Degenerate operators of Tricomi type in L^p–spaces and in spaces of continuous functions

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Abstract

We study elliptic operators L with Dirichlet boundary conditions on a bounded domain Ω whose diffusion coefficients degenerate linearly at $\partial\Omega$ in tangential directions. We compute the domain of L and establish existence, uniqueness and (maximal) regularity of the elliptic and parabolic problems for L in L^p–spaces and in spaces of continuous functions. Moreover, the analytic semigroups generated by L are consistent, positive, compact and exponentially stable.

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1 Introduction

We study elliptic operators L of second order with Dirichlet boundary conditions on a bounded domain Ω whose diffusion coefficients degenerate at $\partial\Omega$ in tangential directions. We aim at a complete theory including existence, uniqueness and (maximal) regularity of the elliptic and parabolic problems for L in L^p–spaces and in spaces of continuous functions. Moreover, we establish consistency, positivity, compactness and exponential stability of the analytic semigroups generated by L. The domain of L is computed explicitly in L^p, $p \in (1, +\infty)$.

We consider symmetric diffusion coefficients which are positive definite at any point in the interior of Ω and only positive semidefinite on the boundary $\partial\Omega$. The degeneracy affects only the tangential variables and is of the order of the distance from $\partial\Omega$. The prototype of this class is the well–known Tricomi operator $L = -y\Delta_x - \partial_y^2$ in the upper halfspace $\{(x, y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\}$. The Tricomi equation has been widely investigated also in view of its applications in transonic gas dynamics.

In an earlier paper [8], some of the authors have studied the analogous questions for the case of complete degeneracy which was also treated in the recent paper [13]. We refer to [8] and [13] for the existing literature on degenerate second order differential operators, but we remark that it is mainly confined to the Hilbert case. We are not aware of results about generation of analytic semigroups in $L^p(\Omega)$ with $p \neq 2$ or $C(\overline{\Omega})$ by operators with tangential degeneracy of first order, where domains are computed explicitly.

Let us present the plan of our paper. In Section 2 we focus our attention on the model problem. We endow the Tricomi operator L with the (best possible) domain

$$D_p^\circ = \left\{u \in W_0^{1,p}(\mathbb{R}_+^{N+1}) \cap W_{\text{loc}}^{2,p}(\mathbb{R}_+^{N+1}) : \partial_y^2 u, |yD_x^2 u|, |\sqrt{y}\,\nabla_x \partial_y u| \in L^p(\mathbb{R}_+^{N+1})\right\},$$

where $p \in (1, \infty)$. By means of the Mikhlin multiplier theorem, J.U. Kim has shown an L^p apriori estimate for this operator, see Theorem 0.1 in [12] which is stated below in Theorem 2.1. Using this and variational estimates, we prove that $(-L, D_p^\circ)$ is densely defined, closed and regularly dissipative. We then have to show that $(\lambda + L)D_p^\circ$ is dense in $L^p(\mathbb{R}_+^{N+1})$ for some $\lambda > 0$ in order to deduce that $(-L, D_p^\circ)$ generates an analytic C_0–semigroup. This range condition is verified approximating the halfspace by strips $S_\varepsilon = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} : \varepsilon < y < \varepsilon^{-1}\}$ for $\varepsilon \in (0, 1/2]$, where one has a uniformly elliptic problem. Due to technical problems, we have to treat the cases $p = 2$, $p > 2$ and $p < 2$ separately. It also follows that the corresponding inhomogeneous parabolic problem has maximal regularity of type L^q, see Corollary 2.14. The section ends with the proof of the generation result for operators with constant coefficients.

In order to deal with the general case of a degenerate operator defined on a bounded smooth domain Ω, we proceed as in the classical setting by using local charts to straighten the boundary of Ω. First, at the beginning of Section 3 we choose a function ϱ such that $\Omega = \{\varrho > 0\}$, $\partial\Omega = \{\varrho = 0\}$, and $\nabla\varrho(\xi)$ is directed along the inward normal vector if $\xi \in \partial\Omega$ (ϱ is an extension of the distance function to $\partial\Omega$). The operator L is of the form

$$L = -\text{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2) - \varrho \sum_{i,j=1}^{N+1} a_{ij}\partial_{ij} - \sum_{i=1}^{N+1} b_i \partial_i, \tag{1.1}$$

where a_{ij}, b_i are continuous functions, a_{ij} satisfy a suitable ellipticity condition (see (H2)) and the vector field $\mathbf{a}$ is C^2 and non tangential on $\partial\Omega$. Hence, the tangential degeneracy of the diffusion is expressed by the properties of $\mathbf{a}$. Second, following an idea in [3], we construct a local change of variables depending on $\mathbf{a}$ and ϱ in such a way that the boundary of $\partial\Omega$ is locally straightened *and* the vectors $\mathbf{a}(\xi)$ are transformed into the last vector of the canonical basis of $\mathbb{R}^{N+1}$. After the change of variables, we thus recover operators having the same form as the model operator. This fact is crucial for the localization arguments in the following two sections leading to our main results.

The main Theorem 4.1 of Section 4 shows that the operator $-L$, now given by (1.1) and endowed with the (optimal) domain

$$D_p(L) = \left\{u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap W^{1,p}_0(\Omega) : \varrho|D^2u|, \mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2u), \sqrt{\varrho}\,|D^2u\,\mathbf{a}| \in L^p(\Omega)\right\},$$

generates an analytic semigroup on $L^p(\Omega)$, $p \in (1,\infty)$. To prove it, besides the localization procedure of Section 3, we employ the technique of freezing the coefficients that allows to apply the results of Section 2.

Section 5 is concerned with the generation of analytic semigroups in $C(\overline{\Omega})$ and $C_0(\Omega)$. The main ingredients of the proofs are the results from Section 4 and the Masuda–Stewart localization technique. However, it is not straightforward to carry out this procedure because of the degeneracy exhibited by the operator. In particular, as a preliminary step we have to prove a quantitative, local version of the Morrey embedding theorem for functions $\varphi \in L^p(Q)$ such that $\partial_y\varphi, \sqrt{y}\,|\nabla_x\varphi| \in L^p(Q)$ for large p, where Q is a parallelepiped in $\mathbb{R}^{N+1}_+$ whose lower base lies on $\mathbb{R}^N \times \{y = 0\}$. Moreover, in applying the Masuda–Stewart technique the required covering must be constructed following the geometry suggested by the degeneracy, which is different from both the classical one and that in [8]. In various corollaries in Sections 4 and 5, we establish additional properties of the analytic semigroups such as consistency, positivity, compactness and exponential stability, as well as maximal regularity in the L^p case.

Notation. We set $\mathbb{R}^{N+1}_+ = \{z = (x,y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\}$ and write $B^+_r(z) = B_r(z) \cap \mathbb{R}^{N+1}_+$ for the balls in $\mathbb{R}^{N+1}_+$. Functions defined on $\mathbb{R}^{N+1}_+$ are extended by 0 to $\mathbb{R}^{N+1}$, and functions on $\mathbb{R}^{N+1}$ are identified with functions on $\mathbb{R}^{N+1}_+$ by restriction. In the whole paper, p denotes a number in $(1,\infty)$. By $C > 0$ we mean a generic constant. The gradient and Hessian on $\mathbb{R}^{N+1}$ are denoted by ∇ and D^2 whereas ∇_x and D^2_x only act in $x \in \mathbb{R}^N$. We denote both by $z_1 \cdot z_2$ and $\langle z_1, z_2\rangle$ the inner product of $z_1, z_2 \in \mathbb{R}^{N+1}$. Given two vectors $a, b \in \mathbb{R}^N$, the symbol $a \otimes b$ denotes the matrix with entries $a_i b_j$.

2 The model problem on a halfspace

We consider the Tricomi operator

$$L = -y\Delta_x - \partial^2_y$$

on the open upper halfspace $\mathbb{R}^{N+1}_+$. The following a priori estimate is established in Theorem 0.1 of [12].

Theorem 2.1 *There exists $M > 0$ such that for every $u \in C^\infty_c(\mathbb{R}^{N+1})$ with $u(x,0) = 0$ it holds*

$$\|yD^2_x u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\partial^2_y u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x\partial_y u\|_{L^p(\mathbb{R}^{N+1}_+)} \le M\,\|Lu\|_{L^p(\mathbb{R}^{N+1}_+)}.$$

We observe that in [12] this theorem is stated with the summand $\|y\Delta_x u\|_{L^p(\mathbb{R}^{N+1}_+)}$ instead of $\|yD^2_x u\|_{L^p(\mathbb{R}^{N+1}_+)}$. The classical Calderón–Zygmund estimate with respect to x then implies the version of the theorem given above. Moreover, in [12] it is allowed that the function u does not vanish at the boundary. In this case, a suitable norm of the restriction of u at $y=0$ is added on the right hand side and the constant M depends on the width L of the strip containing the support of u. But, if $u(x,0)=0$, inspecting the proof in [12] one realizes that M can be taken independent of L. In view of Theorem 2.1, we introduce the spaces

$$\begin{aligned}
D_p &= \left\{u \in W^{1,p}(\mathbb{R}^{N+1}_+) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^{N+1}_+) : |yD^2_x u|, \partial^2_y u, |\sqrt{y}\,\nabla_x \partial_y u| \in L^p(\mathbb{R}^{N+1}_+)\right\},\\
D_p^\circ &= \{u \in D_p : u(\cdot,0)=0 \ \text{ on } \mathbb{R}^N\},
\end{aligned}$$

where the boundary values at $y=0$ are understood in the sense of traces. Endowed with the canonical norm, denoted by $\|\cdot\|_{D_p}$, D_p° and D_p are Banach spaces. We further set

$$\mathcal{D} = \{u \in C_c^\infty(\mathbb{R}^{N+1}) : u(x,0)=0 \ \text{ for all } x \in \mathbb{R}^N\}.$$

The main result of the present section is stated below.

Theorem 2.2 *The operator $(-L, D_p^\circ)$ generates an analytic C_0-semigroup of positive contractions $(T_p(t))_{t\ge 0}$ in $L^p(\mathbb{R}^{N+1}_+)$. Moreover, $T_p(t)f = T_q(t)f$ for all $t \ge 0$, $f \in L^p(\mathbb{R}^{N+1}_+) \cap L^q(\mathbb{R}^{N+1}_+)$, and $1<p,q<\infty$.*

We start by proving the following Lemma which allows us to extend the a priori estimate of Theorem 2.1 to D_p°.

Lemma 2.3 *The space $\mathcal{D}$ is dense in D_p°.*

Proof. Let us first show that the functions in D_p° with compact support in the closure of $\mathbb{R}^{N+1}_+$ are dense in D_p°. Let $u \in D_p^\circ$ and let $\Phi \in C_c^\infty(\mathbb{R}^{N+1})$ be such that $\Phi = 1$ in $B_1(0)$, $\Phi = 0$ in $\mathbb{R}^{N+1} \setminus B_2(0)$ and $0 \le \Phi \le 1$ in $\mathbb{R}^{N+1}$. Set $\Phi_n(z) = \Phi(z/n)$, where $z=(x,y)$. Observe that $|\nabla \Phi_n| \le C/n$, $|D^2\Phi_n| \le C/n^2$ in $B_{2n}(0)\setminus B_n(0)$ and $\nabla\Phi_n = 0$, $D^2\Phi_n = 0$ elsewhere. The functions $u_n := \Phi_n u \in D_p^\circ$ have compact support in the closure of $\mathbb{R}^{N+1}_+$. By dominated convergence, $u_n \to u$ in $W^{1,p}(\mathbb{R}^{N+1}_+)$ and also $\partial^2_y u_n \to \partial^2_y u$ in $L^p(\mathbb{R}^{N+1}_+)$ as $n \to \infty$. We further obtain that the functions

$$\begin{aligned}
yD^2_x u_n &= \Phi_n(yD^2_x u) + (y\nabla_x \Phi_n) \otimes \nabla_x u + \nabla_x u \otimes (y\nabla_x\Phi_n) + u(yD^2_x \Phi_n),\\
\sqrt{y}\,\nabla_x\partial_y u_n &= \Phi_n(\sqrt{y}\,\nabla_x\partial_y u) + \sqrt{y}\,\nabla_x\Phi_n \partial_y u + \sqrt{y}\,\partial_y\Phi_n \nabla_x u + u(\sqrt{y}\,\nabla_x\partial_y\Phi_n)
\end{aligned}$$

converge to $yD^2_x u$ and $\sqrt{y}\,\nabla_x\partial_y u$, respectively, in $L^p(\mathbb{R}^{N+1}_+)$.

Now, let $u \in D_p^\circ$ be such that $\mathrm{supp} u \subseteq \overline{B_R^+(0)}$, for some $R>0$. Denote by $\tilde{u}$ the odd continuation of u with respect to y on $\mathbb{R}^{N+1}$. Then $\tilde{u}$ belongs to $W^{1,p}(\mathbb{R}^{N+1})$ and has compact support in $\mathbb{R}^{N+1}$. Let ρ_n be a standard sequence of mollifiers such that ρ is an even function in each variable. Then $u_n := \rho_n * \tilde{u} \in \mathcal{D}$ and $u_n \to \tilde{u}$ in $W^{1,p}(\mathbb{R}^{N+1})$ as $n\to\infty$. Since $\mathrm{supp}\, u_n \subseteq B_{R+1}(0)$, we have also $\sqrt{y}\,\nabla u_n \to \sqrt{y}\,\nabla u$ in $L^p(\mathbb{R}^{N+1}_+)$. Concerning the second order derivatives we have

$$y\partial_{x_i x_j} u_n = \partial_{x_i}(y(\rho_n * \partial_{x_j}\tilde{u})) = \partial_{x_i}\big(\rho_n * (y\partial_{x_j}\tilde{u}) + (y\rho_n) * \partial_{x_j}\tilde{u}\big)$$

$$= \rho_n * (y\partial_{x_i x_j}\tilde{u}) + (y\partial_{x_i}\rho_n) * \partial_{x_j}\tilde{u}.$$

The first addend clearly converges to $y\partial_{x_i x_j}\tilde{u}$ in $L^p(\mathbb{R}^{N+1})$. For the second term is concerned, a direct computation shows that $(y\partial_{x_i}\rho_n) * \partial_{x_j}\tilde{u} = (y\partial_{x_i}\rho)_n * \partial_{x_j}\tilde{u}$ and therefore it converges to $\partial_{x_j}\tilde{u}\int_{\mathbb{R}^{N+1}} y\partial_{x_i}\rho(x,y)\,dx\,dy$, which is zero. The convergence of $\partial_y^2 u_n = \rho_n * (\partial_y^2\tilde{u})$ to $\partial_y^2\tilde{u}$ in $L^p(\mathbb{R}^{N+1})$ is standard. In order to prove the convergence of the mixed second order derivative, we take advantage of Theorem 2.1. Applying this result to the difference $u_n - u_m$ yields that $(\sqrt{y}\,\partial_{x_k}\partial_y u_n)$ is a Cauchy sequence in $L^p(\mathbb{R}^{N+1}_+)$, $k = 1, \ldots, N$. As a consequence, there exists $v \in L^p(\mathbb{R}^{N+1}_+)$ such that $\sqrt{y}\,\partial_{x_k}\partial_y u_n \to v$ in $L^p(\mathbb{R}^{N+1}_+)$. It is not difficult to see that $v = \sqrt{y}\,\partial_{x_k}\partial_y u$. So we have shown the assertion. □

For $0 < \varepsilon \le 1/2$, we define the strip

$$S_\varepsilon = \{(x,y) : x \in \mathbb{R}^N, \varepsilon < y < \varepsilon^{-1}\}$$

and the spaces

$$\begin{aligned} D^\circ_{p,\varepsilon} &= W^{2,p}(S_\varepsilon) \cap W^{1,p}_0(S_\varepsilon), \\ \mathcal{D}_\varepsilon &= \{u \in C_c^\infty(\mathbb{R}^{N+1}) : u(x,y) = 0 \ \text{ if } y \le \varepsilon \text{ or } y \ge \varepsilon^{-1}\}. \end{aligned}$$

To unify the notation, we use these spaces also for $\varepsilon = 0$ with the agreements

$$S_0 = \mathbb{R}^{N+1}_+, \qquad D^\circ_{p,0} = D^\circ_p, \qquad \mathcal{D}_0 = \mathcal{D}.$$

Clearly, $\mathcal{D}_\varepsilon$ is dense in $D^\circ_{p,\varepsilon}$ for every $\varepsilon > 0$. For $p = 2$ one can easily prove the a priori estimate of Theorem 2.1 in $D^\circ_{2,\varepsilon}$ with a constant M independent of $\varepsilon \in [0, 1/2]$.

