to see that μ_s is a probability measure for every $s < 0$. The set $\{\mu_t : t \in \mathbb{R}\}$ is an evolution system of measures for $G(t, s)$ on $\mathbb{R}$, see the proof of Theorem 5.4 of [22].

We now recall the properties of the evolution semigroup $\mathcal{T}(\cdot)$ (see (1.4)) and the measure ν that we use in this paper. To define it, we use the evolution operator and the evolution system of measures on $\mathbb{R}$ from the above remark.

Proposition 2.4. *Let $p \in [1, +\infty)$. The following properties are satisfied.*

(i) *The measure ν defined in (1.5) is infinitesimally invariant for $\mathcal{T}(\cdot)$; i.e.,*

$$\int_{\mathbb{R}^d} (\mathcal{A}(\cdot)h - D_t h)\, d\nu = 0 \qquad \text{for all} \quad h \in C_c^\infty(\mathbb{R}^{1+d}). \tag{2.3}$$

Moreover, the restriction to $C_c(\mathbb{R}^{1+d})$ of the evolution semigroup $\mathcal{T}(\cdot)$ may be extended to a strongly continuous contraction semigroup $\mathcal{T}_p(\cdot)$ in $L^p(\mathbb{R}^{1+d}, \nu)$. Its generator is denoted by $\mathcal{G}_p$.

(ii) *For any $u \in D(\mathcal{G}_2)$ the following "carré du champs" type inequality holds true:*

$$\eta_0 \int_{\mathbb{R}^{1+d}} |\nabla_x u|^2\, d\nu \le \int_{\mathbb{R}^{1+d}} |Q^{1/2} \nabla_x u|^2\, d\nu \le -\int_{\mathbb{R}^{1+d}} u \mathcal{G}_2 u\, d\nu. \tag{2.4}$$

Proof. We refer the reader to Lemma 6.3(ii) of [22] and Theorem 2.1 of [25] for part (i) and to Corollary 2.16 of [24] for part (ii). The results in [24] are only shown for the case of time-periodic coefficients with slightly different assumptions from our Hypotheses 2.1. We thus sketch the proof of (ii).

We have to replace the space $D(G_\infty)$ used in [24] by the space $\mathcal{D}$ of all $u \in C_b(\mathbb{R}^{1+d})$ belonging to $W_p^{1,2}((-R, R) \times B_R)$ for all $R > 0$ and $1 \le p < +\infty$ such that $\mathcal{G}u := \mathcal{A}(\cdot)u - D_t u$ is contained in $C_b(\mathbb{R}^{1+d})$ and $\text{supp}(u) \subset [-M, M] \times \mathbb{R}^d$ for some $M > 0$. The generator $\mathcal{G}_2$ is the closure of the operator $\mathcal{G}$ defined on $\mathcal{D}$ by

Theorem 2.1 of [25]. Proposition 2.5 of [25] yields

$$u = \int_0^{+\infty} e^{-t}\mathcal{T}(t)(u - \mathcal{G}u)\,dt \tag{2.5}$$

for all $u \in \mathcal{D}$. The gradient estimate (2.2) for $G(t,s)$ directly implies the inequality

$$\|\nabla_x \mathcal{T}(t)h\|_\infty \le C_1(t^{-\frac{1}{2}} \vee 1)\|h\|_\infty \tag{2.6}$$

for $h \in C_b(\mathbb{R}^{1+d})$ and $t > 0$. As in Proposition 2.14 of [24], using (2.5) we infer that $\mathcal{D} \subseteq C_b^{0,1}(\mathbb{R}^{1+d})$ and $\|\nabla_x u\|_\infty \le \tilde{c}\,(\|u\|_\infty + \|\mathcal{G}u\|_\infty)$ for $u \in \mathcal{D}$. Formula (2.4) can now be shown analogously as Proposition 2.15 and Corollary 2.16 in [24], where the first inequality in (2.4) follows from Hypothesis 2.1(ii). □

Remark 2.5. The starting point of the proof of estimate (2.4) is formula (2.3), whose validity can be extended to any function $u \in \mathcal{D}$ (see the proof of Proposition 2.4 for the definition of this space). Plugging $h = u^2$ in (2.3) and estimating $\langle Q\nabla_x u, \nabla_x u\rangle \ge \eta_0|\nabla_x u|^2$ inequality (2.4) formally follows for functions $u \in \mathcal{D}$, which is a core of $D(\mathcal{G}_2)$. But this argument is not correct, since, in general, u^2 does not belong to $\mathcal{D}$ for every $u \in \mathcal{D}$, if the diffusion coefficients are unbounded. This difficulty is overcome by an approximation procedure, where one replaces u^2 by $(u^2\vartheta_n)$, (ϑ_n) being a standard sequence of cutoff functions. Each function $u^2\vartheta_n$ belongs to $\mathcal{D}$ if u is in $\mathcal{D}$. Therefore, $\int_{\mathbb{R}^{1+d}} \mathcal{G}(u^2\vartheta_n)d\nu = 0$ for any $n \in \mathbb{N}$. Estimate (2.4) is obtained by writing explicitly the term $\mathcal{G}(u^2\vartheta_n)$ and then letting n tend to $+\infty$. Hypothesis 2.1(v) is crucial for the convergence of all the integrals obtained by this procedure. This is the only part of the paper where we use it.

Typically, one takes as a Lyapunov function $V(x) = 1 + |x|^{2n}$ for $x \in \mathbb{R}^d$ and some $n \in \mathbb{N}$ or $V(x) = e^{\delta|x|^\beta}$ for $x \in \mathbb{R}^d$ and some $\beta, \delta > 0$, so that Hypothesis 2.1(v) is rather mild. See also the example in Section 5.