Proposition 2.4 *For every $u \in D^\circ_{2,\varepsilon}$ and $0 \le \varepsilon \le 1/2$, we have*

$$\|y\,D_x^2 u\|^2_{L^2(S_\varepsilon)} + \|\partial_y^2 u\|^2_{L^2(S_\varepsilon)} + 2\|\sqrt{y}\,\nabla_x\partial_y u\|^2_{L^2(S_\varepsilon)} = \|Lu\|^2_{L^2(S_\varepsilon)}.$$

Proof. By Lemma 2.3, it suffices to prove the statement for $u \in \mathcal{D}_\varepsilon$. We then obtain

$$\int_{S_\varepsilon}(Lu)^2 = \int_{S_\varepsilon}(y\Delta_x u)^2 + \int_{S_\varepsilon}(\partial_y^2 u)^2 + 2\int_{S_\varepsilon}\partial_y^2 u\,(y\Delta_x u).$$

Notice that the condition $u(x,\varepsilon) = u(x,\varepsilon^{-1}) = 0$ implies that $\nabla_x u(x,\varepsilon) = \nabla_x u(x,\varepsilon^{-1}) = 0$. Integration by parts now leads to

$$\begin{aligned} \int_{S_\varepsilon}\partial_y^2 u\,(y\Delta_x u) &= -\int_{S_\varepsilon}\partial_y^2\nabla_x u \cdot (y\nabla_x u) = \int_{S_\varepsilon} y|\nabla_x\partial_y u|^2 + \frac{1}{2}\int_{S_\varepsilon}\partial_y|\nabla_x u|^2 \\ &= \int_{S_\varepsilon} y|\nabla_x\partial_y u|^2. \end{aligned} \tag{2.1}$$

Moreover, it is easily checked that

$$\int_{S_\varepsilon}(y\Delta_x u)^2 = \int_{S_\varepsilon} y^2\sum_{i,j=1}^N(\partial_{x_i x_j}u)^2,$$

so that the proof is complete. □

Remark 2.5 The computations of the previous proof, see (2.1), yield

$$\|y\, D_x^2 u\|^2_{L^2(\mathbb{R}^{N+1}_+)} + \|\partial_y^2 u\|^2_{L^2(\mathbb{R}^{N+1}_+)} + 2\|\sqrt{y}\,\nabla_x\partial_y u\|^2_{L^2(\mathbb{R}^{N+1}_+)} = \|Lu\|^2_{L^2(\mathbb{R}^{N+1}_+)} + \frac{1}{2}\|\nabla_x u(\cdot,0)\|^2_{L^2(\mathbb{R}^N)}$$

for every $u \in C_c^\infty(\mathbb{R}^{N+1})$. This equality is satisfied also by any function $u \in D_2$ with $\nabla_x\partial_y u \in L^2(\mathbb{R}^{N+1}_+)$. To see this, one can argue by approximation, as in the proof of Lemma 2.3, just replacing $\tilde{u}$ with

$$\hat{u}(x,y) = \begin{cases} u(x,y), & \text{if } y > 0, \\ -3u(x,-y) + 4u(x,-y/2), & \text{if } y < 0. \end{cases}$$

We continue with interpolation inequalities in $D^\circ_{p,\varepsilon}$.

Lemma 2.6 *There exist two constants $C, \eta_0 > 0$ such that for every $u \in D^\circ_{p,\varepsilon}$, $0 \le \varepsilon \le 1/2$ and $0 < \eta \le \eta_0$ the following inequalities hold.*

(i) $\|\partial_y u\|_{L^p(S_\varepsilon)} \le \eta\|\partial_y^2 u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u\|_{L^p(S_\varepsilon)}$

(ii) $\|\partial_{x_i} u\|_{L^p(S_\varepsilon)} \le \eta\big(\|y\partial_{x_i}^2 u\|_{L^p(S_\varepsilon)} + \|\partial_y^2 u\|_{L^p(S_\varepsilon)}\big) + (C/\eta^3)\|u\|_{L^p(S_\varepsilon)}$

(iii) $\|\sqrt{y}\,\partial_{x_i} u\|_{L^p(S_\varepsilon)} \le \eta\|y\partial_{x_i}^2 u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u\|_{L^p(S_\varepsilon)}$.

Proof. Estimate (i) is well–known. Concerning (ii), Lemma 2.7 of [8] yields that

$$\|\partial_{x_i} u\|_{L^p(S_\varepsilon)} \le \eta\|y\partial_{x_i}^2 u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u/y\|_{L^p(S_\varepsilon)}. \tag{2.2}$$

By the one dimensional Hardy inequality applied to $w(y) = u(x,y)\chi_{[\varepsilon,\varepsilon^{-1}]}(y)$, $y \in (0,+\infty)$, and by integration with respect to $x \in \mathbb{R}^N$, we deduce

$$\|u/y\|_{L^p(S_\varepsilon)} \le \frac{p}{p-1}\|\partial_y u\|_{L^p(S_\varepsilon)}. \tag{2.3}$$

Assertion (ii) now follows by combining (2.2) and (2.3) and using (i) with η^2 instead of η for a possibly different value of C. Finally, inequality (iii) is proved in Lemma 2.7 of [8]. □

Theorem 2.1 and Lemmas 2.3 and 2.6 imply the closedness of (L, D°_p).

Proposition 2.7 *The operator (L, D°_p) is closed in $L^p(\mathbb{R}^{N+1}_+)$.*

In the following result we establish the dissipativity and sectoriality of the operator $(-L, D^\circ_{p,\varepsilon})$ for every $0 \le \varepsilon \le 1/2$.

Proposition 2.8 *Let $\mathrm{Re}\,\lambda \ge 0$, $u \in D^\circ_{p,\varepsilon}$, $0 \le \varepsilon \le 1/2$, and $f = \lambda u + Lu$. Set $u^* := \bar{u}|u|^{p-2}$. It then holds*

$$\begin{aligned} (\mathrm{Re}\,\lambda)\|u\|_{L^p(S_\varepsilon)} &\le \|f\|_{L^p(S_\varepsilon)}, \\ \left|\mathrm{Im}\int_{S_\varepsilon}(Lu)u^*\right| &\le \frac{|p-2|}{2\sqrt{p-1}}\left(\mathrm{Re}\int_{S_\varepsilon}(Lu)u^*\right). \end{aligned}$$

Proof. By density, we may assume that $u \in \mathcal{D}_\varepsilon$. In the proof below we suppose that $p \geq 2$. The case $1 < p < 2$ can be treated similarly by a standard regularization of the power $|a|^{p-2}$, cf. Lemma 2.12. Multiplying the equation $\lambda u + Lu = f$ by u^* and integrating by parts on S_ε, all boundary terms vanish and we have

$$\begin{aligned}
\int_{S_\varepsilon} f u^* &= \lambda \|u\|^p_{L^p(S_\varepsilon)} + \int_{S_\varepsilon} y|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\nabla_x u)|^2 + |\mathrm{Im}\,(\bar{u}\nabla_x u)|^2\Big) \\
&\quad + i(p-2)\int_{S_\varepsilon} y|u|^{p-4}\Big(\mathrm{Re}\,(\bar{u}\nabla_x u)(\mathrm{Im}\,(\bar{u}\nabla_x u)\Big) \\
&\quad + \int_{S_\varepsilon} |u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\partial_y u)|^2 + |\mathrm{Im}\,(\bar{u}\partial_y u)|^2\Big) \\
&\quad + i(p-2)\int_{S_\varepsilon} |u|^{p-4}\Big(\mathrm{Re}\,(\bar{u}\,\partial_y u)(\mathrm{Im}\,(\bar{u}\,\partial_y u)\Big).
\end{aligned}$$

Taking the real parts, we obtain

$$\begin{aligned}
\mathrm{Re}\int_{S_\varepsilon} f u^* &= (\mathrm{Re}\,\lambda)\|u\|^p_{L^p(S_\varepsilon)} + \int_{S_\varepsilon} y|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\nabla_x u)|^2 + |\mathrm{Im}\,(\bar{u}\nabla_x u)|^2\Big) \\
&\quad + \int_{S_\varepsilon} |u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\,\partial_y u)|^2 + |\mathrm{Im}\,(\bar{u}\,\partial_y u)|^2\Big) \\
&\geq (\mathrm{Re}\,\lambda)\|u\|^p_{L^p(S_\varepsilon)}
\end{aligned} \tag{2.4}$$

which implies the first part of the statement. Now, choose $\lambda = 0$. We can estimate the imaginary parts as follows:

$$\begin{aligned}
\left|\mathrm{Im}\int_{S_\varepsilon} (Lu)u^*\right| &\leq |p-2|\Big(\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Re}\,(\bar{u}\nabla_x u)|^2\Big)^{\frac{1}{2}}\Big(\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Im}\,(\bar{u}\nabla_x u)|^2\Big)^{\frac{1}{2}} \\
&\quad + |p-2|\Big(\int_{S_\varepsilon} |u|^{p-4}|\mathrm{Re}\,(\bar{u}\,\partial_y u)|^2\Big)^{\frac{1}{2}}\Big(\int_{S_\varepsilon} |u|^{p-4}|\mathrm{Im}\,(\bar{u}\,\partial_y u)|^2\Big)^{\frac{1}{2}} \\
&\leq \frac{|p-2|}{2\sqrt{p-1}}\Big((p-1)\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Re}\,(\bar{u}\nabla_x u)|^2 + \int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Im}\,(\bar{u}\nabla_x u)|^2\Big) \\
&\quad + \frac{|p-2|}{2\sqrt{p-1}}\Big((p-1)\int_{S_\varepsilon} |u|^{p-4}|\mathrm{Re}\,(\bar{u}\partial_y u)|^2 + \int_{S_\varepsilon} |u|^{p-4}|\mathrm{Im}\,(\bar{u}\partial_y u)|^2\Big).
\end{aligned}$$

Using (2.4) with $\lambda = 0$, we deduce the second assertion. □

Remark 2.9 Propositions 2.7 and 2.8 for $\varepsilon = 0$ say that the operator $(-L, D_p^\circ)$ is closed and regularly dissipative in $L^p(\mathbb{R}^{N+1}_+)$ (i.e., $-e^{i\phi}L$ is dissipative for all $\phi \in (-\phi_0, \phi_0)$ and some $\phi_0 \in (0, \pi/2)$. Of course, it is densely defined. According to standard semigroup theory, $(-L, D_p^\circ)$ thus generates a contractive analytic C_0–semigroup if we can show that the range of $\lambda + L$ is dense in $L^p(\mathbb{R}^{N+1}_+)$ for some $\lambda > 0$. This fact will be established separately for the cases $p = 2$, $p > 2$ and $1 < p < 2$.

We now establish Theorem 2.2 in $L^2(\mathbb{R}^{N+1}_+)$. For this purpose, we first note that Proposition 2.4 and Lemma 2.6 imply the following L^2–estimates which are uniform in ε.

Proposition 2.10 *There exists $C > 0$ such that for every $u \in D^\circ_{2,\varepsilon}$ and $0 \le \varepsilon \le 1/2$*

$$\|u\|_{W^{1,2}(S_\varepsilon)} + \|\partial_y^2 u\|_{L^2(S_\varepsilon)} + \|yD_x^2 u\|_{L^2(S_\varepsilon)} + \|\sqrt{y}\,\nabla_x\partial_y u\|_{L^2(S_\varepsilon)} \le C(\|Lu\|_{L^2(S_\varepsilon)} + \|u\|_{L^2(S_\varepsilon)}).$$

Proof of Theorem 2.2 with $p = 2$. It remains to show the range condition. To this aim, we argue as in Proposition 2.9 of [8]. Take $\lambda > 0$ and $f \in L^2(\mathbb{R}^{N+1}_+)$. Then, by Proposition 2.10, there exists a suitable null sequence (ε_n) such that the solutions $u_{\varepsilon_n} \in D^\circ_{2,\varepsilon_n}$ of $\lambda u_{\varepsilon_n} + Lu_{\varepsilon_n} = f$ in S_{ε_n} converge weakly in $W^{2,2}_{\text{loc}}(\mathbb{R}^{N+1}_+)$ to a function u satisfying $\lambda u + Lu = f$ on $\mathbb{R}^{N+1}_+$. Moreover, u belongs to D_2 due to Proposition 2.10 and Fatou's lemma. As in the proof of Proposition 2.9 in [8] one can verify that $u(\cdot, 0) = 0$. In view of Propositions 2.7 and 2.8, the operator $(-L, D^\circ_2)$ generates an analytic C_0-semigroup of contractions $(T_2(t))_{t\ge0}$ in $L^2(\mathbb{R}^{N+1}_+)$. If f is positive, then the approximating functions u_{ε_n} are positive so that u is positive, which implies the positivity of the semigroup. □

We next consider the case $p > 2$.

Proposition 2.11 *For every $\lambda > 0$ and $p > 2$, the range $(\lambda + L)D^\circ_p$ is dense in $L^p(\mathbb{R}^{N+1}_+)$.*

Proof. Let $\lambda > 0$ and $f \in C_c^\infty(\mathbb{R}^{N+1})$. By the case $p = 2$ already discussed, there exists $u \in D^\circ_2$ such that $\lambda u + Lu = f$. We have to show that $u \in D^\circ_p$. This will be done by showing that also the derivatives of u belong to D_2. From the proof of Theorem 2.2 with $p = 2$ given above we know that there exist $\varepsilon_n > 0$ converging to 0 as $n \to +\infty$ such that u is the weak limit in $W^{2,2}_{\text{loc}}(\mathbb{R}^{N+1}_+)$ of u_{ε_n}, where $u_{\varepsilon_n} \in D^\circ_{2,\varepsilon_n}$ satisfies $\lambda u_{\varepsilon_n} + Lu_{\varepsilon_n} = f$ in S_{ε_n}. Fix $k \in \{1, \dots, N\}$. Differentiating with respect to x_k, we find that

$$\begin{cases} \lambda\partial_{x_k}u_{\varepsilon_n} + L(\partial_{x_k}u_{\varepsilon_n}) = \partial_{x_k}f & \text{in } S_{\varepsilon_n} \\ \partial_{x_k}u_{\varepsilon_n} = 0 & \text{on } \partial S_{\varepsilon_n} \end{cases}$$

where $\partial_{x_k}u_{\varepsilon_n}, \partial_{x_k}f \in L^2(S_{\varepsilon_n})$. From elliptic regularity theory, we deduce that $\partial_{x_k}u_{\varepsilon_n} \in D^\circ_{2,\varepsilon_n}$. Further, up to a subsequence, the sequence $\partial_{x_k}u_{\varepsilon_n}$ converges to $\partial_{x_k}u$ strongly in $L^2_{\text{loc}}(\mathbb{R}^{N+1}_+)$. On the other hand, applying the estimate of Proposition 2.10 to $\partial_{x_k}u_{\varepsilon_n}$, we can extract a new subsequence, still denoted by $\partial_{x_k}u_{\varepsilon_n}$, which converges weakly in $W^{2,2}_{\text{loc}}(\mathbb{R}^{N+1}_+)$ to the solution v in D°_2 of $\lambda v + Lv = \partial_{x_k}f$ in $\mathbb{R}^{N+1}_+$, as n tends to $+\infty$. Therefore $v = \partial_{x_k}u$. This implies that $\partial_{x_k}u \in D^\circ_2$, for any $k \in \{1, \dots, N\}$. In particular, $D_x^2 u, \nabla_x\partial_y u, yD_x^3 u, \nabla_x\partial_y^2 u$ and $\sqrt{y}D_x^2\partial_y u$ belong to $L^2(\mathbb{R}^{N+1}_+)$. By iteration, we deduce that any x-derivative of u belongs to D°_2. Next we write

$$\lambda u - \Delta u = g, \qquad u(\cdot, 0) = 0$$

where $g = f + (y-1)\Delta_x u \in W^{1,2}(\mathbb{R}^N \times (0, M))$, for any $M > 0$. From standard regularity theory, we infer that $u \in W^{3,2}(\mathbb{R}^N \times (0, M))$ for any $M > 0$. Take functions $\eta_n \in C^\infty(\mathbb{R})$ such that

$$\eta_n = 1 \text{ in } (-\infty, n], \qquad \eta_n = 0 \text{ in } [n+1, +\infty), \qquad 0 \le \eta_n \le 1,$$
$$\|\eta_n'\|_\infty + \|\eta_n''\|_\infty + \|\eta_n'''\|_\infty \le C$$

for a constant $C > 0$ and all $n \in \mathbb{N}$. By straightforward computations one sees that $v := \partial_y(\eta_n u) \in D_2$ and $\nabla_x\partial_y v \in L^2(\mathbb{R}^{N+1}_+)$. We can thus apply Remark 2.5 and we obtain

$$\|\partial_y^2 v\|_{L^2(\mathbb{R}^{N+1}_+)} \le C\big(\|\partial_y^2 v + y\Delta_x v\|_{L^2(\mathbb{R}^{N+1}_+)} + \|v(\cdot, 0)\|_{W^{1,2}(\mathbb{R}^N)}\big). \tag{2.5}$$

Observe that $(\lambda+L)\partial_y u = \partial_y f + \Delta_x u$. We now estimate the first addend of the right hand side by writing it explicitly

$$\begin{aligned}\partial_y^2 v + y\Delta_x v &= -\eta_n L(\partial_y u) + \eta_n'(y\Delta_x u + 3\partial_y^2 u) + 3\eta_n''\partial_y u + \eta_n''' u \\ &= \eta_n(\lambda\partial_y u - \partial_y f - \Delta_x u) + \eta_n'(y\Delta_x u + 3\partial_y^2 u) + 3\eta_n''\partial_y u + \eta_n''' u.\end{aligned}$$

Due to the previous steps the right hand side can be estimated in terms of $\|u\|_{D_2}$ and $\|f\|_{W^{1,2}(\mathbb{R}_+^{N+1})}$ independently of n. We thus obtain

$$\|\partial_y^2 v + y\Delta_x v\|_{L^2(\mathbb{R}_+^{N+1})} \le C(\|u\|_{D_2} + \|f\|_{W^{1,2}(\mathbb{R}_+^{N+1})}).$$

Moreover,

$$\begin{aligned}\|v(\cdot,0)\|_{W^{1,2}(\mathbb{R}^N)} &= \|(\partial_y u)(\cdot,0)\|_{W^{1,2}(\mathbb{R}^N)} = \|(\partial_y u)(\cdot,0)\|_{L^2(\mathbb{R}^N)} + \|(\nabla_x\partial_y u)(\cdot,0)\|_{L^2(\mathbb{R}^N)} \\ &\le C(\|\partial_y u\|_{L^2(\mathbb{R}_+^{N+1})} + \|\partial_y^2 u\|_{L^2(\mathbb{R}_+^{N+1})} \\ &\quad + \|\nabla_x\partial_y u\|_{L^2(\mathbb{R}_+^{N+1})} + \|\nabla_x\partial_y^2 u\|_{L^2(\mathbb{R}_+^{N+1})}).\end{aligned}$$

Taking into account the estimates from (2.5), it follows that

$$\|\partial_y^3 u\|_{L^2(\mathbb{R}^N\times(0,n))} \le \|\partial_y^2 v\|_{L^2(\mathbb{R}_+^{N+1})} \le C(\|u\|_{D_2} + \|f\|_{W^{1,2}(\mathbb{R}_+^{N+1})} + \|\nabla_x u\|_{D_2}),$$

for some C independent of n. Hence, $\partial_y^3 u \in L^2(\mathbb{R}_+^{N+1})$ which implies that

$$y\Delta_x\partial_y u = \lambda\partial_y u - \partial_y^3 u - \Delta_x u - \partial_y f$$

also belongs to $L^2(\mathbb{R}_+^{N+1})$. Summing up, we have shown that $\partial_y u \in D_2$. It is clear that we can iterate the procedure, and then infer that all derivatives of u belong to D_2. Using Sobolev's embedding, we thus deduce that u, ∇u, $\partial_y^2 u$, $yD_x^2 u$ and $yD_x^2\partial_y u$ belong to $L^p(\mathbb{R}_+^{N+1})$. Lemma 2.6 now yields that $\sqrt{y}\nabla_x\partial_y u \in L^p(\mathbb{R}_+^{N+1})$, and thus $u \in D_p^\circ$. □

In the case $1 < p < 2$ the above argument does not help since here the higher order Sobolev spaces $W^{k,2}(\mathbb{R}_+^{N+1})$ are not embedded into $L^p(\mathbb{R}_+^{N+1})$. However, compactly supported functions $u \in D_2$ of course belong to D_p if $p < 2$. In order to exploit this fact we first prove an estimate for gradient terms.

Lemma 2.12 *Let $1 < p < 2$, $\lambda \ge 0$, $u \in D_{p,\varepsilon}^\circ$, $0 \le \varepsilon \le 1/2$, and $f = \lambda u + Lu$. Then there is a constant $C_p > 0$ not depending on ε and f such that*

$$\|\partial_y u\|_{L^p(S_\varepsilon)} + \|\sqrt{y}\,\nabla_x u\|_{L^p(S_\varepsilon)} \le C_p(\|f\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)}).$$

Proof. By density, we can again limit ourselves to proving the statement for any $u \in \mathcal{D}_\varepsilon$. Let $\delta > 0$ and multiply the equation $\lambda u + Lu = f$ by $u(u^2+\delta)^{\frac{p-2}{2}}$. Integrating by parts over S_ε, we obtain

$$\begin{aligned}\int_{S_\varepsilon} fu(u^2+\delta)^{\frac{p-2}{2}} &= \lambda\int_{S_\varepsilon} u^2(u^2+\delta)^{\frac{p-2}{2}} + (p-1)\int_{S_\varepsilon}(\partial_y u)^2(u^2+\delta)^{\frac{p-2}{2}} \\ &\quad - (p-2)\delta\int_{S_\varepsilon}(\partial_y u)^2(u^2+\delta)^{\frac{p-4}{2}}\end{aligned}$$

$$+ (p-1)\int_{S_\varepsilon} y|\nabla_x u|^2(u^2+\delta)^{\frac{p-2}{2}} - (p-2)\delta\int_{S_\varepsilon} y|\nabla_x u|^2(u^2+\delta)^{\frac{p-4}{2}}.$$

Since $(p-2)\delta < 0$, we infer from Hölder's inequality that

$$(p-1)\int_{S_\varepsilon} \left((\partial_y u)^2 + y|\nabla_x u|^2\right)(u^2+\delta)^{\frac{p-2}{2}} \le \int_{S_\varepsilon} fu(u^2+\delta)^{\frac{p-2}{2}} \le \|f\|_p\, \|(u^2+\delta)^{\frac{1}{2}}\|_p^{p-1}.$$

Hölder's and Young's inequalities now yield

$$\begin{aligned}\int_{S_\varepsilon}(\partial_y u)^p &= \int_{S_\varepsilon}(\partial_y u)^p(u^2+\delta)^{\frac{p(p-2)}{4}}(u^2+\delta)^{\frac{p(2-p)}{4}}\\ &\le \frac{p}{2}\int_{S_\varepsilon}(\partial_y u)^2(u^2+\delta)^{\frac{p-2}{2}} + \frac{2-p}{2}\int_{S_\varepsilon}(u^2+\delta)^{\frac{p}{2}}\\ &\le \|f\|^p_{L^p(S_\varepsilon)} + c_p\,\|(u^2+\delta)^{\frac{1}{2}}\|^p_{L^p(S_\varepsilon)},\end{aligned}$$

and similarly for $\sqrt{y}\nabla_x u$. Letting $\delta \to 0$, the statement follows. □

Proposition 2.13 *For every $\lambda > 0$ and $1 < p < 2$, the range $(\lambda + L)D_p^\circ$ is dense in $L^p(\mathbb{R}_+^{N+1})$.*

Proof. Let $\lambda > 0$ and $f \in C_c^\infty(\mathbb{R}^{N+1})$. For every $\varepsilon > 0$, there is an $u_\varepsilon \in D^\circ_{p,\varepsilon}$ such that $(\lambda + L)u_\varepsilon = f$ on S_ε. Propositions 2.8 and 2.10 and Lemma 2.12 yield

$$\|u_\varepsilon\|_{L^p(S_\varepsilon)} + \|u_\varepsilon\|_{D^\circ_{2,\varepsilon}} + \|\partial_y u_\varepsilon\|_{L^p(S_\varepsilon)} \le C\left(\|f\|_{L^p(\mathbb{R}_+^{N+1})} + \|f\|_{L^2(\mathbb{R}_+^{N+1})}\right),$$

for a constant $C > 0$ independent of ε. Moreover, as $\partial_{x_k} u_\varepsilon$ solves the equation $(\lambda+L)\partial_{x_k}u_\varepsilon = \partial_{x_k} f$, we also have

$$\|\partial_{x_k}u_\varepsilon\|_{L^p(S_\varepsilon)} \le \lambda^{-1}\|\partial_{x_k} f\|_{L^p(\mathbb{R}_+^{N+1})}$$

for every $k \in \{1, \dots, N\}$. By weak compactness, there exists a sequence $\varepsilon_n \to 0$ such that u_{ε_n} converge to some u weakly in $W^{2,2}_{\rm loc}(\mathbb{R}_+^{N+1})$ and in $W^{1,p}(\mathbb{R}_+^{N+1})$. The proof of Theorem 2.2 with $p = 2$ yields that u belongs to D_2° and satisfies $\lambda u + Lu = f$ in $\mathbb{R}_+^{N+1}$. Moreover, $u \in W^{1,p}(\mathbb{R}_+^{N+1})$.

Take $\Phi \in C_c^\infty(\mathbb{R}^{N+1})$ with $\Phi = 1$ in $B_1(0)$, $\Phi = 0$ in $\mathbb{R}^{N+1} \setminus B_2(0)$ and $0 \le \Phi \le 1$ in $\mathbb{R}^{N+1}$. Set $\Phi_n(z) = \Phi(z/n)$, where $z = (x, y)$. For every $n \in \mathbb{N}$, it holds $|\nabla\Phi_n| \le C/n$, $|D^2\Phi_n| \le C/n^2$ in $B_{2n}(0) \setminus B_n(0)$ and $\nabla\Phi_n = 0$, $D^2\Phi_n = 0$ elsewhere. The functions $u_n := \Phi_n u$ belong to D_p° since they are compactly supported. We want to show that u_n converges to u in D_p as $n \to \infty$ which implies the assertion. Due to Proposition 2.7, it suffices to prove that $u_n \to u$ and $Lu_n \to Lu$ in $L^p(\mathbb{R}_+^{N+1})$. The first convergence is clear. To check the second one, we observe that

$$L(u-u_n) = (1-\Phi_n)Lu + 2\partial_y\Phi_n\,\partial_y u + u\partial_{yy}\Phi_n + 2y\nabla_x\Phi_n\cdot\nabla_x u + yu\Delta_x\Phi_n.$$

Since $Lu = f - \lambda u \in L^p(\mathbb{R}_+^{N+1})$ and $u \in W^{1,p}(\mathbb{R}_+^{N+1})$, the properties of Φ_n and dominated convergence easily imply that the functions $L(u - u_n)$ tend to 0 in $L^p(\mathbb{R}_+^{N+1})$. □

Proof of Theorem 2.2. In view of Remark 2.9 and Propositions 2.11 and 2.13, it remains to show positivity and consistency. The proofs of Propositions 2.11 and 2.13 show that the resolvents of $(-L, D_p^\circ)$ coincide on $C_c^\infty(\mathbb{R}^{N+1})$ for all $\lambda > 0$, so that they coincide on $L^p(\mathbb{R}_+^{N+1}) \cap L^q(\mathbb{R}_+^{N+1})$. This fact shows consistency. Positivity then follows from the case $p = 2$ already proved. □

Let $q \in (1,\infty)$, $T>0$ and $J=(0,T)$. We say that a closed, densely defined operator A on a Banach space X has *maximal regularity of type* L^q if for all $f \in L^q(J,X)$ there is a unique solution $u \in L^q(J,D(A)) \cap W^{1,q}(J,X)$ of the Cauchy problem

$$u'(t) = Au(t), \quad t \in J, \qquad u(0)=0.$$

We refer to [6] and [14] for a thorough discussion of this property and for further references. Here we just note that this property does not depend on $T>0$ and $q \in (1,\infty)$ and that A generates an analytic semigroup if it has maximal regularity of type L^q. In our setting we can use that A has maximal regularity of type L^q if it generates a positive and contractive analytic semigroup on an L^p space with $p \in (1,\infty)$. This fact follows from Corollary 5.2 and Theorems 5.3 and 6.1 of [11].

Corollary 2.14 *Let $p,q \in (1,\infty)$. The operator $(-L, D^\circ_p)$ has maximal L^q–regularity.*

As a preparation for the following sections, we further introduce the operator

$$L_0 = -a_0\partial_y^2 - y\sum_{i,j=1}^N a_{ij}\partial_{x_ix_j} + \sum_{i=1}^{N+1} b_i\partial_i u \tag{2.6}$$

with constant coefficients $a_0, a_{ij}, b_i \in \mathbb{R}$ satisfying the conditions $a_0>0$ and $a_{ij}=a_{ji}$ for all $i,j=1,\dots,N$ as well as

$$\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \ge \mu|\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and some $\mu>0$. Set $\mathsf{M} = \max\{|a_{ij}|, |b_i|, a_0, a_0^{-1}, \mu^{-1}\}$. We endow $-L_0$ with the domain D°_p.

Theorem 2.15 *Let $p \in (1,\infty)$. There are constants $\Lambda_p \ge \omega_p \ge 0$ and $C_1 \ge 0$ depending on M, N and p such that for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \omega_p$ and $f \in L^p(\mathbb{R}^{N+1}_+)$ there exists a unique solution $u \in D^\circ_p$ of $\lambda u + L_0 u = f$ such that*

$$|\lambda|\|u\|_{L^p(\mathbb{R}^{N+1}_+)} \le C_1\|f\|_{L^p(\mathbb{R}^{N+1}_+)}, \tag{2.7}$$

$$\|\partial_y^2 u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|yD_x^2 u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x\partial_y u\|_{L^p(\mathbb{R}^{N+1}_+)} \le C_1\|f\|_{L^p(\mathbb{R}^{N+1}_+)}, \tag{2.8}$$

Morover, for $\mathrm{Re}\lambda > \Lambda_p$ we have

$$|\lambda|^{\frac12}\big(\|\partial_y u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x u\|_{L^p(\mathbb{R}^{N+1}_+)}\big) + |\lambda|^{\frac14}\|\nabla_x u\|_{L^p(\mathbb{R}^{N+1}_+)} \le C_1\|f\|_{L^p(\mathbb{R}^{N+1}_+)}.$$

Proof. Assume first that $b_i=0$ for every $i=1,\dots,N+1$ and that $\mathrm{Re}\,\lambda>0$. Let Q be a non-singular $N\times N$ matrix such that $\sum_{i,j=1}^N a_0^{\frac12} a_{ij}\partial_{x_ix_j}\varphi(x) = \Delta\psi(Qx)$ whenever $\varphi(x)=\psi(Qx)$ for $x \in \mathbb{R}^N$. We use the endomorphism of $\mathbb{R}^{N+1}_+$ mapping $z=(x,y)$ to $\zeta=(\xi,\eta)=(Qx, a_0^{-\frac12}y)$. Setting $u(z)=w(\zeta)$ and $f(z)=\phi(\zeta)$, the equation $\lambda u(z)+L_0u(z)=f(z)$ is now equivalent to

$$\lambda w(\zeta) + Lw(\zeta) = \lambda w(\zeta) - \partial_\eta^2 w(\zeta) - \eta\Delta_\xi w(\zeta) = \phi(\zeta),$$

and the first part of the statement follows easily from Theorem 2.2. Applying Theorem 2.2 to w we have

$$\|\partial_y^2 u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|yD_x^2 u\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x\partial_y u\|_{L^p(\mathbb{R}^{N+1}_+)}$$

$$\le C\left(\|Lw\|_{L^p(\mathbb{R}^{N+1}_+)} + \|w\|_{L^p(\mathbb{R}^{N+1}_+)}\right)$$
$$\le C\left(\|f\|_{L^p(\mathbb{R}^{N+1}_+)} + |\lambda|^{-1}\|f\|_{L^p(\mathbb{R}^{N+1}_+)}\right).$$

Therefore estimate (2.8) follows. Finally, by Lemma 2.6 there exist $C, \eta_0 > 0$ such that for every $0 < \varepsilon \le \eta_0$

$$\|\partial_y u\|_{L^p(\mathbb{R}^{N+1}_+)} \le \varepsilon \|\partial_y^2 u\|_{L^p(\mathbb{R}^{N+1}_+)} + \frac{C}{\varepsilon}\|u\|_{L^p(\mathbb{R}^{N+1}_+)}.$$

Taking (2.8) and (2.7) into account, we get

$$\|\partial_y u\|_{L^p(\mathbb{R}^{N+1}_+)} \le C\varepsilon \|f\|_{L^p(\mathbb{R}^{N+1}_+)} + \frac{C}{\varepsilon|\lambda|}\|f\|_{L^p(\mathbb{R}^{N+1}_+)}.$$

Choosing $\varepsilon = |\lambda|^{-1/2}$ yields the desired estimate. The remaining terms can be estimated analogously.