In the time periodic case, the next result was shown in Proposition 3.4 of [24] for $p = 2$ extending a similar result proved in [9] in the autonomous case. In this paper we need it for $p > d$. The proof in our case follows the same lines as [24]. Nevertheless, since Proposition 2.6 is crucial for all our analysis, we provide a proof for the reader's convenience.

Proposition 2.6. *For all $p \in [2, +\infty)$ and $h \in L^p(\mathbb{R}^{1+d}, \nu)$ we have*

$$\lim_{t\to+\infty} \|\,|\nabla_x \mathcal{T}_p(t)h|\,\|_{L^p(\mathbb{R}^{1+d},\nu)} = 0. \tag{2.7}$$

Proof. The estimate (2.2) implies that $|\nabla_x \mathcal{T}(t)h|^p \le C_1^p(t^{-p/2} \vee 1)\mathcal{T}(t)|h|^p$ on $\mathbb{R}^{1+d}$ for all $h \in C_c(\mathbb{R}^{1+d})$, $t > 0$, and $p \in [2, +\infty)$. We now integrate this inequality on $\mathbb{R}^{1+d}$ with respect to the measure ν and use the density of $C_c(\mathbb{R}^{1+d})$ in $L^p(\mathbb{R}^{1+d}, \nu)$. It follows that $\nabla_x \mathcal{T}_p(t)h$ belongs to $L^p(\mathbb{R}^{1+d}, \nu)^d$ and

$$\|\,|\nabla_x \mathcal{T}_p(t)h|\,\|_{L^p(\mathbb{R}^{1+d})} \le C_1(t^{-1/2} \vee 1)\|h\|_{L^p(\mathbb{R}^{1+d},\nu)} \tag{2.8}$$

for all $h \in L^p(\mathbb{R}^{1+d}, \nu)$ and $t > 0$, where we use the contractivity of $\mathcal{T}(t)$ in $L^1(\mathbb{R}^{1+d}, \nu)$. Combined with the Hölder inequality, this estimate and (2.6) yield

$$\begin{aligned}\|\,|\nabla_x \mathcal{T}_p(t)h|\,\|_{L^p(\mathbb{R}^{1+d},\nu)} &\le \|\,|\nabla_x \mathcal{T}(t)h|\,\|_\infty^{1-\frac{2}{p}}\,\|\,|\nabla_x \mathcal{T}_2(t)h|\,\|_{L^2(\mathbb{R}^{1+d},\nu)}^{\frac{2}{p}} \\ &\le C_1\|h\|_\infty \|\,|\nabla_x \mathcal{T}_2(t)h|\,\|_{L^2(\mathbb{R}^{1+d},\nu)}^{\frac{2}{p}}\end{aligned} \tag{2.9}$$

for all $t > 0$ and $h \in C_c(\mathbb{R}^{1+d})$. In view of (2.8) and (2.9), we thus have to show (2.7) only for $p = 2$ since $C_c(\mathbb{R}^{1+d})$ is dense in $L^p(\mathbb{R}^{1+d}, \nu)$. Similarly, it suffices to prove (2.7) with $p = 2$ for functions in the dense subset $D(\mathcal{G}_2^2)$ of $L^2(\mathbb{R}^{1+d}, \nu)$.

Take $u \in D(\mathcal{G}_2^2)$. Then, the function $t \mapsto \|\mathcal{T}_2(t)u\|^2_{L^2(\mathbb{R}^{1+d},\nu)}$ is differentiable on $[0, +\infty)$ with derivative $2\langle \mathcal{T}_2(\cdot)u, \mathcal{G}_2\mathcal{T}_2(\cdot)u\rangle_{L^2(\mathbb{R}^{1+d},\nu)}$. From (2.4), we then deduce

$$2\eta_0 \int_0^t \int_{\mathbb{R}^{1+d}} |\nabla_x \mathcal{T}_2(s)u|^2 \, d\nu \, ds \le -\int_0^t 2\langle \mathcal{T}_2(s)u, \mathcal{G}_2\mathcal{T}_2(s)u\rangle_{L^2(\mathbb{R}^{1+d},\nu)} \, ds$$
$$= \|u\|^2_{L^2(\mathbb{R}^{1+d},\nu)} - \|\mathcal{T}(t)u\|^2_{L^2(\mathbb{R}^{1+d},\nu)} \le \|u\|^2_{L^2(\mathbb{R}^{1+d},\nu)}$$

for $t \ge 0$; i.e., the map $\chi_u := \| \, |\nabla_x \mathcal{T}_2(\cdot)u| \, \|^2_{L^2(\mathbb{R}^{1+d},\nu)}$ belongs to $L^1(0, +\infty)$. The estimate (2.4) also implies that the gradient $\nabla_x : D(\mathcal{G}_2) \to L^2(\mathbb{R}^{1+d}, \nu)^d$ is continuous. Since $u \in D(\mathcal{G}_2^2)$, the function χ_u is thus differentiable and

$$|\chi_u'| = 2\Big| \int_{\mathbb{R}^{1+d}} \langle \nabla_x \mathcal{T}_2(\cdot)u, \nabla_x \mathcal{T}_2(\cdot)\mathcal{G}_2 u\rangle \, d\nu \Big|$$
$$\le 2 \| \, |\nabla_x \mathcal{T}_2(\cdot)u| \, \|_{L^2(\mathbb{R}^{1+d},\nu)} \| \, |\nabla_x \mathcal{T}_2(\cdot)\mathcal{G}_2 u| \, \|_{L^2(\mathbb{R}^{1+d},\nu)} \le \chi_u + \chi_{\mathcal{G}_2 u}$$

in $[0, +\infty)$. Using $\mathcal{G}_2 u \in D(\mathcal{G}_2)$ once more, we conclude that also the derivative χ_u' belongs to $L^1(0, +\infty)$ and so $\chi_u(s)$ vanishes as $s \to +\infty$. □

We conclude this section by a simple convergence lemma for tight sequences of probability measures.