Finally, the general case where first order terms are present in L_0 can be handled by a perturbation argument, since estimates (i) and (ii) of Lemma 2.6 show that the operator $B = \mathbf{b} \cdot \nabla$, with $\mathbf{b} = (b_1, \dots, b_{N+1})$, is a small perturbation of $-a_0 \partial_y^2 - y \sum_{i,j=1}^N a_{ij} \partial_{x_i x_j}$ (see [7, Section III.2]). □

3 The localization procedure

Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ with boundary of class C^2 and let ϱ be a function in $C^2(\mathbb{R}^{N+1})$ such that

$$\Omega = \{\varrho > 0\}, \quad \partial\Omega = \{\varrho = 0\} \text{ and } \nabla\varrho(\xi) = \nu(\xi),\ \xi \in \partial\Omega. \tag{3.1}$$

Here, $\nu(\xi)$ is the inward unitary normal vector to $\partial\Omega$ at ξ. Such a function ϱ can be constructed by extending the distance function from the boundary of Ω. Let us introduce the operator L defined on smooth functions as

$$L\varphi = -\mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2\varphi) - \varrho \sum_{i,j=1}^{N+1} a_{ij}\partial_{ij}\varphi - \sum_{i=1}^{N+1} b_i \partial_i \varphi. \tag{3.2}$$

In the remainder of the paper we shall assume the following conditions on the coefficients.

(H1) $\mathbf{a} = (a_1, \dots, a_{N+1})$ is a vector–valued C^2 function in a neighbourhood of $\overline{\Omega}$ such that at each point $\xi \in \partial\Omega$ the vector $\mathbf{a}(\xi)$ is non tangent at $\partial\Omega$, namely $\mathbf{a}(\xi) \cdot \nu(\xi) \neq 0$.

(H2) a_{ij} are real–valued continuous functions on $\overline{\Omega}$ with $a_{ij} = a_{ji}$ and satisfy the ellipticity conditions

$$\sum_{i,j=1}^{N+1} a_{ij}(\xi)\tau_i\tau_j \ge \mu_0|\tau|^2, \quad \text{for every } \xi \in \partial\Omega,\ \tau \in \mathbb{R}^{N+1} \text{ with } \tau \cdot \mathbf{a}(\xi) = 0,$$
$$\sum_{i,j=1}^{N+1} \Big(a_i(\xi)a_j(\xi) + \varrho(\xi)a_{ij}(\xi)\Big)\zeta_i\zeta_j \ge \mu(\xi)|\zeta|^2, \quad \text{for every } \xi \in \Omega,\ \zeta \in \mathbb{R}^{N+1},$$

for some constant $\mu_0 > 0$ and a suitable function μ with $\inf_K \mu > 0$, for any compact set K contained in Ω.

(H3) b_i are real–valued continuous functions on $\overline{\Omega}$.

Example 3.1 Let us consider $\Omega = B_1(0)$ in $\mathbb{R}^{N+1}$ and choose $\mathbf{a}(\xi) = \xi$, for any $\xi \in \overline{\Omega}$. Set $r = |\xi|$. Then the operator

$$L\varphi = -r^2\partial_r^2\varphi - (1-r^2)\Delta\varphi = -\partial_r^2\varphi - (1-r^2)\frac{N}{r}\partial_r\varphi - \frac{1-r^2}{r^2}\Delta_S\varphi, \qquad r \neq 0$$

where Δ_S denotes the (negative) Laplace-Beltrami operator on $\partial\Omega$, is of the form (3.2) with $\varrho(\xi) = 1 - r^2$. Another simple example is

$$L_1\varphi = -\partial_r^2\varphi - \frac{N}{r}\partial_r\varphi - \frac{1-r^2}{r^2}\Delta_S\varphi, \qquad r \neq 0$$

which differs from L by the first-order bounded perturbation $rN\partial_r$. More generally, any operator which is uniformly elliptic in the interior and can be written near the boundary in the form

$$L\varphi = -\partial_r^2\varphi - (1-r^2)\Delta_S\varphi + B\varphi,$$

where B is a first-order bounded perturbation, satisfies our assumptions.

Without loss of generality, we can assume that

$$\mathsf{m} = \min_{\xi\in\partial\Omega} \mathbf{a}(\xi)\cdot\nu(\xi) > 0 \tag{3.3}$$

and define

$$\mathsf{M} = \max_{1\le i,j\le N+1}\{\|\mathbf{a}\|_\infty, \|a_{ij}\|_\infty, \|b_i\|_\infty\}.$$

Let $\xi_0 \in \partial\Omega$ be fixed. Following [3], in a neighborhood $U = U(\xi_0)$ of ξ_0 we consider functions $\theta_1, \ldots, \theta_N \in C^2(U)$ solving the equation

$$\sum_{i=1}^{N+1} a_i(\xi)\,\partial_i\theta(\xi) = 0, \qquad \xi \in U, \tag{3.4}$$

such that $\nabla\theta_1(\xi_0), \ldots, \nabla\theta_N(\xi_0)$ are linearly independent. Such functions exist by classical results on partial differential equations of first order, see e.g. Theorem 33.3 of [5]. We then define the transformation

$$J : U \to \mathbb{R}^{N+1}, \quad \xi \mapsto (\boldsymbol{\theta}(\xi), \varrho(\xi))$$

where $\boldsymbol{\theta}(\xi) = (\theta_1(\xi), \ldots, \theta_N(\xi))$. Due to (H1), (3.1) and (3.4), the Jacobian matrix of J at ξ_0 is non–singular. Therefore, possibly taking U smaller, we obtain that J is a C^2–diffeormorphism from U onto $J(U)$. It further holds that $J(U\cap\Omega) = J(U)\cap\mathbb{R}^{N+1}_+$ and $J(U\cap\partial\Omega) = J(U)\cap\{y=0\}$. So (U,J) is a local chart. We denote by H the inverse of J. We can cover $\partial\Omega$ by the finite union $V = U_1\cup\cdots\cup U_m$ of open sets of the above type. Thus, below we may always assume that $U(\xi_0) \subset U_i$ for some of the U_i and that J and H are restrictions of the diffeomorphism on U_i. Hence, all the derivatives of J and H up to the second order may be assumed to be bounded by a constant independent of ξ_0. To fix the notation we suppose that for any $k = 1, \ldots, N+1$

$$\|J_k\|_\infty + \|\nabla J_k\|_\infty + \|D^2J_k\|_\infty \le \mathsf{L},$$
$$\|H_k\|_\infty + \|\nabla H_k\|_\infty + \|D^2H_k\|_\infty \le \mathsf{L}.$$

Finally, we can assume that

$$\mathbf{a}(\xi) \cdot \nabla \varrho(\xi) \geq \mathsf{m}/2 \quad \text{for all } \xi \in U \cap \Omega, \tag{3.5}$$

by virtue of (3.3). Such local coordinates have the advantage of transforming all the vectors $\mathbf{a}(\xi)$ at points $\xi \in U \cap \Omega$ into the normal direction at $\{y = 0\}$ by the formula

$$\big(\mathrm{Jac}\, J(\xi)\big)\mathbf{a}(\xi) = \big(\mathbf{a}(\xi) \cdot \nabla \varrho(\xi)\big) e_{N+1}. \tag{3.6}$$

It follows that

$$\big(\mathrm{Jac}\, H(z)\big) e_{N+1} = \frac{\mathbf{a}(\xi)}{\mathbf{a}(\xi) \cdot \nabla \varrho(\xi)} \tag{3.7}$$

for $z = J(\xi)$. Define $\phi(z) = \varrho(Hz)$, for $z \in J(U) \cap \mathbb{R}^{N+1}_+$. Using Taylor's formula with respect to the last variable, for $z = (x, y)$ we find that

$$\phi(z) = \phi(x, y) = \phi(x, 0) + \partial_y \phi(x, 0)\, y + \frac{1}{2}\partial_y^2 \phi(x, t)\, y^2 = y\bigg(\partial_y \phi(x, 0) + \frac{1}{2}\partial_y^2 \phi(x, t)\, y\bigg),$$

for some $t \in (0, y)$. Recalling (3.7), we obtain

$$\partial_y \phi(z) = \big\langle \big(\mathrm{Jac}\, H(z)\big) e_{N+1}, \nabla \varrho(Hz) \big\rangle = \frac{\mathbf{a}(\xi) \cdot \nabla \varrho(\xi)}{\mathbf{a}(\xi) \cdot \nabla \varrho(\xi)} = 1$$

with $\xi = Hz$. Therefore we may write

$$\phi(z) = y\, d(z), \qquad z \in J(U) \cap \mathbb{R}^{N+1}_+, \tag{3.8}$$

where d is a continuous function with $d(x, 0) = 1$ which is bounded from above and below by positive constants independently of ξ_0.

Given a function $u : U \cap \Omega \to \mathbb{R}$, set $Tu = u \circ H$ on $J(U) \cap \mathbb{R}^{N+1}_+$. One can check that

$$\nabla Tu = (\mathrm{Jac}\, H)^* (\nabla u) \circ H.$$

In particular, equality (3.7) yields

$$\partial_y Tu(z) = \langle \nabla Tu(z), e_{N+1} \rangle = \frac{\mathbf{a}(\xi) \cdot \nabla u(\xi)}{\mathbf{a}(\xi) \cdot \nabla \varrho(\xi)}$$

for $\xi = Hz$. The boundedness of the derivatives of H and its inverse implies that T induces isomorphisms from $L^p(U \cap \Omega)$ onto $L^p(J(U) \cap \mathbb{R}^{N+1}_+)$ and from $W^{1,p}(U \cap \Omega)$ onto $W^{1,p}(J(U) \cap \mathbb{R}^{N+1}_+)$, for any $p \in [1, +\infty]$. Let $u \in W^{1,p}(U \cap \Omega) \cap W^{2,p}_{\mathrm{loc}}(U \cap \Omega)$. Due to (3.8), the function $\varrho D^2 u$ belongs to $L^p(U \cap \Omega)$ iff $y D^2(Tu)$ is contained in $L^p(J(U) \cap \mathbb{R}^{N+1}_+)$. Since

$$\begin{aligned}\partial_y^2 Tu &= \big\langle (\mathrm{Jac}\, H)^* (D^2 u)(\mathrm{Jac}\, H) e_{N+1}, e_{N+1} \big\rangle + \text{first order terms} \\ &= \big(\mathbf{a}(\xi) \cdot \nabla \varrho(\xi)\big)^{-2} \mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2 u) + \text{first order terms},\end{aligned}$$

it holds $\mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2 u) \in L^p(U \cap \Omega)$ iff $\partial_y^2(Tu) \in L^p(J(U) \cap \mathbb{R}^{N+1}_+)$. Finally from the expression

$$\sqrt{y}\, \partial_{x_k} \partial_y Tu = \sqrt{y} \big\langle (\mathrm{Jac}\, H)^* (D^2 u)(\mathrm{Jac}\, H) e_{N+1}, e_k \big\rangle + \text{first order terms}$$

$$= \frac{1}{\sqrt{d(J(\xi))}\big(\mathbf{a}(\xi)\cdot\nabla\varrho(\xi)\big)}\big\langle(\sqrt{\varrho}D^2u\,\mathbf{a})(\xi),(\mathrm{Jac}\,H)(J(\xi))e_k\big\rangle + \text{first order terms}$$

it follows that $\sqrt{y}\,\nabla\partial_y Tu \in L^p(J(U)\cap\mathbb{R}^{N+1}_+)$ iff $\sqrt{\varrho}\,D^2u\,\mathbf{a}\in L^p(U\cap\Omega)$. Moreover, in these equivalences also the norms of the respective functions are uniformly equivalent. Moreover, all the operator norms of T and T^{-1} can be estimated by constants independent of ξ_0.

The differential operator L is locally transformed into the operator $\mathcal{L}$ given by

$$\mathcal{L} = -\alpha(z)\partial_y^2 - \phi(z)\sum_{h,k=1}^{N+1}\alpha_{hk}(z)\partial_{hk} - \phi(z)\sum_{k=1}^{N+1}\beta_k(z)\partial_k - \sum_{k=1}^{N+1}\gamma_k(z)\partial_k \tag{3.9}$$

with the coefficients

$$\begin{aligned}
\alpha(z) &= \big(\mathbf{a}(Hz)\cdot\nabla\varrho(Hz)\big)^2,\\
\alpha_{hk}(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz)\partial_{\xi_j}J_h(Hz)\partial_{\xi_i}J_k(Hz),\\
\beta_k(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz)\partial_{\xi_i\xi_j}J_k(Hz),\\
\gamma_k(z) &= \sum_{i,j=1}^{N+1} a_i(Hz)a_j(Hz)\partial_{\xi_i\xi_j}J_k(Hz) + \sum_{i=1}^{N+1} b_i(Hz)\partial_{\xi_i}J_k(Hz).
\end{aligned} \tag{3.10}$$

Notice that the sup–norms of all the coefficients of $\mathcal{L}$ are controlled by constants depending on $\mathsf{M},\mathsf{L},\|\nabla\varrho\|_\infty$ and not deepending on ξ_0. In order to deal with the class of operators introduced in (2.6), we freeze the coefficients of $\mathcal{L}$ at the point $z_0 = J(\xi_0)$ as follows

$$\mathcal{L}^\circ = -\alpha(z_0)\partial_y^2 - y\sum_{h,k=1}^{N}\alpha_{hk}(z_0)\partial_{x_hx_k} - \sum_{k=1}^{N+1}\gamma_k(z_0)\partial_k. \tag{3.11}$$

Remark 3.2 Let us prove that the matrix $\big(\alpha_{hk}(z_0)\big)_{h,k=1}^N$ satisfies the ellipticity condition with a constant independent of ξ_0. Let $\zeta\in\mathbb{R}^N$ and set $\tilde{\zeta} = (\zeta,0)\in\mathbb{R}^{N+1}$. Then, by the definition of $\alpha_{hk}(z_0)$ we have

$$\sum_{h,k=1}^{N}\alpha_{hk}(z_0)\zeta_h\zeta_k = \sum_{i,j=1}^{N+1}a_{ij}(\xi_0)X_iX_j,$$

where $X_i = \sum_{h=1}^{N}\partial_{\xi_i}J_h(\xi_0)\zeta_h$ and thus $X = \big(\mathrm{Jac}\,J(\xi_0)\big)^*\tilde{\zeta}$. In order to apply (H2), we have to show that the vector X is orthogonal to $\mathbf{a}(\xi_0)$. To this aim, using (3.6) we find

$$\langle X,\mathbf{a}(\xi_0)\rangle = \langle\tilde{\zeta},\big(\mathrm{Jac}\,J(\xi_0)\big)\mathbf{a}(\xi_0)\rangle = \big(\mathbf{a}(\xi_0)\cdot\nu(\xi_0)\big)\langle\tilde{\zeta},e_{N+1}\rangle = 0.$$

Therefore

$$\sum_{h,k=1}^{N}\alpha_{hk}(z_0)\zeta_h\zeta_k \ge \mu_0|X|^2 \ge C\mu_0|\zeta|^2,$$

for some constant C independent of ξ_0. *Moreover, estimate* (3.3) *implies that* $\alpha(z_0) \geq \mathsf{m}^2$. *Therefore the operator* $\mathcal{L}^\circ$*, defined by (3.11), satisfies the assertions of Theorem 2.15 with constants* C_1, Λ_p, ω_p *independent of* ξ_0.