Lemma 2.7. *Let $(\tilde{\mu}_n)$ be a tight sequence of probability measures in $\mathbb{R}^d$ and $(g_n) \subset C_b(\mathbb{R}^d)$ be a bounded sequence. The following assertions hold.*

(i) *If g_n tends to zero locally uniformly in $\mathbb{R}^d$ as $n \to +\infty$, then $\int_{\mathbb{R}^d} g_n \, d\tilde{\mu}_n$ vanishes as $n \to +\infty$.*

(ii) *If g_n tends to some $g \in C_b(\mathbb{R}^d)$ locally uniformly in $\mathbb{R}^d$ and $\tilde{\mu}_n$ converges weakly* to a probability measure $\tilde{\mu}$ in $\mathbb{R}^d$ as $n \to +\infty$ (i.e., $\int_{\mathbb{R}^d} f d\tilde{\mu}_n \to \int_{\mathbb{R}^d} f d\tilde{\mu}$, as $n \to +\infty$, for any $f \in C_b(\mathbb{R}^d)$), then $\int_{\mathbb{R}^d} g_n \, d\tilde{\mu}_n$ tends to $\int_{\mathbb{R}^d} g \, d\tilde{\mu}$ as $n \to +\infty$.*

Proof. We only show property (ii), as the first assertion can be treated similarly. By assumption, $M := \sup_{n\in\mathbb{N}}\{\|g_n\|_\infty, \|g\|_\infty\} < +\infty$ and for each $\varepsilon > 0$ there exists a radius $r > 0$ such that $\tilde{\mu}_n(\mathbb{R}^d \setminus B_r) \le \varepsilon$. We can thus estimate

$$\Big| \int_{\mathbb{R}^d} g_n \, d\tilde{\mu}_n - \int_{\mathbb{R}^d} g \, d\tilde{\mu} \Big| \le \int_{B_r} |g_n - g| \, d\tilde{\mu}_n + \int_{\mathbb{R}^d \setminus B_r} |g_n - g| \, d\tilde{\mu}_n + \Big| \int_{\mathbb{R}^d} g \, d\tilde{\mu}_n - \int_{\mathbb{R}^d} g \, d\tilde{\mu} \Big|$$
$$\le \sup_{x\in B_r} |g_n(x) - g(x)| + 2M\varepsilon + \Big| \int_{\mathbb{R}^d} g \, d\tilde{\mu}_n - \int_{\mathbb{R}^d} g \, d\tilde{\mu} \Big|.$$

As $n \to +\infty$, the sum in last line tends to $2M\varepsilon$, and (ii) follows. □

3. Asymptotic behaviour of $G(t, s)$

Throughout this section, $\{\mu_t : t \ge 0\}$ is any tight evolution system of measures for $G(t, s)$, extended to the whole $\mathbb{R}$ as in Remark 2.3. We recall that by Theorem 5.4 in [22] a tight evolution system of measures for $G(t, s)$ does exist. Corollary 3.2 of [7] yields that there exists a positive function $\rho : \mathbb{R}^{1+d} \to \mathbb{R}$ such that $\rho(t, \cdot)$ is the density of μ_t with respect to the Lebesgue measure for every $t \in \mathbb{R}$. In Corollary 3.3 we will see that actually there exists only one tight evolution system of measures for $G(t, s)$.

To begin with, we use Hypothesis 2.1(i) to prove a lower bound on the densities $\rho(t,\cdot)$, which is crucial in our analysis.

Lemma 3.1. *For each $k \in \mathbb{N}$ there exists a number $\delta_k > 0$ such that $\rho(\tau, x) \geq \delta_k$ for all $\tau \geq 0$ and $|x| \leq k$.*

Proof. Let $D_k = \{(t,s,x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \overline{B_k} \times \overline{B_k} : t > s\}$ for every $k \in \mathbb{N}$. By $g_k : D_k \to [0,+\infty)$ we denote the Green's function of the parabolic problem

$$\begin{aligned} D_t u(t,x) &= \mathcal{A}(t)u(t,x), \qquad (t,x) \in (s,+\infty) \times B_k, \\ u(t,x) &= 0, \qquad (t,x) \in (s,+\infty) \times \partial B_k, \\ u(s,x) &= f(x), \qquad x \in B_k, \end{aligned}$$

as constructed in Theorem 3.16 of [16] and its corollaries. The proof of Proposition 2.4 in [22] yields $g \geq g_k$ on D_k for each $k \in \mathbb{N}$, where g is Green's function in Proposition 2.2(i). Since the family $\{\mu_t : t \geq 0\}$ is tight, there is a radius $k_0 \in \mathbb{N}$ such that $\mu_t(\overline{B_{k_0}}) \geq 1/2$ for all $t \geq 0$. Throughout the proof, the integer $k \geq k_0$ is arbitrary, but fixed. We claim that there exists a number $\delta_k > 0$ such that

$$g_{k+2}(\tau+1,\tau,x,y) \geq 2\delta_k \qquad \text{for all} \quad \tau \geq 0,\ x,y \in \overline{B_k}. \tag{3.1}$$