In the next sections we shall use a suitable covering of $\overline{\Omega}$, constructed as follows. For every $\xi_0 \in \partial\Omega$, let (U_{ξ_0}, J_{ξ_0}) be the local chart constructed at the beginning of the section. Given $\varepsilon > 0$, choose a ball $B_{r(\xi_0)}(\xi_0) \subset U_{\xi_0}$ such that if $z \in J_{\xi_0}(B_{r(\xi_0)}(\xi_0)) \cap \mathbb{R}^{N+1}_+$, then

$$\begin{aligned} |\alpha(z) - \alpha(z_0)| &< \varepsilon, \\ |d(z)\alpha_{hk}(z) - \alpha_{hk}(z_0)| &< \varepsilon, \qquad h,k = 1,\dots,N+1 \\ |\phi(z)| + |\sqrt{y}\,d(z)| &< \varepsilon, \\ |\gamma_k(z) - \gamma_k(z_0)| &< \varepsilon, \qquad k = 1,\dots,N+1 \end{aligned} \tag{3.12}$$

where $z_0 = J_{\xi_0}(\xi_0)$, α_{hk}, γ_k are given in (3.10) and d, ϕ in (3.8). Set $\mathscr{F}_\varepsilon = \{B_{r(\xi)}(\xi) : \xi \in \partial\Omega\}$. By a suitable covering argument (see e.g. [1, Theorem 2.18]), recalling that $\partial\Omega$ is compact, we can extract a finite subcovering $\mathscr{G}_\varepsilon = \{B_{r(\xi_i)}(\xi_i) : i = 1,\dots,m\}$ such that at most c_N among the balls of $\mathscr{G}_\varepsilon$ overlap. Here c_N is a natural number which depends only on the dimension. Set $U_i = B_{r(\xi_i)}(\xi_i)$, $J_i = J_{\xi_i|B_{r(\xi_i)}(\xi_i)}$ and $\widetilde{U}_i = J_i(U_i)$, $z_i = J_i(\xi_i)$.

We shall see that the arbitrariness of ε will play an important role in the proofs of the main results.

4 Generation in L^p on bounded domains

Let $1 < p < \infty$. We introduce the domain

$$D_p(L) = \left\{u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap W^{1,p}_0(\Omega) : \varrho D^2 u, \mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2 u), \sqrt{\varrho}\, D^2 u\, \mathbf{a} \in L^p(\Omega)\right\},$$

which is a Banach space with respect to the canonical norm

$$\|u\|_{D_p(L)} = \|u\|_{W^{1,p}(\Omega)} + \|\varrho D^2 u\|_{L^p(\Omega)} + \|\mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2 u)\|_{L^p(\Omega)} + \|\sqrt{\varrho}\, D^2 u\, \mathbf{a}\|_{L^p(\Omega)}.$$

The main result of this section is stated in the next theorem.

Theorem 4.1 *Under assumptions (H1), (H2) and (H3) the operator* $(-L, D_p(L))$ *generates an analytic semigroup in* $L^p(\Omega)$ *for* $p \in (1,\infty)$*. In particular, there exists* $\omega_p > 0$ *such that*

$$\sup_{\mathrm{Re}\lambda \geq \omega_p} \|\lambda(\lambda + L)^{-1}\| < +\infty.$$

We shall use the following interpolative estimates, whose proof is based on the use of the local charts introduced in Section 3 and on the estimates in Lemma 2.6 (see also [8, Lemma 3.3]).

Lemma 4.2 *There exist* $\varepsilon_0, C > 0$ *such that for every* $0 < \varepsilon \leq \varepsilon_0$ *and every* $u \in D_p(L)$

$$\begin{aligned} \|\mathbf{a} \cdot \nabla u\|_{L^p(\Omega)} &\leq \varepsilon \|u\|_{D_p(L)} + \frac{C}{\varepsilon}\, \|u\|_{L^p(\Omega)} \\ \|\sqrt{\varrho}\, \nabla u\|_{L^p(\Omega)} &\leq \varepsilon \|u\|_{D_p(L)} + \frac{C}{\varepsilon}\, \|u\|_{L^p(\Omega)} \\ \|\nabla u\|_{L^p(\Omega)} &\leq \varepsilon \|u\|_{D_p(L)} + \frac{C}{\varepsilon^3}\, \|u\|_{L^p(\Omega)}. \end{aligned} \tag{4.1}$$

Proof of Theorem 4.1. We first construct a right inverse of $\lambda+L$ satisfying the sectioriality estimate. In a second step the injectivity of $\lambda+L$ is established.

Step 1. We claim that there exist $\omega_p^1, C>0$ such that for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\ge\omega_p^1$ and $f\in L^p(\Omega)$ there is $u\in D_p(L)$ satisfying $\lambda u+Lu=f$ and $|\lambda|\,\|u\|_{L^p(\Omega)}\le C\,\|f\|_{L^p(\Omega)}$. Consider the open covering $\{U_1,\dots,U_m\}$ of $\partial\Omega$ satisfying (3.12) with ε to be determined. Let U_0 be an open set with boundary of class C^2 such that $U_0\subset\subset\Omega$ and $\{U_0,U_1,\dots,U_m\}$ is a covering of $\overline{\Omega}$. Let $H_i=J_i^{-1}$ and $\widetilde{U}_i=J_i(U_i)$ for $i\in\{1,\cdots,m\}$. We define

$$T_i: L^p(U_i)\to L^p(\widetilde{U}_i),\qquad T_i\varphi=\varphi\circ H_i \tag{4.2}$$

Set $\Omega_i=U_i\cap\Omega$. We consider T_i also on $L^p(\Omega_i)$. Let $\{\eta_i^2\}_{i=0}^m$ with $0\le\eta_i\le 1$ be a partition of unity subordinate to $U_0,U_1,\cdots,U_m$. To simplify the notation, in the constant C below (that may change from line to line) the dependence on U_i and η_i is made explicit by writing a subscript i, whereas we omit the dependence on the other quantities $N,p,\mathsf{m},\mu,\mathsf{M},\mathsf{L}$ and the set Ω.

Let $f\in L^p(\Omega)$ be fixed. Since the operator L is nondegenerate in U_0, it is well-known that if $\lambda\in\mathbb{C}$ and $\operatorname{Re}\lambda\ge\lambda_0$, for a suitable $\lambda_0\in\mathbb{R}$, then there exists a unique solution $u_0\in W^{2,p}(U_0)\cap W_0^{1,p}(U_0)$ of the equation $\lambda u_0+Lu_0=\eta_0 f$. Set $R_0(\lambda)f=\eta_0u_0$ and extend it by 0. Then $R_0(\lambda)f\in D_p(L)$ and

$$(\lambda+L)R_0(\lambda)f=\eta_0^2f+[L,\eta_0]u_0=\eta_0^2f+E_0f,$$

where $[L,\eta_0]$ denotes the commutator between L and the multiplicative operator by η_0. It is easily seen that

$$\|E_0f\|_{L^p(\Omega)}\le\frac{C_0}{|\lambda|^{1/2}}\|f\|_{L^p(U_0)}, \tag{4.3}$$

where the constant C_0 depends on U_0.

Now, fix $i\ge 1$. Denote by $\mathcal{L}_i$, $\mathcal{L}_i^\circ$ the operators obtained from $\mathcal{L}$, $\mathcal{L}^\circ$, defined in (3.9), (3.11), replacing J,H,z_0 with J_i,H_i,z_i, respectively. By Theorem 2.15 and Remark 3.2, for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\Lambda_p$, there exists a unique solution $v_i\in D_p^\circ$ of $\lambda v_i+\mathcal{L}_i^\circ v_i=T_i(\eta_i f)$ in $\mathbb{R}^{N+1}_+$ with

$$\begin{aligned}\|v_i\|_{L^p(\mathbb{R}^{N+1}_+)}&\le\frac{C}{|\lambda|}\|T_i(\eta_if)\|_{L^p(\mathbb{R}^{N+1}_+)}\\ \|\partial_yv_i\|_{L^p(\mathbb{R}^{N+1}_+)}+\|\sqrt{y}\,\nabla_xv_i\|_{L^p(\mathbb{R}^{N+1}_+)}&\le\frac{C}{|\lambda|^{1/2}}\|T_i(\eta_if)\|_{L^p(\mathbb{R}^{N+1}_+)}\\ \|v_i\|_{D_p}&\le C\|T_i(\eta_if)\|_{L^p(\mathbb{R}^{N+1}_+)}.\end{aligned} \tag{4.4}$$

We set

$$R_i(\lambda)f=T_i^{-1}\big(T_i(\eta_i)v_i\big)$$

and extend this function by 0 to Ω. Then $R_i(\lambda)f$ belongs to $D_p(L)$ and has compact support contained in Ω_i. By the identity $L=T_i^{-1}\mathcal{L}_iT_i$ holding in $L^p(\Omega_i)$, we easily get

$$(\lambda+L)R_i(\lambda)f=T_i^{-1}(\lambda+\mathcal{L}_i)\big(T_i(\eta_i)v_i\big)=\eta_i^2f+B_if+E_if$$

on Ω_i, where

$$B_if=T_i^{-1}\big(T_i(\eta_i)(\mathcal{L}_i-\mathcal{L}_i^\circ)v_i\big)\quad\text{ and }\quad E_if=T_i^{-1}\big([\mathcal{L}_i,T_i(\eta_i)]v_i\big).$$

We now estimate the L^p-norms of $B_i f$ and $E_i f$. It holds

$$
\begin{aligned}
(\mathcal{L}_i - \mathcal{L}_i^\circ) v_i(z) &= -\Big(\alpha^i(z) + \phi^i(z)\alpha^i_{N+1\,N+1}(z) - \alpha^i(z_i)\Big)\partial_y^2 v_i \\
&\quad - \sum_{h,k=1}^{N} y\Big(d^i(z)\alpha^i_{hk}(z) - \alpha^i_{hk}(z_i)\Big)\partial_{x_h x_k} v_i - 2\sqrt{y}\, d^i(z)\sum_{h=1}^{N}\sqrt{y}\,\alpha^i_{h\,N+1}(z)\partial_{x_h y} v_i \\
&\quad - \phi^i(z)\sum_{k=1}^{N+1}\beta^i_k(z)\partial_k v_i - \sum_{k=1}^{N+1}(\gamma^i_k(z) - \gamma^i_k(z_i))\partial_k v_i
\end{aligned}
\tag{4.5}
$$

for every $z \in \widetilde{U}_i \cap \mathbb{R}^{N+1}_+$, where the superscript i means that the corresponding function is relative to (U_i, J_i) and the function d was defined in (3.8). Therefore (3.12) yields

$$
\|B_i f\|_{L^p(\Omega)} \le C\,\|(\mathcal{L}_i - \mathcal{L}_i^\circ)v_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}^{N+1}_+)} \le C\varepsilon\|v_i\|_{D_p}.
$$

By (4.4) it turns out that

$$
\|B_i f\|_{L^p(\Omega)} \le C\varepsilon\|f\|_{L^p(\Omega_i)}. \tag{4.6}
$$

Concerning $E_i f$, we have

$$
\begin{aligned}
\big[\mathcal{L}_i, T_i(\eta_i)\big]v_i = &- \alpha^i(z)v_i\,\partial_y^2 T_i(\eta_i) - \phi^i(z)v_i\sum_{h,k=1}^{N+1}\alpha^i_{hk}(z)\partial_{hk}T_i(\eta_i) \\
&- \phi^i(z)v_i\sum_{k=1}^{N+1}\beta^i_k(z)\partial_k T_i(\eta_i) - v_i\sum_{k=1}^{N+1}\gamma^i_k(z)\partial_k T_i(\eta_i) \\
&- 2\alpha^i(z)\partial_y v_i\partial_y T_i(\eta_i) - 2\phi^i(z)\sum_{h,k=1}^{N+1}\alpha^i_{hk}(z)\partial_h T_i(\eta_i)\partial_k v_i
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|E_i f\|_{L^p(\Omega)} &\le C\|[\mathcal{L}_i, T_i(\eta_i)]v_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}^{N+1}_+)} \\
&\le C_i\Big(\|v_i\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\partial_y v_i\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x v_i\|_{L^p(\mathbb{R}^{N+1}_+)}\Big).
\end{aligned}
$$

The estimates (4.4) then lead to

$$
\|E_i f\|_{L^p(\Omega)} \le \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}. \tag{4.7}
$$

Setting $R(\lambda)f = \sum_{i=0}^{m} R_i(\lambda)f$ and $S(\lambda)f = E_0 f + \sum_{i=1}^{m}\big(B_i f + E_i f\big)$ we have

$$
(\lambda + L)R(\lambda)f = f + S(\lambda)f. \tag{4.8}
$$

Estimates (4.3), (4.6) and (4.7) imply that

$$
\|S(\lambda)f\|_{L^p(\Omega)} \le \sum_{i=1}^{m} C\varepsilon\|f\|_{L^p(\Omega_i)} + \sum_{i=0}^{m}\frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}.
$$

Since at most c_N among the U_i's overlap, we get

$$\|S(\lambda)f\|_{L^p(\Omega)} \leq c_N C\varepsilon \|f\|_{L^p(\Omega)} + \sum_{i=0}^{m} \frac{C_i}{|\lambda|^{1/2}} \|f\|_{L^p(\Omega_i)}.$$

Now, choose $\varepsilon > 0$ sufficiently small and $|\lambda|$ large enough to get $\|S(\lambda)\| \leq 1/2$. This shows that there exists $\omega_p^1 > 0$ such that for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \omega_p^1$, $I + S(\lambda) : L^p(\Omega) \to L^p(\Omega)$ is invertible with inverse $V(\lambda)$ satisfying $\|V(\lambda)\| \leq 2$. By (4.8), with $V(\lambda)f$ instead of f, we derive that $u = R(\lambda)V(\lambda)f$ belongs to $D_p(L)$ and solves the equation $\lambda u + Lu = f$. It further follows that

$$\|u\|_{L^p(\Omega)} \leq \sum_{i=0}^{m} \|R_i(\lambda)V(\lambda)f\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|}\|V(\lambda)f\|_{L^p(\Omega)} \leq \frac{2C}{|\lambda|}\|f\|_{L^p(\Omega)}. \tag{4.9}$$

Step 2. Using the results and the notation of the first step, for any $u \in D_p(L)$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > \max\{0, \omega_p^1\}$ we can write

$$\begin{aligned} R_i(\lambda)(\lambda + L)u &= \eta_i^2 u + F_i u + G_i u, \quad i \geq 1, \\ R_0(\lambda)(\lambda + L)u &= \eta_0^2 u + Hu \end{aligned}$$

where

$$\begin{aligned} F_i u &= T_i^{-1}\Big(T_i(\eta_i)(\lambda + \mathcal{L}_i^\circ)^{-1}(\mathcal{L}_i - \mathcal{L}_i^\circ)T_i(\eta_i u)\Big) \\ G_i u &= T_i^{-1}\Big(T_i(\eta_i)(\lambda + \mathcal{L}_i^\circ)^{-1}T_i\big([\eta_i, L]u\big)\Big), \\ Hu &= \eta_0(\lambda + L_0)^{-1}([L, \eta_0]u), \end{aligned}$$

and L_0 denotes the realization of L in $L^p(U_0)$ with Dirichlet boundary conditions. Summing over i, it turns out that

$$\sum_{i=0}^{m} R_i(\lambda)(\lambda + L)u = u + \sum_{i=1}^{m}(F_i u + G_i u) + Hu.$$