To prove the claim, we rewrite the operators $\mathcal{A}(t)$ in divergence form and apply Theorem 9(iii) in [3] with $\Omega' = B_{k+1}$, $\Omega = B_{k+2}$ and $T = 8$ to the operators $\mathcal{L}(t) = D_t - \operatorname{div}(\tilde{Q}(t+\tau,\cdot)\nabla_x) - \langle \tilde{b}(t+\tau), \nabla_x\rangle$ on $(0,1] \times B_{k+2}$ for $\tau \geq 0$. Here the coefficients $\tilde{q}_{ij} = \tilde{q}_{ji}$ belong to $C_b(\mathbb{R}_+ \times \mathbb{R}^d)$ and satisfy $\tilde{q}_{ij} = q_{ij}$ on $\mathbb{R}_+ \times B_{k+2}$ as well as $\langle \tilde{Q}(t,x)\xi, \xi\rangle \geq \eta_0/2$ for all $i,j \in \{1,\dots,d\}$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\xi \in \partial B_1$. The drift coefficients $\tilde{b}_i$ are continuous extensions of $b_i - \sum_{j=1}^d D_i q_{ij}$ to $\mathbb{R}^{1+d}$, such that on $\tilde{b}_i = b_i - \sum_{j=1}^d D_i q_{ij}$ on $\mathbb{R}_+ \times B_{k+2}$ and $\tilde{b}_i = 0$ on $\mathbb{R}_+ \times \mathbb{R} \setminus B_{k+3}$ for $k \geq k_0$ and $i,j \in \{1,\dots,d\}$. By the uniqueness statement in Theorem 6 of [3], the map $g_{k+2}(\cdot+\tau,\tau,\cdot,\cdot)$ is the Green's function of $\mathcal{L}(t)$ on $(0,1] \times B_{k+2}$. Theorem 9(iii) of [3] now implies that

$$g_{k+2}(t+\tau,\tau,x,y) \geq C_1 t^{-d/2} \exp(-C_2 t^{-1}|x-y|^2) \tag{3.2}$$

for all $x,y \in B_{k+1}$, $t \in (0, \min\{8, (d(y,\partial B_{k+1}))^2\}]$ and $\tau \geq 0$. The constants C_1 and C_2 depend on η_0, $\sup_{t\geq 0} \|q_{ij}(t,\cdot)\|_{L^\infty(B_{k+2})}$, $\sup_{t\geq 0} \|b_j(t,\cdot)\|_{L^p(B_{k+2})}$ and on $\sup_{t\geq 0} \|D_i q_{ij}(t,\cdot)\|_{L^p(B_{k+2})}$ for all $i,j \in \{1,\dots,d\}$ and some $p > d$. (Note that these suprema are finite due to Hypotheses 2.1(i).) If $y \in B_k$, then $d(y,\partial B_{k+1}) \geq 1$. Hence, we can take $t = 1$ in (3.2), and (3.1) follows.

We can now complete the proof. Take a Borel set $B \subset \overline{B_k}$ and some $\tau \geq 0$. From (1.2), (2.1) and (3.1), we deduce

$$\begin{aligned} \int_B \rho(\tau,x)\,dx &= \int_{\mathbb{R}^d} \int_B g(\tau+1,\tau,x,y)\rho(\tau+1,x)\,dy\,dx \\ &\geq \int_{\overline{B_k}} \int_B g_{k+2}(\tau+1,\tau,x,y)\rho(\tau+1,x)\,dy\,dx \\ &\geq 2\delta_k\lambda(B) \int_{\overline{B_k}} \rho(\tau+1,x)\,dx = 2\delta_k\lambda(B)\mu_{\tau+1}(B_k) \geq \delta_k\lambda(B), \end{aligned}$$

where λ is the Lebesgue measure. This lower bound yields the assertion. ☐

We now establish our main result on the convergence of $G(t,s)$.

Theorem 3.2. *Let $s \geq 0$, $p \in [1, +\infty)$, and $\{\mu_t : t \geq 0\}$ be a tight evolution system of measures for $G(t,s)$. The following assertions are true.*

(i) $\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}$ *tends to* 0 *as* $t \to +\infty$ *for each* $f \in L^p(\mathbb{R}^d, \mu_s)$.

(ii) *For each* $f \in C_b(\mathbb{R}^d)$, $G(t,s)f$ *tends to* $m_s(f)$ *locally uniformly in* $\mathbb{R}^d$ *as* $t \to +\infty$.

Proof. (i) First of all, we observe that it suffices to prove the assertion for $s \in \mathbb{R}_+ \setminus \mathcal{N}$, where $\mathcal{N}$ is a null set. Indeed, if $s \in \mathcal{N}$, we fix any $s_* \in \mathbb{R}_+ \setminus \mathcal{N}$ such that $s_* > s$. If $f \in L^p(\mathbb{R}^d, \mu_s)$, then the function $g = G(s_*, s)f$ belongs to $L^p(\mathbb{R}^d, \mu_{s_*})$ and since $G(t,s)f = G(t, s_*)g$ and $m_s(f) = m_{s_*}(g)$,

$$\lim_{t\to+\infty} \|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)} = \lim_{t\to+\infty} \|G(t,s_*)g - m_{s_*}(g)\|_{L^p(\mathbb{R}^d,\mu_t)} = 0.$$

Moreover, it suffices to prove the assertion for each $f \in C_c^\infty(\mathbb{R}^d)$ and all s outside a null set $\mathcal{N}(f)$. Indeed, taking a sequence $(f_n) \in C_c^\infty(\mathbb{R}^d)$, which is dense in all spaces $L^p(\mathbb{R}^d, \mu_r)$ for $r \geq 0$, we find a common null set $\mathcal{N}$ for all $n \in \mathbb{N}$. By an approximation argument and (1.3), we then obtain the assertion for all $s \notin \mathcal{N}$ and $f \in L^p(\mathbb{R}^d, \mu_s)$. Finally, we can assume that $p > d$, since for $p \in [1, d]$ Hölder's inequality shows that $\|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)} \leq \|G(t,s)f - m_s(f)\|_{L^{2d}(\mathbb{R}^d,\mu_t)}$ for all $f \in C_c^\infty(\mathbb{R}^d)$ and $t > s$. Thus, we let $f \in C_c^\infty(\mathbb{R}^d)$ and $p > d$.

Fix a positive sequence (t_n) diverging to $+\infty$, and functions α_m in $C_c^\infty(\mathbb{R})$ such that $0 \leq \alpha_m \leq 1$ in $\mathbb{R}^d$ and $\alpha_m = 1$ on $[-m, m]$ for each $m \in \mathbb{N}$. We extend again $G(t,s)$ and μ_t to $\mathbb{R}$ as in Remark 2.3. Proposition 2.6 implies that

$$\int_{\mathbb{R}} \left\| \rho(s+t_n, \cdot)\alpha_m(s)^p \, |\nabla_x G(s+t_n, s)f|^p \right\|_{L^1(\mathbb{R}^d)} ds = \| \, |\nabla_x \mathcal{T}(t_n)(\alpha_m f)| \, \|^p_{L^p(\mathbb{R}^{1+d},\nu)}$$

tends to 0 as $n \to +\infty$ for each $m \in \mathbb{N}$. There thus exist null sets $\mathcal{N}_m \subset [-m, m]$ and subsequences $(t_n^{(m)})$ diverging to $+\infty$, with $t_k^{(m+1)} \in (t_n^{(m)})_n$ for all $k, m \in \mathbb{N}$, such that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \rho(s + t_n^{(m)}, \cdot) \, |\nabla_x G(s + t_n^{(m)}, s)f|^p \, dx = 0$$

for all $m \in \mathbb{N}$ and $s \in \mathbb{R}_+ \setminus \mathcal{N}_m$. We can thus determine a diagonal sequence (t_{n_j}) such that

$$\lim_{j\to+\infty} \int_{\mathbb{R}^d} \rho(s + t_{n_j}, \cdot)|\nabla_x G(s + t_{n_j}, s)f|^p \, dx \tag{3.3}$$

for each $s \in \mathbb{R}_+ \setminus \mathcal{N}$, where $\mathcal{N} = \bigcup_{m\in\mathbb{N}} \mathcal{N}_m$ is a null set.

Fix $s \in \mathbb{R}_+ \setminus \mathcal{N}$. We use Lemma 3.1 with $\tau = s + t_n$. For every $k \in \mathbb{N}$, it provides a number $\delta_k > 0$ such that $\rho(s + t_n, x) \geq \delta_k$ for $n \in \mathbb{N}$ and $|x| \leq k$. This lower bound and (3.3) yield

$$\lim_{j\to+\infty} \| \, |\nabla_x G(s + t_{n_j}, s)f| \, \|_{L^p(B_k)} = 0 \tag{3.4}$$

for each $k \in \mathbb{N}$. Observing that $\|G(s + t_{n_j}, s)f\|_{L^p(B_k)} \leq c_k^{1/p} \|G(s + t_{n_j}, s)f\|_\infty \leq \|f\|_\infty$ for some positive constant c_k (see Proposition 2.2(i)), we then find constants $\tilde{c}_k > 0$ such that $\|G(s + t_{n_j}, s)f\|_{W^{1,p}(B_k)} \leq \tilde{c}_k$ for all $j \in \mathbb{N}$. Since $p > d$, $W^{1,p}(B_k)$ is compactly embedded in $C(\overline{B_k})$. By a diagonal argument, there exists a function $g(s, \cdot) \in C(\mathbb{R}^d)$ such that $G(s + t_{n_j}, s)f$ converges to $g(s, \cdot)$ locally uniformly in $\mathbb{R}^d$, up to a subsequence. In particular, $\|g(s, \cdot)\|_\infty \leq \|f\|_\infty$.

On the other hand, $\nabla_x G(s+t_{n_j},s)f$ tends to 0 in $L^p(B_R)^d$ as $n\to+\infty$, for every $R>0$, due to (3.4). The weak gradient $\nabla_x g(s,\cdot)$ thus vanishes, and hence $g(s)$ is constant in x. To prove that this constant is $m_s(f)$, it suffices to observe that

$$m_s(f)-g(s)=\int_{\mathbb{R}^d}(f-g(s))\,d\mu_s=\int_{\mathbb{R}^d}G(s+t_{n_j},s)(f-g(s))\,d\mu_{s+t_{n_j}}$$

and use Lemma 2.7(i) with $\tilde{\mu}_n=\mu_{s+t_{n_j}}$. As a result, $g(s)=m_s(f)$ and $G(s+t_{n_j},s)f$ tends to $m_s(f)$ locally uniformly as $j\to+\infty$, for $s\in\mathbb{R}_+\setminus\mathcal{N}$. Since $G(s+t_{n_j},s)m_s(f)\mathbb{1}=m_s(f)\mathbb{1}$ by Proposition 2.2(i), from Lemma 2.7(i) we infer

$$\lim_{j\to+\infty}\|G(s+t_{n_j},s)(f-m_s(f))\|_{L^p(\mathbb{R}^d,\mu_{s+t_{n_j}})}=0.$$

Finally, the function $h=\|G(\cdot,s)f-m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}$ is decreasing in $[s,+\infty)$ since

$$\begin{aligned}h(t_2)&=\|G(t_2,s)(f-m_s(f))\|_{L^p(\mathbb{R}^d,\mu_{t_2})}=\|G(t_2,t_1)G(t_1,s)(f-m_s(f))\|_{L^p(\mathbb{R}^d,\mu_{t_2})}\\&\le\|G(t_1,s)(f-m_s(f))\|_{L^p(\mathbb{R}^d,\mu_{t_1})}=h(t_1)\end{aligned}$$

for $s\le t_1<t_2$, where we have used property (i) in Proposition 2.2 and (1.3). We conclude that $\lim_{t\to+\infty}\|G(t,s)f-m_s(f)\|_{L^p(\mathbb{R}^d,\mu_t)}=0$.