Let $u \in D_p(L)$ be such that $(\lambda + L)u = 0$. The above identity yields

$$u = -\sum_{i=1}^{m}\big(F_i u + G_i u\big) - Hu. \tag{4.10}$$

We claim that $u = 0$. To prove this, we need to estimate the norms of u in $D_p(L)$ and in $L^p(\Omega)$. To shorten the notation we set

$$\begin{aligned} \|\cdot\|_{p,i} &= \|\cdot\|_{L^p(\Omega_i)} \\ \|\cdot\|_{D_{p,i}} &= \|\cdot\|_{W^{1,p}(\Omega_i)} + \|\varrho D^2(\cdot)\|_{p,i} + \|\mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2(\cdot))\|_{p,i} + \|\sqrt{\varrho}\, D^2(\cdot)\,\mathbf{a}\|_{p,i} \end{aligned}$$

As Hu is supported in U_0, its norm in $D_p(L)$ is equivalent to the $W^{2,p}$-norm, therefore the classical L^p estimates yield

$$\|Hu\|_{D_p(L)} \leq C_0\|[L, \eta_0]u\|_{p,0}.$$

Since $[L,\eta_0]$ is a first-order operator, for every $\delta>0$ there exists $C_\delta>0$ such that

$$\|Hu\|_{D_p(L)} \le C_0\delta\|u\|_{D_{p,0}} + C_\delta\|u\|_{p,0}. \tag{4.11}$$

On the other hand

$$\|Hu\|_{L^p(\Omega)} \le \frac{C_0}{|\lambda|}\|u\|_{D_{p,0}}. \tag{4.12}$$

Here, C_0 denotes a suitable constant depending on η_0. Let us estimate F_iu and G_iu for every $i\ge 1$. Set

$$f_i = (\mathcal{L}_i - \mathcal{L}_i^\circ)T_i(\eta_i u), \quad g_i = T_i([\eta_i, L]u)$$

and

$$\varphi_i = (\lambda+\mathcal{L}_i^\circ)^{-1}f_i, \quad \psi_i = (\lambda+\mathcal{L}_i^\circ)^{-1}g_i.$$

We have

$$\begin{aligned}\|F_iu\|_{D_p(L)} &\le C\|T_i(\eta_i)\varphi_i\|_{D_p} \\ &\le C\|\varphi_i\|_{D_p} + C_i\Big(\|\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\partial_y\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)}\Big),\end{aligned} \tag{4.13}$$

where C_i depends on $\|\nabla\eta_i\|_\infty, \|D^2\eta_i\|_\infty$ and Ω. Theorem 2.15, Remark 3.2 and (3.12) further imply

$$\|\varphi_i\|_{D_p} \le C\|f_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le C\varepsilon\|\eta_i u\|_{D_{p,i}}$$

$$\|\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le \frac{C}{|\lambda|}\|f_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le \frac{C\varepsilon}{|\lambda|}\|\eta_i u\|_{D_{p,i}}.$$

Similarly, for $\mathrm{Re}\,\lambda>\Lambda_p$ we derive

$$\|\partial_y\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)} + \|\sqrt{y}\,\nabla_x\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le \frac{C}{|\lambda|^{1/2}}\|f_i\|_p \le \frac{C\varepsilon}{|\lambda|^{1/2}}\|\eta_i u\|_{D_{p,i}}. \tag{4.14}$$

Using

$$\|\eta_i u\|_{D_{p,i}} \le \|u\|_{D_{p,i}} + C_i(\|u\|_{p,i} + \|\nabla u\|_{p,i}),$$

we arrive at

$$\|F_iu\|_{D_p(L)} \le \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|u\|_{D_{p,i}} + C_i(\|u\|_{p,i} + \|\nabla u\|_{p,i}). \tag{4.15}$$

For the L^p norm of F_iu we further obtain the better estimate

$$\|F_iu\|_{L^p(\Omega)} \le C\|\varphi_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le \frac{C}{|\lambda|}\|f_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le \frac{C_i}{|\lambda|}\|u\|_{D_{p,i}}. \tag{4.16}$$

The estimates for G_iu are similar. Replacing φ_i, f_i with ψ_i, g_i, respectively, in (4.13), (4.14) and observing that

$$\|g_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le C_i(\|u\|_{p,i} + \|\nabla u\|_{p,i}),$$

we infer

$$\|G_iu\|_{D_p(L)} \le C_i(\|u\|_{p,i} + \|\nabla u\|_{p,i}), \tag{4.17}$$

and

$$\|G_iu\|_{L^p(\Omega)} \le \frac{C}{|\lambda|}\|g_i\|_{L^p(\mathbb{R}^{N+1}_+)} \le \frac{C_i}{|\lambda|}\|u\|_{D_{p,i}}. \tag{4.18}$$

Formulae (4.10), (4.11), (4.15) and (4.17) now yield

$$\|u\|_{D_p(L)} \leq \sum_{i=1}^{m} \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}} \right) \|u\|_{D_{p,i}} + \sum_{i=1}^{m} C_i(\|u\|_{p,i} + \|\nabla u\|_{p,i})$$
$$+ C_0 \delta \|u\|_{D_{p,0}} + C_\delta \|u\|_{p,0}.$$

At this point, as in the last part of the first step, we take sufficiently small $\varepsilon, \delta > 0$ and sufficiently large $|\lambda|$ to conclude

$$\|u\|_{D_p(L)} \leq C(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}).$$

The interpolative estimate (4.1) further implies

$$\|u\|_{D_p(L)} \leq C\|u\|_{L^p(\Omega)}.$$

Moreover, from (4.10), (4.12), (4.16) and (4.18) it follows that

$$\|u\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|} \|u\|_{D_p(L)}.$$

Combining the last two estimates we obtain

$$\|u\|_{D_p(L)} \leq \frac{C}{|\lambda|} \|u\|_{D_p(L)}.$$

If $|\lambda|$ is large enough, u must be 0. Therefore, there exists $\omega_p \geq \omega_p^1$ such that $\lambda + L : D_p(L) \to L^p(\Omega)$ is injective for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\, \lambda \geq \omega_p$. Taking into account the first step and (4.9) we have proved that $\lambda + L$ is bijective from $D_p(L)$ onto $L^p(\Omega)$ with $\|\lambda(\lambda + L)^{-1}\| \leq C$ for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq \omega_p$. □

We now discuss further properties of the generator $(-L, D_p(L)$ and its semigroup $(T_p(t))_{t \geq 0}$, see also Corollary 5.4. Taking $\varepsilon = |\lambda|^{-1/4}$ in Lemma 4.2, we first deduce the following estimate from the sectoriality of L.

Corollary 4.3 *Assume that (H1), (H2) and (H3) hold and that $p \in (1, \infty)$. There exist $C, \gamma_p > 0$ such that for every $\mathrm{Re}\, \lambda \geq \gamma_p$ and $u \in D_p(L)$ we have*

$$\|\nabla u\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|^{1/4}} \|\lambda u + Lu\|_{L^p(\Omega)}.$$

Corollary 4.4 *Assume that (H1), (H2) and (H3) hold and that $1 < p < q < +\infty$. Then the following assertions hold.*

(i) We have $T_p(t)f = T_q(t)f$ for every $f \in L^q(\Omega)$ and $t \geq 0$. Therefore, we simply write $T(t)$ instead of $T_p(t)$.

(ii) $T(t)$ is compact for $t > 0$ and the spectra and the eigenspaces of $(L, D_p(L))$ and $(L, D_q(L))$ coincide.

(iii) $T(t)$ is positive for $t \geq 0$.

Proof. The consistency of the semigroups $(T_p(t))_{t\geq 0}$ and $(T_q(t))_{t\geq 0}$ follows from the consistency of the corresponding resolvents which is an immediate consequence of the inclusion $D_q(L) \subset D_p(L)$. The resolvent is compact since $D_p(L) \hookrightarrow W^{1,p}(\Omega)$ by Corollary 4.3 and Ω is bounded. The analyticity of $T(t)$ thus yields the compactness of the semigroup. In this situation it is known that the remaining assertions in (ii) are true, cf. [2, Proposition 2.6]. To prove (iii), it suffices to show that $u = (\lambda + L)^{-1} f \in D_p(L)$ is positive for all $\lambda \geq \omega_p$, $p > N + 1$ and positive $f \in C(\overline{\Omega})$. In this case u is continuous by Sobolev's embedding and it vanishes at the boundary. If there were a $z_0 \in \Omega$ with $u(z_0) < 0$, then u would have an interior minimum $u(z_1) < 0$. Hence, $Lu(z_1) = f(z_1) - \lambda u(z_1) > 0$. But this inequality contradicts Bony's maximum principle, [4, Theorem 1], and so $u \geq 0$ as needed. □

Corollary 4.5 *Let (H1), (H2) and (H3) hold and that $p \in (1, \infty)$. If the coefficients of L are $C_b^2(\Omega)$, then $L + \omega_p'$ is accretive on $D_p(L)$ for some $\omega_p' \geq 0$. Moreover, $(-L, D_p(L))$ has maximal regularity of type L^q.*

Proof. We rewrite L in divergence form obtaining first order coefficients with bounded derivatives. The accretivity of the shifted operator then follows easily. As in Corollary 2.14, the second assertion is then a consequence of the results in [11]. □

5 Generation in spaces of continuous functions on a bounded domain

In this section we shall prove that the operator $-L$ defined in (3.2) and endowed with the domain

$$D_0(L) = \left\{ u \in C(\overline{\Omega}) \cap \bigcap_{1\leq p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega) \,\middle|\, \mathbf{a}\cdot\nabla u,\ \sqrt{\varrho}\,\nabla u,\ Lu \in C(\overline{\Omega}),\ u_{|\partial\Omega} = 0 \right\},$$

generates an analytic semigroup in $C(\overline{\Omega})$. The main ingredients will be the localization procedure already implemented in the previous section and a suitable adaptation of the Masuda-Stewart method to the model operator in the halfspace.
Let $\overline{z} = (\overline{x}, \overline{y}) \in \mathbb{R}^{N+1}_+$ and $r, s, \kappa > 0$. Let us introduce the cubes

$$\mathcal{C}(\overline{x}) = \prod_{i=1}^{N} [\overline{x}_i, \overline{x}_i + r], \qquad \mathcal{C}_\kappa(\overline{x}) = \prod_{i=1}^{N} \left[\overline{x}_i - \frac{r}{2}\kappa, \overline{x}_i + \left(\frac{\kappa}{2} + 1\right) r\right]$$

and the parallelepipeds

$$\mathcal{Q}(\overline{z}) = \mathcal{C}(\overline{x}) \times [\overline{y}, \overline{y} + s], \qquad \mathcal{Q}_\kappa(\overline{z}) = \mathcal{C}_\kappa(\overline{x}) \times \left[\overline{y} - \frac{s}{2}\kappa, \overline{y} + \left(\frac{\kappa}{2} + 1\right) s\right]. \tag{5.1}$$

Set $\mathcal{Q}^+_\kappa(\overline{z}) = \mathcal{Q}_\kappa(\overline{z}) \cap \mathbb{R}^{N+1}_+$. We start with a lemma collecting all the relevant properties of weighted spaces we need in the sequel, relying on Grisvard's paper [10]. Notice that in [10] the weighted spaces involved are slightly different from ours, but we shall show that we may use Grisvard's results. We fix a parallelepiped $Q = \mathcal{Q}(\overline{z})$ with $\overline{z} = (\overline{x}, 0)$ and side lengths r, s, set $C = \mathcal{C}(\overline{x})$ and, following the notation in [10], we introduce the weighted spaces

$$W^{1,p}_{p/2}(Q) = \{u \in W^{1,p}_{\mathrm{loc}}(Q) :\ \sqrt{y}u, \sqrt{y}\nabla u \in L^p(Q)\},$$

$$\overset{\circ}{W}{}^{1,p}_{p/2}(Q) = \{u \in W^{1,p}_{p/2}(Q) : \gamma u = 0\},$$

endowed with the obvious norm, where γ is the trace operator defined according to Lemma 5.1(ii) below.

Lemma 5.1 *Let $p > 2$ and $Q = \mathcal{Q}(\overline{z})$ be a parallelepiped with $\overline{z} = (\overline{x}, 0)$ and side lengths $r, s > 0$. The following statements hold:*

(i) the space $C^\infty(\overline{Q})$ is dense in $W^{1,p}_{p/2}(Q)$;

(ii) the trace operator $\gamma : W^{1,p}_{p/2}(Q) \to L^p(C)$ is well-defined and continuous;

(iii) the following Hardy-type inequality holds in $\overset{\circ}{W}{}^{1,p}_{p/2}(Q)$:

$$\left\| \frac{w}{\sqrt{y}} \right\|_{L^p(Q)} \leq \frac{2p}{p-2} \|\sqrt{y}\, \partial_y w\|_{L^p(Q)};$$

(iv)

$$W^{1,p}_{p/2}(Q) = \{u \in W^{1,p}_{\text{loc}}(Q) : \; u, \sqrt{y}\nabla u \in L^p(Q)\}.$$

Proof. Observe that (i) follows from Théorème 1.4, (ii) from Propositions 1.1' and 1.2 and (iii) from Théorème 1.2 in [10]. Concerning (iv), we have only to show that if $u \in W^{1,p}_{p/2}(Q)$, then u belongs to $L^p(Q)$ or, using (i), that there exists $C > 0$ such that for every $u \in C^\infty(\overline{Q})$

$$\|u\|_{L^p(Q)} \leq C \|u\|_{W^{1,p}_{p/2}(Q)}.$$

Splitting $u = u_1 + u_2$ with u_1, u_2 vanishing for y close to $0, s$, respectively, and noticing that the assertion is trivial for u_1, we may confine to functions $u \in C^\infty(\overline{Q})$ vanishing for $y = s$. Hence

$$u(x, y) = \int_s^y \partial_y u(x, \tau) d\tau = \int_s^y \partial_y u(x, \tau) \tau^{-1/2} \tau^{1/2} d\tau$$

and using Hölder's inequality

$$|u(x, y)|^p \leq \int_0^s |\partial_y u(x, \tau)|^p \tau^{p/2} d\tau \left(\int_0^s \tau^{-p'/2} d\tau \right)^{p-1}.$$

Integrating with respect to x we obtain $\|u\|_{L^p(Q)} \leq C_s \|\sqrt{y} \partial_y u\|_{L^p(Q)}$. □

Lemma 5.2 *Let $p > 2(N+1)$ and $\varphi \in W^{1,p}_{p/2}(\mathcal{Q}(\overline{z}))$ where $\overline{z} = (\overline{x}, 0)$ and $\mathcal{Q}(\overline{z})$ with side lengths $r, s > 0$. Then $\varphi \in C(\mathcal{Q}(\overline{z}))$ and there is $C_{r,s} > 0$ such that*

$$\|\varphi\|_{L^\infty(\mathcal{Q}(\overline{z}))} \leq C_{r,s} \Big(\|\varphi\|_{L^p(\mathcal{Q}(\overline{z}))} + \|\sqrt{y}\, \nabla \varphi\|_{L^p(\mathcal{Q}(\overline{z})))} \Big).$$