(ii) Fix $f\in C_b(\mathbb{R}^d)$, $s\in\mathbb{R}_+$, $R>0$ and $p>d$. Since $C_b(\mathbb{R}^d)\subset L^p(\mathbb{R}^d,\mu_s)$, $\|G(t+s,s)f-m_s(f)\|_{L^p(\mathbb{R}^d,\mu_{t+s})}$ tends to 0 as $t\to+\infty$, by the first part of the proof. Taking Lemma 3.1 into account, we can estimate

$$\|G(t+s,s)f-m_s(f)\|_{L^p(B_R)}\le\delta_R^{-1/p}\|G(t+s,s)f-m_s(f)\|_{L^p(\mathbb{R}^d,\mu_{t+s})}$$

for all $t\ge0$ and some positive constant δ_R. Hence, $\|G(t+s,s)f-m_s(f)\|_{L^p(B_R)}$ tends to 0 as $t\to+\infty$. In particular, there exists a positive constant $C_1=C_1(R)$ such that

$$\|G(t+s,s)f-m_s(f)\|_{L^p(B_R)}\le C_1 \tag{3.5}$$

for all $t\ge0$. Moreover, the gradient estimate (2.2) implies that

$$\|\,|\nabla_x G(t+s,s)f|\,\|_{L^p(B_R)}\le C_2\|f\|_\infty \tag{3.6}$$

for all $t\ge1$ and some positive constant $C_2=C_2(R)$. From (3.5) and (3.6) we deduce that the family of functions $\{G(t+s,s)f-m_s(f):t\ge1\}$ is bounded in $W^{1,p}(B_R)$ and, consequently, in $C^\beta(B_R)$ for some $\beta\in(0,1)$ since $p>d$. By the Arzelà-Ascoli theorem, from any sequence (t_n) diverging to $+\infty$ we can extract a subsequence (t_{n_k}) such that $G(t_{n_k}+s,s)f-m_s(f)$ converges uniformly in B_R to zero as $k\to+\infty$, since it tends to zero in $L^p(B_R)$. This shows that $G(t+s,s)f-m_s(f)$ tends to 0, uniformly in B_R, as $t\to+\infty$. □

Corollary 3.3. *$G(t,s)$ has exactly one tight evolution system of measures.*

Proof. Let $\{\mu_t^{(1)}:t\ge0\}$ and $\{\mu_t^{(2)}:t\ge0\}$ be two evolution systems of measures with corresponding means $m_t^{(i)}$. Fix $s\in\mathbb{R}_+$ and $f\in C_b(\mathbb{R}^d)$. Then, Theorem 3.2(ii) shows that, as $t\to+\infty$, $G(t,s)f$ converges both to $m_s^{(1)}(f)$ and $m_s^{(1)}(f)$, locally uniformly in $\mathbb{R}^d$. Hence, $m_s^{(1)}(f)=m_s^{(2)}(f)$ and, consequently, $\mu_s^{(1)}=\mu_s^{(2)}$. □

4. Converging coefficients

In this section, we consider coefficients that converge as $t \to +\infty$ by introducing the next additional hypothesis.

Hypothesis 4.1. *The coefficients $q_{i,j}$ and b_i belong to $C_b^{\alpha/2,\alpha}(\mathbb{R}_+ \times B_R)$ for all $i, j \in \{1, \cdots, d\}$ and $R > 0$ and $Q(t, \cdot)$ and $b(t, \cdot)$ converge pointwise to maps $Q_\infty : \mathbb{R}^d \to \mathbb{R}^{d^2}$ and $b_\infty : \mathbb{R}^d \to \mathbb{R}^d$, respectively, as $t \to +\infty$.*

Remark 4.2. Hypotheses 2.1 and 4.1 imply that $Q_\infty \in C^\alpha_{\rm loc}(\mathbb{R}^d; \mathbb{R}^{d^2})$ and $b_\infty \in C^\alpha_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$ satisfy the t–independent analogues of Hypotheses 2.1(ii) and (iii). The evolution operator generated by $\mathcal{A}_\infty$ is a semigroup $\{T(t) : t \geq 0\}$ which admits a single invariant measure μ_∞ having a density $\rho_\infty > 0$ with respect to the Lebesgue measure. (See e.g. Theorems 8.1.15 and 8.1.20 of [23] or [26].)

As in Section 3, $\{\mu_t : t \geq 0\}$ is any tight evolution system of measures with densities $\rho(t, \cdot)$. Under the additional Hypothesis 4.1, we show that the densities $\rho(t, \cdot)$ converge to ρ_∞ and we derive a variant of Theorem 3.2.

Proposition 4.3. *The densities $\rho(t, \cdot)$ converge to ρ_∞ locally uniformly in $\mathbb{R}^d$ and in $L^1(\mathbb{R}^d)$ as $t \to +\infty$.*

Proof. We first prove local uniform convergence. It suffices to show that every sequence (s_n) diverging to $+\infty$ admits a subsequence such that $\rho(s_{n_j}, \cdot)$ converges to ρ_∞ locally uniformly on $\mathbb{R}^d$ as $j \to +\infty$. As in the proof of Theorem 6.2 of [2] we see that μ_t weakly* converges to μ_∞ as $t \to +\infty$. Because of Proposition 2.4(ii) and Hypothesis 2.1(i), Corollary 3.9 of [7] yields that ρ is contained in $C^\beta((s, s+1) \times B_R)$ for every $s, R > 0$ and some $\beta > 0$. The proofs given there also yield that the norms of ρ in these spaces are bounded by a constant $C = C(R)$ independent of s. See also [19]. As a result, ρ belongs to $C_b^\beta([0, +\infty) \times B_R)$ for every $R > 0$. The Arzelà-Ascoli theorem now provides a sequence (t_n) diverging to $+\infty$ such that the density $\rho(t_n, \cdot)$ of the measure μ_{t_n} converges to a function $g \in C(\mathbb{R}^d)$ locally uniformly in $\mathbb{R}^d$ as $n \to +\infty$. The weak* convergence of μ_t to μ_∞ thus yields

$$\int_{\mathbb{R}^d} f \rho_\infty \, dx = \int_{\mathbb{R}^d} f \, d\mu_\infty = \lim_{t_k \to +\infty} \int_{\mathbb{R}^d} f \, d\mu_{t_k} = \lim_{t_k \to +\infty} \int_{\mathbb{R}^d} f \rho(t_k, \cdot) \, dx = \int_{\mathbb{R}^d} f g \, dx$$

for every $f \in C_c^\infty(\mathbb{R}^d)$. Hence, $\rho_\infty = g$ and the local uniform convergence is shown.

To prove the L^1-convergence, let $\varepsilon > 0$. By the tightness, there is a radius $R > 0$ such that $\mu_t(\mathbb{R}^d \setminus B_R),\ \mu_\infty(\mathbb{R}^d \setminus B_R) \leq \varepsilon$ for all $t \geq 0$. From the first part of the proof we deduce

$$\begin{aligned}\limsup_{t\to+\infty} \|\rho(t, \cdot) - \rho_\infty\|_{L^1(\mathbb{R}^d)} &= \limsup_{t\to+\infty} \left[\|\rho(t, \cdot) - \rho_\infty\|_{L^1(B_R)} + \|\rho(t, \cdot) - \rho_\infty\|_{L^1(\mathbb{R}^d \setminus B_R)}\right] \\ &\leq \limsup_{t\to+\infty} \mu_t(\mathbb{R}^d \setminus B_R) + \mu_\infty(\mathbb{R}^d \setminus B_R) \\ &\leq 2\varepsilon.\end{aligned}$$

□

Theorem 4.4. *Let $s \geq 0$, $p \in [1, +\infty)$ and $f \in C_b(\mathbb{R}^d)$. Then, $G(t, s)f$ tends to $m_s(f)$ in $L^p(\mathbb{R}^d, \mu_\infty)$ as $t \to +\infty$.*

Proof. The result follows from Proposition 4.3, Theorem 3.2(i) and the estimates

$$\|G(t, s)f - m_s(f)\|^p_{L^p(\mathbb{R}^d, \mu_\infty)}$$

$$\leq \int_{\mathbb{R}^d} |\rho_\infty - \rho(t,\cdot)|\, |G(t,s)(f-m_s(f))|^p\, dx + \|G(t,s)f - m_s(f)\|^p_{L^p(\mathbb{R}^d,\mu_t)}$$
$$\leq 2^p \|f\|_\infty^p \, \|\rho_\infty - \rho(t,\cdot)\|_{L^1(\mathbb{R}^d)} + \|G(t,s)f - m_s(f)\|^p_{L^p(\mathbb{R}^d,\mu_t)}. \qquad \square$$

5. An example

We consider the family of operators $\mathcal{A}(t)$ defined on smooth functions φ by

$$(\mathcal{A}(t)\varphi)(x) = (1+|x|^2)^\gamma \sum_{i,j=1}^d q_{ij}^{(0)}(t,x) D_{ij}\varphi(x) - b^{(0)}(t)(1+|x|^2)^r \langle x, \nabla_x \varphi(x)\rangle$$

for $t \geq 0$ and $x \in \mathbb{R}^d$, under the following assumptions.

(i) $q_{ij}^{(0)} = q_{ji}^{(0)}$ belong to $C^{\alpha/2,1+\alpha}_{\rm loc}(\mathbb{R}_+ \times \mathbb{R}^d) \cap C_b(\mathbb{R}_+; C^1_b(\mathbb{R}^d))$ for some $\alpha \in (0,1)$ and for all $R > 0$ and $i,j \in \{1,\dots,d\}$. Moreover, $\langle Q^{(0)}(t,x)\xi, \xi\rangle \geq \eta_0$ in $\mathbb{R}^{1+d}$ for some positive constant η_0 and every $\xi \in \partial B_1$;

(ii) The function $b \in C^{\alpha/2}_{\rm loc}(\mathbb{R}_+) \cap C_b(\mathbb{R}_+)$ satisfies $\beta := \inf_{t\geq 0} b^{(0)}(t) > 0$.

(iii) $r > \gamma - 1$ and $\gamma \geq 0$.