Moreover, there is $C > 0$ such that

$$\|\varphi\|_{L^\infty(\mathcal{Q}(\overline{z}))} \leq C\, r^{-\frac{N}{p}} s^{-\frac{1}{p}} \Big(\|\varphi\|_{L^p(\mathcal{Q}(\overline{z}))} + s \|\partial_y \varphi\|_{L^p(\mathcal{Q}(\overline{z}))} + \frac{r}{\sqrt{s}} \|\sqrt{y}\, \nabla_x \varphi\|_{L^p(\mathcal{Q}(\overline{z}))} \Big), \quad (5.2)$$

if $\partial_y \varphi$ is p-summable-

Proof. First we prove that there exists $C > 0$ such that for any $\varphi \in C^1(\mathcal{Q}_1)$

$$|\varphi(0,0)| \le C\big(\|\varphi\|_{L^p(\mathcal{Q}_1)} + \|\sqrt{y}\,\nabla\varphi\|_{L^p(\mathcal{Q}_1)}\big), \tag{5.3}$$

where $\mathcal{Q}_1$ denotes the unit cube $[0,1]^{N+1}$. Integrating the identity

$$\varphi(x,y) - \varphi(0,0) = \int_0^1 \nabla\varphi(tx,ty)\cdot(x,y)dt$$

over $\mathcal{Q}_1$, we have

$$\begin{aligned}\Big|\iint_{\mathcal{Q}_1}\varphi(x,y)dxdy - \varphi(0,0)\Big| &\le \sqrt{N+1}\int_0^1\iint_{\mathcal{Q}_1}|\nabla\varphi(tx,ty)|dxdy\,dt\\ &= \sqrt{N+1}\int_0^1 t^{-N-1}\iint_{t\mathcal{Q}_1}|\nabla\varphi(\xi,\eta)|d\xi d\eta\,dt\\ &\le \sqrt{N+1}\Big(\iint_{\mathcal{Q}_1}|\sqrt{\eta}\,\nabla\varphi(\xi,\eta)|^p d\xi d\eta\Big)^{\frac{1}{p}}\int_0^1 t^{-N-1}\Big(\iint_{t\mathcal{Q}_1}\frac{1}{\eta^{q/2}}d\xi d\eta\Big)^{\frac{1}{q}}dt\\ &\le C\|\sqrt{y}\,\nabla\varphi\|_{L^p(\mathcal{Q}_1)}\end{aligned}$$

since $p > 2(N+1)$, where $\frac{1}{p}+\frac{1}{q}=1$. Therefore (5.3) follows. By a standard shifting and rescaling argument estimate (5.3) takes the following form

$$|\varphi(x_0,y_0)| \le C\sigma^{-\frac{N+1}{p}}\Big(\|\varphi\|_{L^p(\mathcal{Q}_\sigma(x_0,y_0))} + \sqrt{\sigma}\,\|\sqrt{y}\,\nabla\varphi\|_{L^p(\mathcal{Q}_\sigma(x_0,y_0))}\Big) \tag{5.4}$$

in the cube $\mathcal{Q}_\sigma(x_0,y_0) = (x_0,y_0)+\sigma\mathcal{Q}_1$ for any $(x_0,y_0)\in\mathbb{R}^{N+1}_+$. Of course, on the left hand side of (5.4) we may write the values of the function φ in the other vertices of $Q_\sigma(x_0,y_0)$, keeping the right hand side unchanged.

We next divide $\mathcal{Q}_1$ in 2^{N+1} cubes with side length $\frac{1}{2}$ and let $\mathcal{Q}^i$ be any of these cubes. Therefore every $(x,y)\in\mathcal{Q}^i$ is the vertex of a cube $\mathcal{Q}^*$ of side length $\frac{1}{2}$ contained in $\mathcal{Q}_1$. Applying estimate (5.4) in $\mathcal{Q}^*$ we obtain

$$|\varphi(x,y)| \le C\Big(\|\varphi\|_{L^p(\mathcal{Q}_1)} + \|\sqrt{y}\,\nabla\varphi\|_{L^p(\mathcal{Q}_1)}\Big).$$

Since (x,y) and $\mathcal{Q}^i$ are arbitrary, we have

$$\|\varphi\|_{L^\infty(\mathcal{Q}_1)} \le C\Big(\|\varphi\|_{L^p(\mathcal{Q}_1)} + \|\sqrt{y}\,\nabla\varphi\|_{L^p(\mathcal{Q}_1)}\Big)$$

for $\varphi\in C^1(\mathcal{Q}_1)$ and, using Lemma 5.1(i), (iv), for every $\varphi \in W^{1,p}_{p/2}(\mathcal{Q}_1)$. Hence

$$\begin{aligned}\|\varphi\|_{L^\infty(\mathcal{Q}_1)} &\le C\Big(\|\varphi\|_{L^p(\mathcal{Q}_1)} + \|\sqrt{y}\,\nabla\varphi\|_{L^p(\mathcal{Q}_1)}\Big)\\ &\le C\Big(\|\varphi\|_{L^p(\mathcal{Q}_1)} + \|\partial_y\varphi\|_{L^p(\mathcal{Q}_1)} + \|\sqrt{y}\,\nabla_x\varphi\|_{L^p(\mathcal{Q}_1)}\Big)\end{aligned}$$

if $\varphi\in W^{1,p}_{p/2}(\mathcal{Q}_1)$. Estimate (5.2) then follows by shifting and rescaling the cube $\mathcal{Q}_1$. □

We are ready to state and prove the main result of the section.

Theorem 5.3 *Assume that (H1), (H2) and (H3) hold. Then the operator $(-L, D_0(L))$ generates an analytic semigroup $T(\cdot)$ in $C(\overline{\Omega})$. It further holds*

$$\|\mathbf{a}\cdot\nabla u\|_{L^\infty(\Omega)} + \|\sqrt{\varrho}\,\nabla u\|_{L^\infty(\Omega)} \le C|\lambda|^{-\frac{1}{2}}\|f\|_{L^\infty(\Omega)} \tag{5.5}$$

for every $Re\lambda > \omega_0$, $f \in C(\overline{\Omega})$, $u = (\lambda+L)^{-1}f$, and some $\omega_0 \ge 0$. Moreover this semigroup is contractive, positive, compact, exponentially stable, and it is the restriction of the semigroups on $L^p(\Omega)$ obtained in Theorem 4.1.

Proof. Let $\{U_1,\dots,U_m\}$ be a covering of $\partial\Omega$ satisfying (3.12) with $\varepsilon > 0$ to be chosen. Let $U_0 \subset\subset \Omega$ be an open set with boundary of class C^2 such that $\{U_0, U_1,\dots,U_m\}$ is a covering of $\overline{\Omega}$. Finally, let $\{\eta_i\}_{i=0,\dots,m}$ be a partition of unity corresponding to this covering.

Take $f \in C(\overline{\Omega})$. Fix $p > 3N+2$ and choose $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \ge \omega_p$, where ω_p is given by Theorem 4.1. Let u be the unique solution in $D_p(L)$ of the equation $\lambda u + Lu = f$. By straightforward computations one can check that $u_i := \eta_i u$ solves the equation

$$\lambda u_i + Lu_i = \eta_i f - h_i \tag{5.6}$$

with

$$\begin{aligned} h_i = {}& \mathrm{tr}(\mathbf{a}\otimes\mathbf{a}\,D^2\eta_i)u + \varrho\Big(\sum_{j,k=1}^{N+1} a_{jk}\partial_{jk}\eta_i\Big)u + \sum_{k=1}^{N+1} b_k\partial_k\eta_i u \\ & + 2\big(\mathbf{a}\cdot\nabla\eta_i\big)\big(\mathbf{a}\cdot\nabla u\big) + 2\varrho\sum_{j,k=1}^{N+1} a_{jk}\partial_j\eta_i\partial_k u. \end{aligned} \tag{5.7}$$

Let us first deal with the case $i = 0$. Since L is nondegenerate in U_0, Theorem 3.1.19 in [15] gives constants $K_p, \lambda_p > 0$ such that

$$|\lambda|\|u_0\|_{L^\infty(U_0)} + |\lambda|^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(U_0)} \le K_p|\lambda|^{\frac{N+1}{2p}} \sup_{\xi\in\overline{U}_0}\|\eta_0 f - h_0\|_{L^p(B_\xi)}$$

if $\mathrm{Re}\lambda \ge \lambda_p$, where $B_\xi = U_0 \cap B(\xi, |\lambda|^{-\frac{1}{2}})$. Using also $\|\eta_0 f\|_{L^p(B_\xi)} \le \|f\|_{L^p(B(\xi,|\lambda|^{-\frac{1}{2}})} \le |\lambda|^{-\frac{N+1}{2p}}\|f\|_{L^\infty(\Omega)}$, we derive

$$\begin{aligned} |\lambda|\|u_0\|_{L^\infty(U_0)} + |\lambda|^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(U_0)} &\le K_p\|f\|_{L^\infty(\Omega)} + K_p|\lambda|^{\frac{N+1}{2p}}\|h_0\|_{L^p(\Omega)} \\ &\le K_p\|f\|_{L^\infty(\Omega)} + C|\lambda|^{\frac{N+1}{2p}}(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}) \\ &\le K_p\|f\|_{L^\infty(\Omega)} + C|\lambda|^{\frac{N+1}{2p}}(|\lambda|^{-1}\|f\|_{L^p(\Omega)} + |\lambda|^{-\frac{1}{4}}\|f\|_{L^p(\Omega)}) \end{aligned}$$

from Theorem 4.1 and Corollary 4.3. Choosing $|\lambda| \ge 1$ and estimating $\|f\|_{L^p(\Omega)}$ by $\|f\|_{L^\infty(\Omega)}$, we are led to

$$|\lambda|\|u_0\|_{L^\infty(U_0)} + |\lambda|^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(U_0)} \le C\|f\|_{L^\infty(\Omega)}. \tag{5.8}$$

Let $i \ge 1$ and set $w_i = T_i(u_i)$, T_i being the operator defined in (4.2). Then $w_i \in D_p^\circ$ and $\mathrm{supp}\, w_i \subset \widetilde{U}_i \cap \mathbb{R}_+^{N+1}$. Lemma 5.2 implies that $w_i, \partial_y w_i, \sqrt{y}\,\nabla_x w_i \in C(\mathbb{R}_+^{N+1})$. Moreover, (5.6) is transformed into

$$\lambda w_i + \mathcal{L}_i w_i = T_i(\eta_i f) - T_i(h_i),$$

where the transformed operator $\mathcal{L}_i$ is given in formula (3.9) adapted to the local chart (U_i, J_i). Therefore

$$\lambda w_i + \mathcal{L}_i^\circ w_i = (\mathcal{L}_i^\circ - \mathcal{L}_i) w_i + T_i(\eta_i f) - T_i(h_i), \tag{5.9}$$

where $\mathcal{L}_i^\circ$ is the operator obtained by freezing the coefficients of $\mathcal{L}_i$ according to (3.11).

Let $\overline{z} \in \mathbb{R}_+^{N+1}$ and consider the parallelepipeds introduced in (5.1), with r, s, κ to be chosen below. Take a smooth cutoff function θ such that $\theta = 1$ on $\mathcal{Q}(\overline{z})$, $\theta = 0$ on $\mathbb{R}^{N+1} \setminus \mathcal{Q}_\kappa(\overline{z})$, $0 \le \theta \le 1$ and

$$\|\partial_y \theta\|_\infty \le \frac{C}{\kappa s}, \quad \|\partial_y^2 \theta\|_\infty \le \frac{C}{\kappa^2 s^2}, \quad \|\nabla_x \theta\|_\infty \le \frac{C}{\kappa r}, \quad \|D_x^2 \theta\|_\infty \le \frac{C}{\kappa^2 r^2}$$

for a constant $C > 0$ independent of $\overline{z}$ and r, s, κ. From now on, for the sake of simplicity, we write $\mathcal{Q}$ and $\mathcal{Q}_\kappa$ instead of $\mathcal{Q}(\overline{z})$ and $\mathcal{Q}_\kappa(\overline{z})$, respectively. Set $v_i = \theta w_i$. It is easily seen that $v_i \in D_p^\circ$ and solves the equation

$$\lambda v_i + \mathcal{L}_i^\circ v_i = \theta(\lambda w_i + \mathcal{L}_i^\circ w_i) - g_i$$

where

$$\begin{aligned} g_i = \alpha^i(z_i)\, w_i\, \partial_y^2 \theta + y\, w_i \sum_{h,k=1}^{N} \alpha_{hk}^i(z_i) \partial_{x_h x_k} \theta + w_i \sum_{k=1}^{N+1} \gamma_k^i(z_i) \partial_k \theta \\ + 2\alpha^i(z_i) \partial_y \theta \partial_y w_i + 2y \sum_{h,k=1}^{N} \alpha_{hk}^i(z_i) \partial_{x_h} \theta \partial_{x_k} w_i. \end{aligned}$$

If $\mathrm{Re}\lambda \ge \Lambda_p$, we can apply the estimates of Theorem 2.15 to v_i (recalling Remark 3.2) and obtain

$$\begin{aligned} |\lambda| \|w_i\|_{L^p(\mathcal{Q})} &+ |\lambda|^{\frac{1}{2}} \Big(\|\partial_y w_i\|_{L^p(\mathcal{Q})} + \|\sqrt{y}\, \nabla_x w_i\|_{L^p(\mathcal{Q})} \Big) + |\lambda|^{\frac{1}{4}} \|\nabla w_i\|_{L^p(\mathcal{Q})} \\ &+ \|\partial_y^2 w_i\|_{L^p(\mathcal{Q})} + \|y D_x^2 w_i\|_{L^p(\mathcal{Q})} + \|\sqrt{y}\, \nabla_x \partial_y w_i\|_{L^p(\mathcal{Q})} \\ \le C \Bigg(& \|\lambda w_i + \mathcal{L}_i^\circ w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa^2 s^2} \|w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa^2 r^2} \|y\, w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa r} \|w_i\|_{L^p(\mathcal{Q}_\kappa^+)} \\ &+ \frac{1}{\kappa s} \|w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa s} \|\partial_y w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa r} \|y \nabla_x w_i\|_{L^p(\mathcal{Q}_\kappa^+)} \Bigg) \\ \le C & \|\lambda w_i + \mathcal{L}_i^\circ w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + C(\kappa+1)^{\frac{N+1}{p}} r^{\frac{N}{p}} s^{\frac{1}{p}} \Bigg(\frac{1}{\kappa^2 s^2} \|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa^2 r^2} \|y\, w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} \\ &+ \frac{1}{\kappa r} \|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa s} \|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa s} \|\partial_y w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} + \frac{1}{\kappa r} \|y \nabla_x w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} \Bigg). \end{aligned} \tag{5.10}$$

Let $|\lambda|, \kappa \ge 1$. We consider the subsets of $\mathbb{R}_+^{N+1}$ given by

$$\mathbf{A} = \{(x,0) : x \in \mathbb{R}^N\} \quad \text{and} \quad \mathbf{B} = \{(x,y) : x \in \mathbb{R}^N, y \ge |\lambda|^{-\frac{1}{2}}\}.$$

If $\overline{z} \in \mathbf{A}$, we choose

$$s = |\lambda|^{-\frac{1}{2}}, \qquad r = |\lambda|^{-\frac{3}{4}}.$$

Notice that the previous choice implies $r = s^{\frac{3}{2}}$, according to the characteristics of the Tricomi equation in two variables. Since $w_i, \partial_y w_i, \sqrt{y}\, \nabla_x w_i$ belong to $W_{p/2}^{1,p}(Q)$, we can use Lemma 5.2 to estimate

$$|\lambda|\, \|w_i\|_{L^\infty(\mathcal{Q})} + |\lambda|^{\frac{1}{2}} \big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q})} + \|\sqrt{y}\, \nabla_x w_i\|_{L^\infty(\mathcal{Q})} \big) \tag{5.11}$$

$$\le Cr^{-\frac{N}{p}}s^{-\frac{1}{p}}\bigg(|\lambda|\|w_i\|_{L^p(\mathcal{Q})}+|\lambda|^{\frac{1}{2}}\Big(\|\partial_y w_i\|_{L^p(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x w_i\|_{L^p(\mathcal{Q})}\Big)$$
$$+\|\partial_y^2 w_i\|_{L^p(\mathcal{Q})}+\|yD_x^2 w_i\|_{L^p(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\mathcal{Q})}+\|y^{-\frac{1}{2}}\nabla_x w_i\|_{L^p(\mathcal{Q})}\bigg).$$