Let $\delta \in (0, 2(r+1-\gamma))$. Then every smooth and positive function $V : \mathbb{R}^d \to \mathbb{R}$ with $V(x) = e^{|x|^\delta}$ for $x \in \mathbb{R}^d \setminus B_1$ satisfies Hypothesis 2.1(iii). Indeed, we have

$$\begin{aligned}(\mathcal{A}(t)V)(x) &= \delta V(x)|x|^\delta \big[(\delta |x|^{\delta-4} + (\delta-2)|x|^{-4})(1+|x|^2)^\gamma \langle Q^{(0)}(t,x)x, x\rangle \\ &\quad + {\rm Tr}(Q^{(0)}(t,x))(1+|x|^2)^\gamma |x|^{-2} - b^{(0)}(t)(1+|x|^2)^r\big] \\ &\leq \delta V(x)|x|^\delta h(x),\end{aligned}$$

for $t \geq 0$ and $|x| \geq 1$, where $h(x) = c\,|x|^{\delta-2}(1+|x|^2)^\gamma - \beta(1+|x|^2)^r$ tends to $-\infty$ as $|x| \to +\infty$ and $c > 0$ is a constant depending on the bounds of Q^0. One easily checks the other conditions in Hypothesis 2.1 and the additional condition in Theorem 3.2(ii). Finally, Hypothesis 4.1 is satisfied if $q_{ij} \in C^{\alpha/2,\alpha}_b(\mathbb{R}_+ \times B_R)$ for all $R > 0$, $b \in C^{\alpha/2}(\mathbb{R}_+)$, and $Q^{(0)}(t,\cdot)$ and $b(t)$ converge to $Q^{(0)}_\infty$ and b_∞, respectively, as $t \to +\infty$.

References

1. L. Angiuli, *Pointwise gradient estimates for evolution operators associated with Kolmogorov operators*, Arch. Math. (Basel) **101** (2013), 159-170.
2. L. Angiuli, L. Lorenzi, A. Lunardi, *Hypercontractivity and asymptotic behaviour in nonautonomous Kolmogorov equations*, Comm. Partial Differential Equations **38** (2013), 2049–2080.
3. D.G. Aronson. *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 607–694.
4. D.G. Aronson, P. Besala, *Parabolic equations with unbounded coefficients*, J. Differential Equations **3** (1967), 1–14.
5. D.G. Aronson and P. Besala *Uniqueness of the positive solutions of parabolic equations with unbounded coefficients*, Colloq. Math. **18** (1967), 126–135.
6. W. Bodanko, *Sur le problème de Cauchy et les problèmes de Fourier pour les équations paraboliques dans un domaine non borné*, Ann. Polon. Math. **18** (1966), 79–94.
7. V.I. Bogachev, N.V. Krylov, M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusion under minimal conditions*, Comm. Partial Differential Equations **26** (2001), 2037–2080.
8. C. Chicone, Y. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, American Mathematical Society, 1999.
9. G. Da Prato, B. Goldys, *Elliptic operators on $\mathbb{R}^d$ with unbounded coefficients*, J. Differential Equations **172** (2001), 333–358.

10. G. Da Prato, A. Lunardi, *On the Ornstein–Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. (1995) **131**, 94–114.
11. G. Da Prato, A. Lunardi, *Ornstein-Uhlenbeck operators with time periodic coefficients*, J. Evol. Equ. **7** (2007), 587–614.
12. G. Da Prato, M. Röckner, *A note on evolution systems of measures for time-dependent stochastic differential equations*, Proceedings of the 5th Seminar on Stochastic Analysis, Random Fields and Applications, Ascona 2005, Progr. Probab. **59**, Birkhäuser Verlag, 2008, 115–122.
13. G. Da Prato. and J. Zabczyk, *Ergodicity for infinite-dimensional systems*, London Mathematical Society Lecture Note Series, **229**, Cambridge University Press, Cambridge, 1996
14. W. Feller, *Diffusion processes in one dimension*, Trans. Amer. Math. Soc., **77** (1954), 1–31.
15. M. Friedlin, Some remarks in the Smoluchowski-Kramers approximation, J. Stat. Physics, **117** (2004), 617–634.
16. A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, 1964.
17. M. Geissert, A. Lunardi, *Invariant measures and maximal L^2 regularity for nonautonomous Ornstein-Uhlenbeck equations*, J. Lond. Math. Soc. (2) **77** (2008), 719–740.
18. M. Geissert, A. Lunardi, *Asymptotic behavior and hypercontractivity in nonautonomous Ornstein-Uhlenbeck equations*, J. Lond. Math. Soc. (2) **79** (2009), 85–106.
19. N.V. Krylov, *Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces*, J. Funct. Anal. **183** (2001), 1–41.
20. M. Krzyżański, *Sur la solution fondamentale de l'équation linéaire normale du type parabolique dont le dernier coefficient est non borné. I*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **32** (1962), 326–330.
21. M. Krzyżański, *Sur la solution fondamentale de l'équation linéaire normale du type parabolique dont le dernier coefficient est non borné. II*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **32** (1962), 471–476.
22. M. Kunze, L. Lorenzi, A. Lunardi, *Nonautonomous Kolmogorov parabolic equations with unbounded coefficients*, Trans. Amer. Math. Soc. **362** (2010), 169–198.
23. L. Lorenzi, M. Bertoldi, Analytical Methods for Markov Semigoups, Chapman Hall/CRC Press, 2006.
24. L. Lorenzi, A. Lunardi, A. Zamboni, *Asymptotic behavior in time periodic parabolic problems with unbounded coefficients*, J. Differential Equations **249** (2010), 3377–3418.
25. L. Lorenzi, A. Zamboni, *Cores for parabolic operators with unbounded coefficients*, J. Differential Equations **246** (2009), 2724–2761.
26. G. Metafune, D. Pallara, M. Wacker, *Feller semigroups on* $\mathbb{R}^N$, Semigroup Forum **65** (2002), 159–205.
27. J. Prüss, A. Rhandi, R. Schnaubelt, *The domain of elliptic operators on* $L^p(\mathbb{R}^d)$ *with unbounded drift coefficients*, Houston J. Math. **32** (2006), 563–576.

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