We have to estimate the last term in the inequality above. Since $w_i\in D_p^\circ$, by Lemma 2.3 there are $w_i^n\in\mathcal{D}$ such that $w_i^n\to w_i$ in D_p. In particular $\nabla w_i^n\to\nabla w_i$ and $\sqrt{y}\nabla_x\partial_y w_i^n\to\sqrt{y}\nabla_x\partial_y w_i$ in $L^p(\mathcal{Q})$ and pointwise. Since $\nabla_x w_i^n\in\mathcal{D}$, we may apply Lemma 5.1(iii) to get

$$\left\|\frac{\nabla_x w_i^n}{\sqrt{y}}\right\|_{L^p(\mathcal{Q})}\le\frac{2p}{p-2}\|\sqrt{y}\,\nabla_x\partial_y w_i^n\|_{L^p(\mathcal{Q})}.$$

Letting $n\to\infty$ and using Fatou's lemma on the left hand side we see that the above estimate holds for w_i. Combining (5.11) with (5.10), we thus find

$$|\lambda|\,\|w_i\|_{L^\infty(\mathcal{Q})}+|\lambda|^{\frac{1}{2}}\big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathcal{Q})}\big)+$$
$$+r^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|\partial_y^2 w_i\|_{L^p(\mathcal{Q})}+\|yD_x^2 w_i\|_{L^p(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\mathcal{Q})}\Big)$$
$$\le C\,r^{-\frac{N}{p}}s^{-\frac{1}{p}}\|\lambda w_i+\mathcal{L}_i^\circ w_i\|_{L^p(\mathcal{Q}_\kappa^+)}+C(\kappa+1)^{\frac{N+1}{p}}\bigg(\frac{1}{\kappa^2 s^2}\|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}+\frac{1}{\kappa^2 r^2}\|y\,w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}$$
$$+\frac{1}{\kappa r}\|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}+\frac{1}{\kappa s}\|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}+\frac{1}{\kappa s}\|\partial_y w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}+\frac{1}{\kappa r}\|y\nabla_x w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}\bigg).$$

Since $y\le(\frac{\kappa}{2}+1)|\lambda|^{-\frac{1}{2}}$ in $\mathcal{Q}_\kappa^+$ and $|\lambda|,\kappa\ge 1$, we arrive at

$$\begin{aligned}&|\lambda|\,\|w_i\|_{L^\infty(\mathcal{Q})}+|\lambda|^{\frac{1}{2}}\big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathcal{Q})}\big)\\&\quad+r^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|\partial_y^2 w_i\|_{L^p(\mathcal{Q})}+\|yD_x^2 w_i\|_{L^p(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\mathcal{Q})}\Big)\\&\le C\,r^{-\frac{N}{p}}s^{-\frac{1}{p}}\|\lambda w_i+\mathcal{L}_i^\circ w_i\|_{L^p(\mathcal{Q}_\kappa^+)}\\&\quad+C(\kappa+1)^{\frac{N+1}{p}}\bigg(\frac{|\lambda|}{\kappa}\|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}+\frac{|\lambda|^{\frac{1}{2}}}{\sqrt{\kappa}}\big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}+\|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}\big)\bigg).\end{aligned}\tag{5.12}$$

If $\overline{z}\in\mathbf{B}$, we choose

$$s=|\lambda|^{-\frac{1}{2}},\qquad r=|\lambda|^{-\frac{1}{2}}\overline{y}^{\frac{1}{2}}.$$

The classical Sobolev embedding yields

$$\|\phi\|_{L^\infty(\mathcal{Q})}\le Cr^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|\phi\|_{L^p(\mathcal{Q})}+s\|\partial_y\phi\|_{L^p(\mathcal{Q})}+r\|\nabla_x\phi\|_{L^p(\mathcal{Q})}\Big)$$

for any $\phi\in W^{1,p}(\mathcal{Q})$. Recalling that $y\ge\overline{y}$ in $\mathcal{Q}$ and the choice of r, we infer

$$\|\phi\|_{L^\infty(\mathcal{Q})}\le Cr^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|\phi\|_{L^p(\mathcal{Q})}+s\|\partial_y\phi\|_{L^p(\mathcal{Q})}+|\lambda|^{-\frac{1}{2}}\|\sqrt{y}\,\nabla_x\phi\|_{L^p(\mathcal{Q})}\Big).$$

We apply these estimates to $w_i,\partial_y w_i$ and $\sqrt{y}\,\partial_{x_k}w_i$, $k=1,\dots,N$ and obtain

$$|\lambda|\,\|w_i\|_{L^\infty(\mathcal{Q})}+|\lambda|^{\frac{1}{2}}\big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathcal{Q})}\big)$$
$$\le Cr^{-\frac{N}{p}}s^{-\frac{1}{p}}\bigg(|\lambda|\|w_i\|_{L^p(\mathcal{Q})}+|\lambda|^{\frac{1}{2}}\Big(\|\partial_y w_i\|_{L^p(\mathcal{Q})}+\|\sqrt{y}\,\nabla_x w_i\|_{L^p(\mathcal{Q})}\Big)$$

$$+\|\partial_y^2 w_i\|_{L^p(\mathcal{Q})} + \|yD_x^2 w_i\|_{L^p(\mathcal{Q})} + \|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\mathcal{Q})} + \|y^{-\frac{1}{2}}\nabla_x w_i\|_{L^p(\mathcal{Q})}\Big).$$

Here, the last term can be absorbed since

$$\|y^{-\frac{1}{2}}\nabla_x w_i\|_{L^p(\mathcal{Q})} \le |\lambda|^{\frac{1}{4}}\|\nabla_x w_i\|_{L^p(\mathcal{Q})} \le |\lambda|^{\frac{1}{2}}\|\sqrt{y}\,\nabla_x w_i\|_{L^p(\mathcal{Q})}$$

because of $y \ge \overline{y} \ge s = |\lambda|^{-\frac{1}{2}}$. Therefore we can continue as before. Noticing that $\overline{y} \ge s = |\lambda|^{-\frac{1}{2}}$ and $y \le \overline{y} + \frac{\kappa+2}{2}s \le \frac{\kappa+4}{2}\,\overline{y}$ in $\mathcal{Q}_\kappa^+$ we derive again (5.12). Now (5.9) and (5.12) lead to

$$\begin{aligned}
&|\lambda|\,\|w_i\|_{L^\infty(\mathcal{Q})} + |\lambda|^{\frac{1}{2}}\big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q})} + \|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathcal{Q})}\big)+\\
&\qquad + r^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|\partial_y^2 w_i\|_{L^p(\mathcal{Q})} + \|yD_x^2 w_i\|_{L^p(\mathcal{Q})} + \|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\mathcal{Q})}\Big)\\
&\quad \le C\,r^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|(\mathcal{L}_i^\circ - \mathcal{L}_i)w_i\|_{L^p(\mathcal{Q}_\kappa^+)} + \|T_i(h_i)\|_{L^p(\mathcal{Q}_\kappa^+)}\Big)\\
&\qquad + C(\kappa+1)^{\frac{N+1}{p}}\Bigg(\|T_i(\eta_i f)\|_{L^\infty(\mathcal{Q}_\kappa^+)} + \frac{|\lambda|}{\kappa}\|w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)} + \frac{|\lambda|^{\frac{1}{2}}}{\sqrt{\kappa}}\big(\|\partial_y w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}\\
&\qquad\qquad + \|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathcal{Q}_\kappa^+)}\big)\Bigg)
\end{aligned}\tag{5.13}$$

for all $\overline{z} \in \mathbf{A}\cup\mathbf{B}$. Taking in (5.13) the supremum over $\overline{z} \in \mathbf{A}\cup\mathbf{B}$ and fixing a sufficiently large $\kappa \ge 1$, we get

$$\begin{aligned}
&|\lambda|\,\|w_i\|_{L^\infty(\mathbb{R}_+^{N+1})} + |\lambda|^{\frac{1}{2}}\big(\|\partial_y w_i\|_{L^\infty(\mathbb{R}_+^{N+1})} + \|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathbb{R}_+^{N+1})}\big)+\\
&\qquad + r^{-\frac{N}{p}}s^{-\frac{1}{p}}\Big(\|\partial_y^2 w_i\|_{L^p(\mathbb{R}_+^{N+1})} + \|yD_x^2 w_i\|_{L^p(\mathbb{R}_+^{N+1})} + \|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\mathbb{R}_+^{N+1})}\Big)\\
&\quad \le C\Big(r^{-\frac{N}{p}}s^{-\frac{1}{p}}\|(\mathcal{L}_i^\circ - \mathcal{L}_i)w_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}_+^{N+1})} + |\lambda|^{\frac{3N+2}{4p}}\|T_i(h_i)\|_{L^p(\mathbb{R}_+^{N+1})} + \|f\|_{L^\infty(\Omega)}\Big).
\end{aligned}\tag{5.14}$$

Let us study the right hand side of (5.14). Recalling (4.5) and (3.12) we have

$$\begin{aligned}
\|(\mathcal{L}_i^\circ - \mathcal{L}_i)w_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}_+^{N+1})} \le C\varepsilon\Big(&\|\partial_y^2 w_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}_+^{N+1})} + \|yD_x^2 w_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}_+^{N+1})}\\
&+ \|\sqrt{y}\,\nabla_x\partial_y w_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}_+^{N+1})} + \|\nabla w_i\|_{L^p(\widetilde{U}_i\cap\mathbb{R}_+^{N+1})}\Big).
\end{aligned}$$

Hence, choosing a small $\varepsilon > 0$, we can get rid of the terms with the second order derivatives in (5.14). Moreover, Corollary 4.3 yields

$$\begin{aligned}
\|\nabla w_i\|_{L^p(\mathbb{R}_+^{N+1})} &\le C(\|u_i\|_{L^p(\Omega)} + \|\nabla u_i\|_{L^p(\Omega)}) \le C_i(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)})\\
&\le \frac{C}{|\lambda|^{1/4}}\|f\|_{L^p(\Omega)} \le \frac{C}{|\lambda|^{1/4}}\|f\|_{L^\infty(\Omega)}.
\end{aligned}$$

Because of (5.7), we can estimate

$$\|T_i(h_i)\|_{L^p(\mathbb{R}_+^{N+1})} \le C|\lambda|^{-1/4}\|f\|_{L^\infty(\Omega)}$$

in the same way. Since $p > 3N+2$, we can now deduce from (5.14) that

$$|\lambda|\,\|w_i\|_{L^\infty(\mathbb{R}_+^{N+1})} + |\lambda|^{\frac{1}{2}}\big(\|\partial_y w_i\|_{L^\infty(\mathbb{R}_+^{N+1})} + \|\sqrt{y}\,\nabla_x w_i\|_{L^\infty(\mathbb{R}_+^{N+1})}\big) \le C\|f\|_{L^\infty(\Omega)}.$$

It follows that

$$|\lambda|\,\|u_i\|_{L^\infty(\Omega\cap U_i)} + |\lambda|^{\frac12}\big(\|\mathbf{a}\cdot\nabla u_i\|_{L^\infty(\Omega\cap U_i)} + \|\sqrt{\varrho}\,\nabla u_i\|_{L^\infty(\Omega\cap U_i)}\big) \le C\|f\|_{L^\infty(\Omega)}.$$

Recalling (5.8), we conclude that $u, \mathbf{a}\cdot\nabla u, \sqrt{\varrho}\,\nabla u \in C(\overline{\Omega})$ and

$$|\lambda|\,\|u\|_{L^\infty(\Omega)} + |\lambda|^{\frac12}\big(\|\mathbf{a}\cdot\nabla u\|_{L^\infty(\Omega)} + \|\sqrt{\varrho}\,\nabla u\|_{L^\infty(\Omega)}\big) \le C\|f\|_{L^\infty(\Omega)}.$$

Finally, since $u, Lu \in L^q(\Omega)$ for every $1 < q < \infty$ and L is nondegenerate in the interior, local elliptic regularity implies that $u \in W^{2,q}_{\text{loc}}(\Omega)$, see e.g. [9, Lemma 9.16].

We have established that there is ω_0 such that for every $\text{Re}\lambda \ge \omega_0$ and $f \in C(\overline{\Omega})$, there exists a solution $u \in D_0(L)$ of $\lambda u + Lu = f$ satisfying $\|u\|_{L^\infty(\Omega)} \le C|\lambda|^{-1}\|f\|_{L^\infty(\Omega)}$ and (5.5). Now assume that $\lambda u + Lu = f$ holds for some $\lambda > 0$, $u \in D_0(L)$ and a real $f \in C(\overline{\Omega})$. Set $v = u - \lambda^{-1}\|f\|_\infty$. Then $\lambda v + Lv = f - \|f\|_\infty \le 0$ on Ω and $v \le 0$ on $\partial\Omega$. If $v(z_0) > 0$ for some $z_0 \in \Omega$, then v has a maximum $v(z_1) > 0$ in Ω. Since $v \in W^{2,q}_{\text{loc}}(\Omega)$ for any $q \in (1,\infty)$, we can apply Bony's maximum principle, [4, Theorem 1], which implies that $Lv(z_1) \ge 0$. This is impossible, and thus $u \le \lambda^{-1}\|f\|_\infty$. The same argument works for $-u$ and thus $|u| \le \lambda^{-1}\|f\|_\infty$ on Ω. This means that $(-L, D_0(L))$ is dissipative in $C(\overline{\Omega})$. Hence, $\lambda + L : D_0(L) \to C(\overline{\Omega})$ is invertible for all $\text{Re}\,\lambda > 0$ and $(-L, D_0(L))$ generates a contractive analytic semigroup $T_\infty(\cdot)$ on $C(\overline{\Omega})$.

By construction, the resolvents of $(-L, D_0(L))$ and $(-L, D_p(L))$ coincide on $C(\overline{\Omega})$ for all $p > 3N + 2$ and sufficiently large $\lambda > 0$. Taking into account Corollary 4.4, we conclude that $T_\infty(\cdot)$ is the restriction of the semigroups $T_p(\cdot)$ on $L^p(\Omega)$ generated by $(-L, D_p(L))$ for each $p \in (1,\infty)$. In particular, $T_\infty(\cdot)$ is positive. We further have seen that $D_0(L) \subset D_p(L)$ for $p > 3N+2$ so that $D_0(L)$ is embedded into $W^{1,p}(\Omega)$ for these p by Corollary 4.3, which in turn is compactly embedded into $C(\overline{\Omega})$. Hence, $T_\infty(t)$ is compact for each $t > 0$ because the semigroup is analytic.

Since $T_\infty(\cdot)$ is compact, positive and bounded, the exponential stability of $T_\infty(\cdot)$ is equivalent to the injectivity of $(-L, D_0(L))$. (Use e.g. Theorem VI.1.10 and Corollary IV.3.12 of [7].) Let $Lu = 0$ for some $u \in D_0(L)$. Take $\varepsilon > 0$ and a smoooth function $v > 0$ on $\overline{\Omega}$ such that $-Lv > 0$ on Ω (e.g., $v(z) = e^{sx_1} + \cdots + e^{sx_N} + e^{sy}$ for a large $s > 0$). If $u + \varepsilon v$ had a maximum $z_0 \in \Omega$, then $-L(u+\varepsilon v)(z_0) \le 0$ by [4] which is impossible. Hence, $u + \varepsilon v$ takes its maximum at the boundary. The same holds for the minimum. Letting $\varepsilon \to 0$, we deduce $u = 0$. □

Corollary 5.4 *Assume that (H1), (H2) and (H3) hold. Then the semigroup $T(\cdot)$ in $L^p(\Omega)$ for $p \in (1,\infty)$ constructed in Theorem 4.1 is exponentially stable and has the same spectrum and eigenspaces as its restriction to $C(\overline{\Omega})$.*

Proof. The second assertion can be shown as in Corollary 4.4. Thus the first assertion follows from Theorem 5.3. □

Corollary 5.5 *Assume that (H1), (H2) and (H3) hold. Then the semigroup $T(\cdot)$ in $C(\overline{\Omega})$ constructed in Theorem 5.3 leaves invariant $C_0(\Omega)$ and its restriction to $C_0(\Omega)$ is an analytic C_0–semigroup. Moreover, the restriction is contractive, positive, compact and exponentially stable.*

Proof. Since $C_c^\infty(\Omega) \subset D_0(L) \subset C_0(\Omega)$, the closure of $D_0(L)$ is $C_0(\Omega)$. Hence, $T(\cdot)$ leaves invariant $C_0(\Omega)$ and is strongly continuous on $C_0(\Omega)$. The other claims are then clear. □

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